# **Types**

#### 12 lectures for CST Part II by Andrew Pitts

{www.cl.cam.ac.uk/teaching/1516/Types/>

"One of the most helpful concepts in the whole of programming is the notion of type, used to classify the kinds of object which are manipulated. A significant proportion of programming mistakes are detected by an implementation which does type-checking before it runs any program. Types provide a taxonomy which helps people to think and to communicate about programs."

R. Milner, Computing Tomorrow (CUP, 1996), p264

"The fact that companies such as Microsoft, Google and Mozilla are investing heavily in systems programming languages with stronger type systems is not accidental – it is the result of decades of experience building and deploying complex systems written in languages with weak type systems."

> T. Ball and B. Zorn, *Teach Foundational Language Principles*, Viewpoints, Comm. ACM (2014) 58(5) 30–31

Type systems channel TCS into PLS & Verification

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type systems as checkable documentation of programmer intentions

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  goes back to FORTRAN.

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Whole-language safety.
 PL "meta-theory" - properties of all legal progs
 E.g. §4 of this course
 Requires formal math/logic methods

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# Formal type systems

- Constitute the precise, mathematical characterisation of informal type systems (such as occur in the manuals of most typed languages.)
- Basis for type soundness theorems: "any well-typed program cannot produce run-time errors (of some specified kind)."
- Can decouple specification of typing aspects of a language from algorithmic concerns: the formal type system can define typing independently of particular implementations of type-checking algorithms.

# Typical type system judgement

is a relation between typing environments ( $\Gamma$ ), program phrases (e) and type expressions ( $\tau$ ) that we write as

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We consider *structural* type systems, in which there is a language of type expressions built up using type constructs (e.g. *int list*  $\rightarrow$  *int* in ML).

(By contrast, in *nominal* type systems, type expressions are just unstructured names.)

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C/Java-style:

bar foo

# Type checking, typeability, and type inference

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- ► Type-checking problem: given Γ, e, and τ, is Γ ⊢ e : τ derivable in the type system?
- *Typeability* problem: given  $\Gamma$  and e, is there any  $\tau$  for which  $\Gamma \vdash e : \tau$  is derivable in the type system?

Solving the second problem usually involves devising a *type inference algorithm* computing a  $\tau$  for each  $\Gamma$  and e (or failing, if there is none).

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**Type preservation.** If  $\Gamma \vdash e : \tau$  and  $dom(\Gamma) \subseteq dom(s)$  and  $\langle e, s \rangle \rightarrow \langle e', s' \rangle$ , then  $\Gamma \vdash e' : \tau$  and  $dom(\Gamma) \subseteq dom(s')$ .

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Hence well-typed programs don't get stuck: **Safety.** If  $\Gamma \vdash e : \tau$ ,  $dom(\Gamma) \subseteq dom(s)$  and  $\langle e, s \rangle \rightarrow^* \langle e', s' \rangle$ , then either e' is a value, or there exist e'', s'' such that  $\langle e', s' \rangle \rightarrow \langle e'', s'' \rangle$ .

#### Outline of the rest of the course

- ► ML polymorphism. Principal type schemes and type inference. [2]
- Polymorphic reference types. The pitfalls of combining ML polymorphism with reference types. [1]
- Polymorphic lambda calculus (PLC). Explicit versus implicitly typed languages. PLC syntax and reduction semantics. Examples of datatypes definable in the polymorphic lambda calculus. [3]
- Dependent types. Dependent function types. Pure type systems. System F-omega. [2]
- Propositions as types. Example of a non-constructive proof. The Curry-Howard correspondence between intuitionistic second-order propositional calculus and PLC. The calculus of Constructions. Inductive types. [3]

new material this year

 Overloading (or ad hoc polymorphism): same symbol denotes operations with unrelated implementations. (E.g. + might mean both integer addition and string concatenation.)

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- Parametric polymorphism (generics): same expression belongs to a family of structurally related types.
   E.g. in Standard ML, length function

fun length nil = 0 | length(x::xs) = 1 + (length xs)

has type  $\tau list \rightarrow int$  for all types  $\tau$ .

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we introduce *type variables*  $\alpha$  (i.e. variables for which types may be substituted) and write

*length* :  $\forall \alpha \ (\alpha \ list \rightarrow int)$ .

 $\forall \alpha \ (\alpha \ list \rightarrow int)$  is an example of a *type scheme*.

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Overall, the expression has type *bool list*.

Forms of hypothesis in typing judgements

► *Ad hoc* (overloading):

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if f: bool \rightarrow bool
and f: bool list \rightarrow bool list,
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Parametric:

if  $f: \forall \alpha \ (\alpha \rightarrow \alpha)$ , then (f true) :: (f nil) : boollist.

Appropriate if expression behaviour is uniform for different type instantiations.

ML uses parametric hypotheses (type schemes) in its typing judgements.

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and maps each  $x_i$  to the type scheme  $\sigma_i$  for  $i = 1 \dots n$ .)

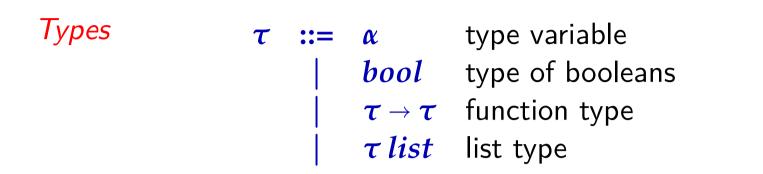
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- M is a Mini-ML expression
- au is a Mini-ML type.



Types $\tau ::= \alpha$ type variable|booltype of booleans| $\tau \rightarrow \tau$ function type| $\tau list$ list type

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When  $A = \{\alpha_1, \dots, \alpha_n\}$  (mutually distinct type variables) we write  $\forall A(\tau)$  as  $\forall \alpha_1, \dots, \alpha_n(\tau)$ .

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When  $A = \{\}$  is empty, we write  $\forall A(\tau)$  just as  $\tau$ . In other words, we regard the set of types as a subset of the set of type schemes by identifying the type  $\tau$  with the type scheme  $\forall \{\}(\tau)$ .