Datatypes in PLC [Sect. 4.4]

- define a suitable PLC type for the data
- define suitable PLC expressions for values & operations on the data
- show PLC expressions have correct typings & computational behaviour

Example : finite lists [p48 →]

Iteratively defined functions on finite lists

$A^* \triangleq$ finite lists of elements of the set A

Notation :

empty list : $Nil \in A^*$

cons : $$\frac{x \in A \quad l \in A^*}{x :: l \in A^*}$$

Iteratively defined functions on finite lists

$A^* \triangleq$ finite lists of elements of the set A

Given a set B, an element $x' \in B$, and a function $f : A \to B \to B$, the *iteratively defined function* $\mathit{listIter}\, x'\, f$ is the unique function $g : A^* \to B$
satisfying:

$$g\, \mathit{Nil} = x'$$
$$g\, (x :: \ell) = f\, x\, (g\, \ell)$$

for all $x \in A$ and $\ell \in A^*$.

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$g\,\mathit{Nil} = x'$

$g(x_1 :: \mathit{Nil}) = f\, x_1\, x'$

$g(x_2 :: x_1 :: \mathit{Nil}) = f\, x_2\, (f\, x_1\, x')$

$g(x_n :: \cdots :: x_1 :: \mathit{Nil}) = f\, x_n\, (\cdots (f\, x_1\, x') \cdots)$

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For each $\ell \in A^*$
is a function

$x' f \longmapsto \mathit{listIter}\, x'\, f$

$B \to (A \to B \to B) \to B$

which is "polymorphic" in B (& A)

Polymorphic lists

$$\alpha\ \mathit{list} \triangleq \forall \alpha'\ (\alpha' \to (\alpha \to \alpha' \to \alpha') \to \alpha')$$

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List iteration in PLC

$$\mathit{iter} \triangleq \Lambda\alpha, \alpha'(\lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha'(\\ \lambda \ell : \alpha\ \mathit{list}\ (\ell\ \alpha'\ x'\ f)))$$

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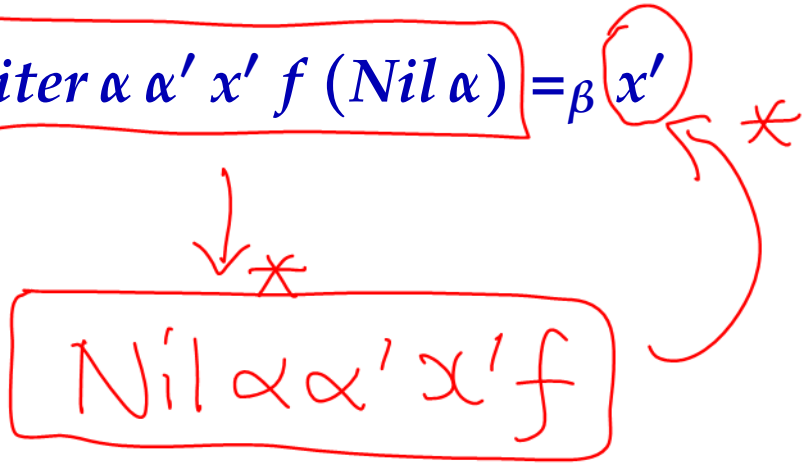

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$(Cons\ \alpha\ x\ \ell)\ \alpha'\ x'\ f \to^* f\ x\ (\ell\ \alpha'\ x'\ f)$

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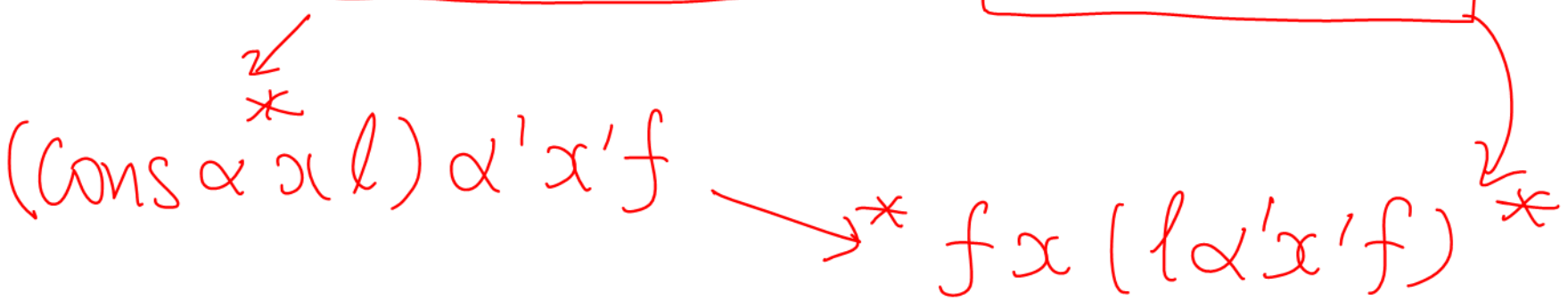

