Mini-ML type-inference problem:

(b) given Γ & M, does there exist σ such that $\Gamma \vdash M : \sigma$ holds?

Two examples involving self-application

$$M \triangleq \texttt{let}\ f = \lambda x_1\,(\lambda x_2\,(x_1))\ \texttt{in}\ f\ f$$

$$M' \triangleq (\lambda f\,(f\ f))\ \lambda x_1\,(\lambda x_2\,(x_1))$$

Are M and M' typeable in the Mini-ML type system?

(in the empty typing environment)

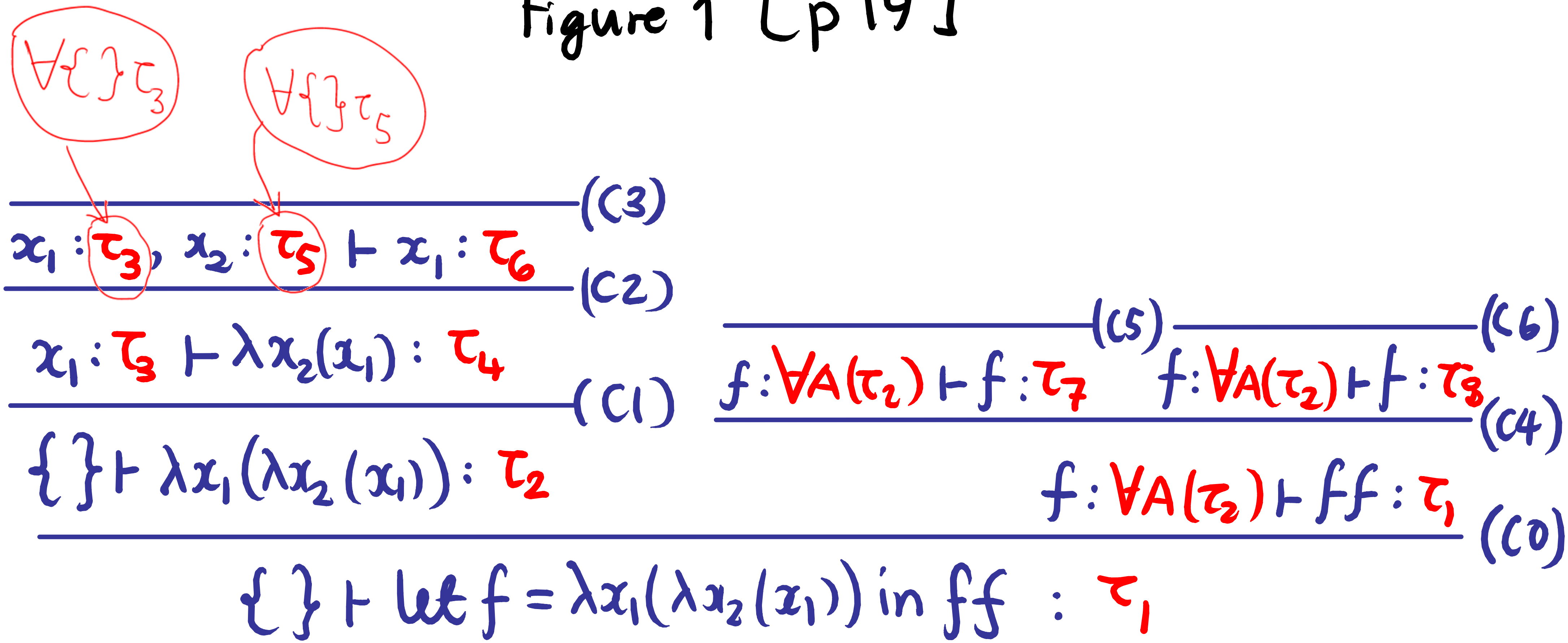
Figure 1 [p15]

$$\dfrac{\dfrac{\dfrac{}{x_1 : \quad , x_2 : \quad \vdash x_1 :}\ (C3)}{x_1 : \quad \vdash \lambda x_2(x_1) :}\ (C2)}{\{\} \vdash \lambda x_1(\lambda x_2(x_1)) :}\ (C1)$$

$$\dfrac{\dfrac{}{f : \quad \vdash f :}\ (C5) \qquad \dfrac{}{f : \quad \vdash f :}\ (C6)}{f : \quad \vdash f f :}\ (C4)$$

$$\dfrac{\{\} \vdash \lambda x_1(\lambda x_2(x_1)) : \qquad f : \quad \vdash f f :}{\{\} \vdash \text{let } f = \lambda x_1(\lambda x_2(x_1)) \text{ in } f f \ :}\ (C0)$$

