Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

$(X \rightharpoonup Y) = \{ f \subseteq X \times Y \mid f \text{ is functional} \}$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Def $\underline{graph}(f) = \{(x, f(x)) \in X \times Y \mid x \in \underline{dom}(f)\}$

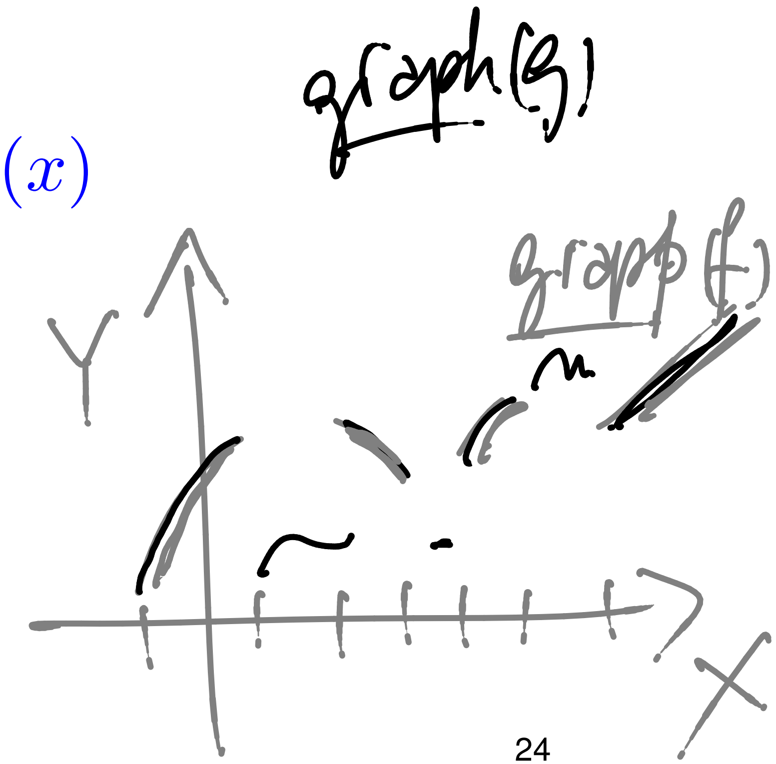
Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$



Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Example: $f_{[\![B]\!],[\![C]\!]} : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$

is monotone.

Least Elements

provide least amount of information on your data domain.

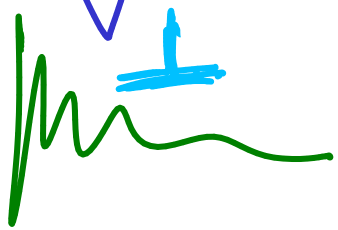
Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\ d \sqsubseteq x\ .$$

Every domain of computation has a least element.

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.



- Note also that a poset may not have least element.

<u>Example</u>: $(X \to Y), \sqsubseteq$

$\mathbb{N}, \leq$

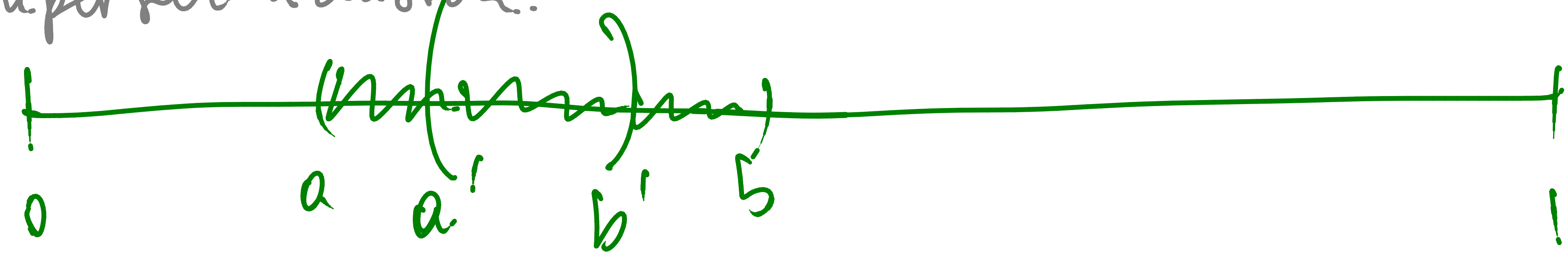
$\mathbb{Z}, \leq$ — does not have $\perp$

$[0,1], \leq$

$\mathbb{I}[0,1]$ underlying set consists of all those open intervals in $[0,1]$ (that is, (a,b) s.t. $a<b$)

with order given by superset inclusion.

$(a,b) \sqsubseteq (a',b')$



$[\![\text{while } B \text{ do } C]\!] =_{def} \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$

WANT

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

define it from the uniqueness property of least elements

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

what we are interested in

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \;\Rightarrow\; \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

Definition by (least) universal property.

Proof principle

$$(\mathit{lfp}\,2) \quad \frac{f(x) \sqsubseteq x}{\underline{\mathit{fix}}(f) \sqsubseteq x}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

<u>loop invariance</u>

$$\frac{f_{[\![B]\!],[\![C]\!]}([\![P]\!]) \sqsubseteq [\![P]\!]}{\underline{\mathit{fix}}(f_{[\![B]\!],[\![C]\!]}) = [\![\mathtt{while}\ B\ \mathtt{do}\ C]\!] \sqsubseteq [\![P]\!]}$$

Proof principle

1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Recall we wanted $[\![\text{while } B \text{ do } C]\!]$ to be a fixed point of $f_{[\![B]\!],[\![C]\!]}$. The definition

$$[\![\text{while } B \text{ do } C]\!] = \underline{fix}\left(f_{[\![B]\!],[\![C]\!]}\right)$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

achieves

$$f_{[\![B]\!][\![C]\!]}\left([\![\text{while } B \text{ do } C]\!]\right) \sqsubseteq [\![\text{while } B \text{ do } C]\!]$$

What about $\sqsupseteq$? So that we have $=$.

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

$$(\mathrm{lfp}2)\ \frac{f(x) \sqsubseteq x}{\mathrm{fix}(f) \sqsubseteq x}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\checkmark}{f(\mathrm{fix}\, f) \sqsubseteq \mathrm{fix}(f)}\ (\mathrm{lfp}1)}{f(f(\mathrm{fix}\, f)) \sqsubseteq f(\mathrm{fix}\, f)}\ (\mathrm{mon})}{\mathrm{fix}\, f \sqsubseteq f(\mathrm{fix}\, f)}\ (\mathrm{lfp}2) \qquad \dfrac{\checkmark}{f(\mathrm{fix}\, f) \sqsubseteq \mathrm{fix}\, f}\ (\mathrm{lfp}1)}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}\ (A)$$