FACT Given a closed PLC type τ

$$\{\text{closed } \beta\text{-normal forms of type } \tau\text{list}\}$$
$$\cong$$
$$\{\text{closed } \beta\text{-normal forms of type } \tau\}^*$$

$$nil \longleftrightarrow \beta NF\,(Nil\,\tau)$$

$$N_1 :: nil \longleftrightarrow \beta NF(Cons\,\tau\,(N_1\,(Nil\,\tau)))$$

$$N_2 :: N_1 :: nil \longleftrightarrow \beta NF(Cons\,\tau\,(N_2\,(Cons\,\tau\,(N_1\,(Nil\,\tau)))))$$

etc

"Algebraic" data types in ML

datatype $(\alpha_1,\ldots,\alpha_n)$ alg = C_1 of τ_1 | ... | C_m of τ_m

types τ_1 --- τ_m built up from $\alpha_1 \ldots \alpha_n$ and the type $(\alpha_1 \ldots \alpha_n)$ alg using unit, _ * _ & previously declared alg. datatypes

"Algebraic" data types in ML

datatype $(\alpha_1, \ldots, \alpha_n)$ alg = C_1 of τ_1 | ... | C_m of τ_m

Eg.

datatype bool = T of unit | F of unit

datatype α list = Nil of unit |
Cons of $\alpha * \alpha$ list

E.g. of a non-algebraic ML datatype

```
datatype nTree = Leaf
               | Node of (nat -> nTree)
```

[Fig.5, p50]

PLC encodings of ML algebraic datatypes

ML	PLC
$\alpha_1 * \alpha_2$	$\forall\alpha((\alpha_1 \to \alpha_2 \to \alpha) \to \alpha)$
datatype (α_1,α_2)sum = Inl of α_1 | Inr of α_2	$\forall\alpha((\alpha_1 \to \alpha) \to (\alpha_2 \to \alpha) \to \alpha)$
datatype nat = Zero | Succ of nat	$\forall\alpha(\alpha \to (\alpha \to \alpha) \to \alpha)$
datatype binTree = Leaf | Node of binTree * binTree	$\forall\alpha(\alpha \to (\alpha \to \alpha \to \alpha) \to \alpha)$
unit	$\forall\alpha(\alpha \to \alpha)$

[Fig.5, p50]

PLC encodings of ML algebraic datatypes

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datatype binTree = Leaf | Node of binTree * binTree	$\forall\alpha(\alpha \to (\alpha \to \alpha \to \alpha) \to \alpha)$
unit	$\forall\alpha(\alpha \to \alpha)$

Standard ML signatures and structures

```
signature QUEUE =
  sig
    type 'a queue
    exception Empty
    val empty : 'a queue
    val insert : 'a * 'a queue -> 'a queue
    val remove : 'a queue -> 'a * 'a queue
end

structure Queue =
  struct
    type 'a queue = 'a list * 'a list
    exception Empty
    val empty = (nil, nil)
    fun insert (f, (front,back)) = (f::front, back)
    fun remove (nil, nil) = raise Empty
      | remove (front, nil) = remove (nil, rev front)
      | remove (front, b::back) = (b, (front, back))
  end
```

PLC + existential types

Types

$t ::= \cdots \mid \exists\, \alpha\, (\tau)$

Expressions

$M ::= \cdots \mid \mathtt{pack}\,(\tau, M) : \exists\, \alpha\, (\tau) \mid \mathtt{unpack}\, M\, \mathtt{as}\, (\alpha, x)\, \mathtt{in}\, M : \tau$

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Typing rules

$$(\exists\textbf{intro})\ \frac{\Gamma \vdash M : \tau[\tau'/\alpha]}{\Gamma \vdash (\mathtt{pack}\,(\tau', M) : \exists\, \alpha\, (\tau)) : \exists\, \alpha\, (\tau)}$$

$$(\exists\textbf{elim})\ \frac{\Gamma \vdash E : \exists\, \alpha\, (\tau) \qquad \Gamma, x : \tau \vdash M' : \tau'}{\Gamma \vdash (\mathtt{unpack}\, E\, \mathtt{as}\, (\alpha, x)\, \mathtt{in}\, M' : \tau') : \tau'}\ \text{if } \alpha \notin ftv(\Gamma, \tau')$$

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Reduction

$$\mathtt{unpack}\,(\mathtt{pack}\,(\tau', M) : \exists\, \alpha\, (\tau))\, \mathtt{as}\, (\alpha, x)\, \mathtt{in}\, M' : \tau' \rightarrow M'[\tau'/\alpha, M/x]$$

Existential types in PLC

$$\exists\,\alpha\,(\tau) \triangleq \forall\beta\,((\forall\alpha\,(\tau \to \beta)) \to \beta)$$

$$\texttt{pack}\,(\tau', M) : \exists\,\alpha\,(\tau) \triangleq \Lambda\beta\,(\lambda y : \forall\alpha\,(\tau \to \beta)\,(y\,\tau'\,M))$$

$$\texttt{unpack}\,E\,\texttt{as}\,(\alpha, x)\,\texttt{in}\,M' : \tau' \triangleq E\,\tau'\,(\Lambda\alpha\,(\lambda x : \tau\,(M')))$$

(where $\beta \notin ftv(\alpha\,\tau\,\tau'\,M\,M')$)

Existential types in PLC

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(where $\beta \notin ftv(\alpha\,\tau\,\tau'\,M\,M')$)

These definitions satisfy the typing and reduction rules on the previous slide (exercise).