Figure 1 [p 19]


∀{}τ3
∀{}τ5
(C3)
x1 : τ3, x2 : τ5 ⊢ x1 : τ6
(C2)
x1 : τ3 ⊢ λx2(x1) : τ4
(C1)
{} ⊢ λx1(λx2(x1)) : τ2
(C5)
(C6)
f : ∀A(τ2) ⊢ f : τ7
f : ∀A(τ2) ⊢ f : τ8
(C4)
f : ∀A(τ2) ⊢ ff : τ1
(C0)
{} ⊢ let f = λx1(λx2(x1)) in ff : τ1

Constraints generated while inferring a type for
$\texttt{let}\ f = \lambda x_1\,(\lambda x_2\,(x_1))\ \texttt{in}\ f\ f$

$$A = ftv(\tau_2) \qquad \text{(C0)}$$

$$\tau_2 = \tau_3 \to \tau_4 \qquad \text{(C1)}$$

$$\tau_4 = \tau_5 \to \tau_6 \qquad \text{(C2)}$$

$$\forall \{\,\}\,(\tau_3) \succ \tau_6, \text{ i.e. } \tau_3 = \tau_6 \qquad \text{(C3)}$$

$$\tau_7 = \tau_8 \to \tau_1 \qquad \text{(C4)}$$

$$\forall A\,(\tau_2) \succ \tau_7 \qquad \text{(C5)}$$

$$\forall A\,(\tau_2) \succ \tau_8 \qquad \text{(C6)}$$

$$\tau_2 \overset{(C1)}{=} \tau_3 \to \tau_4 \overset{(C2)}{=} \tau_3 \to (\tau_5 \to \tau_6) \overset{(C3)}{=} \tau_6 \to (\tau_5 \to \tau_6)$$

$$\tau_2 \overset{(C1)}{=} \tau_3 \to \tau_4 \overset{(C2)}{=} \tau_3 \to (\tau_5 \to \tau_6) \overset{(C3)}{=} \tau_6 \to (\tau_5 \to \tau_6)$$

Take $\left.\begin{array}{l} \tau_6 = \alpha_1 \\ \tau_5 = \alpha_2 \end{array}\right\}$ type variables.

So $A = ftv(\tau_2) = ftv(\alpha_1 \to (\alpha_2 \to \alpha_1)) = \{\alpha_1, \alpha_2\}$

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(C5) : $\forall\{\alpha_1, \alpha_2\}(\alpha_1 \to (\alpha_2 \to \alpha_1)) > \tau_7 \overset{(C4)}{=} \tau_8 \to \tau_1$

(C6) : " " " $> \tau_8$

so $\begin{cases} \tau_8 \to \tau_1 = \tau_9 \to (\tau_{10} \to \tau_9) \\ \tau_8 = \tau_{11} \to (\tau_{12} \to \tau_{11}) \end{cases}$ for some $\tau_9, \tau_{10}, \tau_{11}, \tau_{12}$

$$\tau_2 \overset{(C1)}{=\!=} \tau_3 \to \tau_4 \overset{(C2)}{=\!=} \tau_3 \to (\tau_5 \to \tau_6) \overset{(C3)}{=\!=} \tau_6 \to (\tau_5 \to \tau_6)$$

Take $\left.\begin{array}{l} \tau_6 = \alpha_1 \\ \tau_5 = \alpha_2 \end{array}\right\}$ type variables.

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So $\left\{\begin{array}{l} \tau_8 = \tau_9 \ \& \ \tau_1 = (\tau_{10} \to \tau_9) \\ \tau_8 = \tau_{11} \to (\tau_{12} \to \tau_{11}) \end{array}\right.$ for some $\tau_9, \tau_{10}, \tau_{11}, \tau_{12}$

So $\tau_1 = \tau_{10} \to \tau_9 = \tau_{10} \to \tau_8 = \tau_{10} \to (\tau_{11} \to (\tau_{12} \to \tau_{11}))$

Thus

$$\{\} \vdash (\text{let } f = \lambda x_1(\lambda x_2(x_1)) \text{ in } f f) : \tau_{10} \to (\tau_{11} \to (\tau_{12} \to \tau_{11}))$$

holds for any $\tau_{10}, \tau_{11}, \tau_{12}$

So

$$\vdash (\text{let } f = \lambda x_1(\lambda x_2(x_1)) \text{ in } f f) : \forall \alpha_1, \alpha_2, \alpha_3 (\alpha_1 \to (\alpha_2 \to (\alpha_3 \to \alpha_2)))$$

Two examples involving self-application

$$M \triangleq \texttt{let } f = \lambda x_1 \, (\lambda x_2 \, (x_1)) \texttt{ in } f \, f$$

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Are M and M' typeable in the Mini-ML type system?

[Page 17]

The constraints generated from trying to type

$$(\lambda f(f f))\ \lambda x_1(\lambda x_2(x_1))$$

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$$\tau_7 \overset{(C13)}{=} \tau_4 \overset{(C12)}{=} \tau_6 \overset{(C11)}{=} \tau_7 \to \tau_5$$

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these cannot be equal – they have different numbers of the symbol "→" in them

Two examples involving self-application

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This is not typeable

Principal type schemes for closed expressions

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eg $\forall \alpha_1, \alpha_2, \alpha_3\, (\alpha_1 \to (\alpha_2 \to (\alpha_3 \to \alpha_2)))$ is principal type scheme

let $f = \lambda x_1 (\lambda x_2 (x_1))$ in $f f$

Theorem (Hindley; Damas-Milner)

Theorem. If the closed Mini-ML expression M is typeable (i.e. $\vdash M : \sigma$ holds for some type scheme σ), then there is a principal type scheme for M.

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Theorem. If the closed Mini-ML expression M is typeable (i.e. $\vdash M : \sigma$ holds for some type scheme σ), then there is a principal type scheme for M.

Indeed, there is an algorithm which, given any closed Mini-ML expression M as input, decides whether or not it is typeable and returns a principal type scheme if it is.

An ML expression with a principal type scheme hundreds of pages long

$$
\begin{array}{l}
\texttt{let}\ pair = \lambda x\,(\lambda y\,(\lambda z\,(z\,x\,y)))\ \texttt{in} \\
\quad \texttt{let}\ x_1 = \lambda y\,(pair\,y\,y)\ \texttt{in} \\
\quad\quad \texttt{let}\ x_2 = \lambda y\,(x_1(x_1\,y))\ \texttt{in} \\
\quad\quad\quad \texttt{let}\ x_3 = \lambda y\,(x_2(x_2\,y))\ \texttt{in} \\
\quad\quad\quad\quad \texttt{let}\ x_4 = \lambda y\,(x_3(x_3\,y))\ \texttt{in} \\
\quad\quad\quad\quad\quad \texttt{let}\ x_5 = \lambda y\,(x_4(x_4\,y))\ \texttt{in} \\
\quad\quad\quad\quad\quad\quad x_5(\lambda y\,(y))
\end{array}
$$

Unification of ML types

There is an algorithm mgu which when input two Mini-ML types τ_1 and τ_2 decides whether τ_1 and τ_2 are *unifiable*, i.e. whether there exists a type-substitution $S \in \mathbf{Sub}$ with

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By convention $mgu(\tau_1, \tau_2) = FAIL$ if (and only if) τ_1 and τ_2 are not unifiable.

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(For type schemes σ and σ', with $\sigma' = \forall A'\,(\tau')$ say, we define $\boxed{\sigma \succ \sigma'}$ to mean $A' \cap ftv(\sigma) = \{\}$ and $\sigma \succ \tau'$.)

Example typing problem and solutions

Typing problem

$$\boxed{x : \forall\alpha\,(\beta \to (\gamma \to \alpha)) \vdash x\,\mathtt{true} : ?}$$

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- $S_4 = \{\beta \mapsto bool, \gamma \mapsto bool\}, \sigma_3 = \forall\{\,\}\,(bool \to bool)$

Both (S_1, σ_1) and (S_2, σ_2) are in fact principal solutions.

Properties of the Mini-ML typing relation with respect to substitution and type scheme specialisation

- If $\Gamma \vdash M : \sigma$, then for any type substitution $S \in \mathbf{Sub}$

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Requirements for a principal typing algorithm, pt

pt operates on typing problems $\Gamma \vdash M : ?$ (consisting of a typing environment Γ and a Mini-ML expression M).

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- If $\Gamma \vdash M : ?$ has no solution, then $pt(\Gamma \vdash M : ?)$ returns $FAIL$.

How the principal typing algorithm *pt* works

$$pt(\Gamma \vdash M : ?) = (S, \tau) \mid FAIL$$

- Call *pt* recursively following the structure of M and guided by the typing rules, bottom-up.
- Thread substitutions sequentially and compose them together when returning from a recursive call.
- When types need to agree to satisfy a typing rule, use *mgu* (and *pt* returns *FAIL* only if *mgu* does).
- When types are unknown, generate a fresh type variable.

Some of the clauses in a definition of $\boldsymbol{pt}$

Function abstractions: $pt(\Gamma \vdash \lambda x\,(M) : ?) \triangleq$
$\quad \textbf{let } \alpha = \textbf{fresh in}$
$\quad \textbf{let } (S, \tau) = pt(\Gamma, x : \alpha \vdash M : ?) \textbf{ in } (S, S(\alpha) \rightarrow \tau)$

Some of the clauses in a definition of *pt*

Function abstractions: $pt(\Gamma \vdash \lambda x\,(M) : ?) \triangleq$

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Function applications: $pt(\Gamma \vdash M_1\,M_2 : ?) \triangleq$

$$\begin{array}{l}\textbf{let } (S_1, \tau_1) = pt(\Gamma \vdash M_1 : ?) \textbf{ in}\\ \textbf{let } (S_2, \tau_2) = pt(S_1\,\Gamma \vdash M_2 : ?) \textbf{ in}\\ \textbf{let } \alpha = \textbf{fresh in}\\ \textbf{let } S_3 = mgu(S_2\,\tau_1, \tau_2 \to \alpha) \textbf{ in } (S_3 S_2 S_1, S_3(\alpha))\end{array}$$

Mini-ML type system, III

$$(\mathbf{fn})\ \frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x\,(M) : \tau_1 \to \tau_2}\ \text{if } x \notin dom(\Gamma)$$

$$(\mathbf{app})\ \frac{\Gamma \vdash M : \tau_1 \to \tau_2 \qquad \Gamma \vdash N : \tau_1}{\Gamma \vdash M\,N : \tau_2}$$

$$pt(\Gamma \vdash M_1 : ?) = (S_1, \tau_1)$$

$$S_1 \Gamma \vdash M_1 : \tau_1$$

$$\hookrightarrow pt(\Gamma' \vdash M_1 M_2 : ?) =$$

$pt(\Gamma \vdash M_1 : ?) = (S_1, \tau_1)$

$pt(S_1 \Gamma \vdash M_2 : ?) = (S_2, \tau_2)$

+slide 28

$S_2 S_1 \Gamma \vdash M_1 : S_2 \tau_1$

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$\hookrightarrow pt(\Gamma' \vdash M_1 M_2 : ?) =$

$pt(\Gamma \vdash M_1 : ?) = (S_1, \tau_1)$

$pt(S_1\Gamma \vdash M_2 : ?) = (S_2, \tau_2)$

$mgu(S_2\tau_1, \tau_2 \to \alpha) = S_3$

+ slide 28

$S_3\tau_2 \to S_3\alpha$

$\|$

$S_3S_2S_1\Gamma \vdash M_1 : S_3S_2\tau_1$

$S_3S_2S_1\Gamma \vdash M_2 : S_3\tau_2$

$\hookrightarrow pt(\Gamma \vdash M_1M_2 : ?) = (S_3S_2S_1, S_3\alpha)$

$pt(\Gamma \vdash M_1 : ?) = (S_1, \tau_1)$

$pt(S_1\Gamma \vdash M_2 : ?) = (S_2, \tau_2)$

$mgu(S_2\tau_1, \tau_2 \to \alpha) = S_3$

$$S_3 S_2 \tau_1 = S_3\tau_2 \to S_3\alpha$$

$$\frac{S_3 S_2 S_1 \Gamma \vdash M_1 : S_3\tau_2 \to S_3\alpha \qquad S_3 S_2 S_1 \Gamma' \vdash M_2 : S_3\tau_2}{S_3 S_2 S_1 \Gamma' \vdash M_1 M_2 : S_3\alpha}\ (app)$$

$pt(\Gamma' \vdash M_1 M_2 : ?) = (S_3 S_2 S_1,\ S_3\alpha)$