Last time

$$\Gamma \vdash M : ?$$

This time

$$\Gamma \vdash A$$

A suggestive notation

$$A \to B$$
$$\forall \alpha . A$$
$$\exists \alpha . A$$

A suggestive notation

$$A \to B$$
$$\forall \alpha . A$$
$$\exists \alpha . A$$
$$A \times B$$
$$A + B$$

A suggestive notation

$$A \to B$$
$$\forall \alpha . A$$
$$\exists \alpha . A$$
$$A \wedge B$$
$$A \vee B$$

A suggestive notation

$$A \to B$$
$$\forall \alpha . A$$
$$\exists \alpha . A$$
$$A \wedge B$$
$$A \vee B$$

Types *correspond to* **propositions**

A suggestive notation

$$A \to B$$
$$\forall \alpha . A$$
$$\exists \alpha . A$$
$$A \wedge B$$
$$A \vee B$$

Types *correspond to* **propositions**

(Part 1 of the **Curry-Howard** correspondendence)

What logic?

$\lambda^{\rightarrow}$

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

System F

$$\forall \alpha . A \qquad \exists \alpha . A$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

System F corresponds to **second-order propositional logic**

$$\forall \alpha . A \qquad \exists \alpha . A$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

System F corresponds to **second-order propositional logic**

$$\forall\alpha.A \qquad \exists\alpha.A$$

System Fω

$$\lambda\alpha.A \qquad A\ B$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

System F corresponds to **second-order propositional logic**

$$\forall \alpha . A \qquad \exists \alpha . A$$

System Fω corresponds to **higher-order propositional logic**

$$\lambda \alpha . A \qquad A\ B$$

What logic?

$\lambda^{\rightarrow}$ corresponds to **propositional logic**

$$\mathcal{B} \qquad A \rightarrow B \qquad A \wedge B \qquad A \vee B$$

System F corresponds to **second-order propositional logic**

$$\forall \alpha.A \qquad \exists \alpha.A$$

System Fω corresponds to **higher-order propositional logic**

$$\lambda \alpha.A \qquad A\ B$$

What about **first-order logic**?

Propositional vs predicate

Propositional logic

$$P \to Q$$

$$(\forall P. P \to P) \to (\exists Q. Q \to Q)$$

Predicate logic (FOPL)

$$P(x)$$

$$\forall x \in A. P(x)$$

Lambda and logic cubes

$$
\begin{tikzcd}
 & F\omega \arrow[rr] & & \lambda C \\
F \arrow[ur] \arrow[rr] & & \lambda P2 \arrow[ur] & \\
 & \lambda\underline{\omega} \arrow[uu] \arrow[rr] & & \lambda P\underline{\omega} \arrow[uu] \\
\lambda^{\to} \arrow[uu] \arrow[ur] \arrow[rr] & & \lambda P \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Lambda and logic cubes

$$
\begin{tikzcd}
 & F\omega \arrow[rr] & & \lambda C \\
F \arrow[ur] \arrow[rr] & & \lambda P2 \arrow[ur] & \\
 & \lambda\underline{\omega} \arrow[uu] \arrow[rr] & & \lambda P\underline{\omega} \arrow[uu] \\
\lambda^{\to} \arrow[uu] \arrow[ur] \arrow[rr] & & \lambda P \arrow[uu] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{PROP}\omega \arrow[rr] & & \mathrm{PRED}\omega \\
\mathrm{PROP}2 \arrow[ur] \arrow[rr] & & \mathrm{PRED}2 \arrow[ur] & \\
 & \mathrm{PROP}\underline{\omega} \arrow[uu] \arrow[rr] & & \mathrm{PRED}\underline{\omega} \arrow[uu] \\
\mathrm{PROP} \arrow[uu] \arrow[ur] \arrow[rr] & & \mathrm{PRED} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

More suggestive notation

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

More suggestive notation

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \qquad\qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

More suggestive notation

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \qquad\qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Terms *correspond to* **proofs**

More suggestive notation

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \qquad\qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Terms *correspond to* **proofs**

(Part 2 of the **Curry-Howard** correspondendence)

Inference rules for $\to$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{tvar} \qquad\qquad \frac{A \in \Gamma}{\Gamma \vdash A}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B}\ \to\text{-intro} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}\ \to\text{-elim} \qquad\qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Inference rules for $\times$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B}\ \times\text{-intro} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{fst}\ M : A}\ \times\text{-elim-1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim-1}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{snd}\ M : B}\ \times\text{-elim-2} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim-2}$$

Classical vs intuitionistic logic

Classical logic

Emphasis on **truth**

Truth values: $\top$, $\bot$

$A \vee \neg A$ always holds

Intuitionistic logic

Emphasis on **proof**

Proofs inhabit propositions

$A \vee \neg A$ doesn't hold in general

Brouwer-Heyting-Kolmogorov (BHK) interpretation

A proof of $A \to B$:

a function that builds a proof of B from a proof of A.

A proof of $A \wedge B$:

a pair of a proof of A and a proof of B.

$\neg A$

means $A \to \bot$

$\bot$

has no proof

Continuing the correspondence

Types *correspond to* **propositions**

Programs *correspond to* **proofs**

Continuing the correspondence

Types *correspond to* **propositions**

Programs *correspond to* **proofs**

Evaluation *corresponds to* **proof simplification**

Continuing the correspondence

Types *correspond to* **propositions**

Programs *correspond to* **proofs**

Evaluation *corresponds to* **proof simplification**

(The three-part **Curry-Howard** correspondendence)

Who should care?

Language designers

e.g. *linear logic*: restrictions on structural rules
corresponds to a language with resource management guarantees

Logicians

since results about programming languages transfer "for free"
e.g. strong normalization implies consistency

Authors (and users) of proof assistants

e.g. Coq and other tools based on type theory

Programmers?

Logical equivalences

$$\forall\beta.(\forall\alpha.(P\alpha \to \beta)) \to \beta \qquad \leftrightarrow \qquad \exists\alpha.P\alpha$$

$$\forall\beta.(P \to \beta) \wedge (Q \to \beta) \to \beta \qquad \leftrightarrow \qquad P \vee Q$$

Proof: we must show

$$\forall\beta.(\forall\alpha.(P\alpha \to \beta)) \to \beta \vdash \exists\alpha.P\alpha$$

$$\exists\alpha.P\alpha \vdash \forall\beta.(\forall\alpha.(P\alpha \to \beta)) \to \beta$$

etc.

A proof

Let $\Gamma = \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta$. Then

$$
\dfrac{
\dfrac{\dfrac{}{\Gamma \vdash \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta}}{\Gamma \vdash (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim}
\qquad
\dfrac{\dfrac{\dfrac{}{\Gamma, \alpha, P\alpha \vdash \exists\alpha.P\alpha}\ \exists\text{-intro}}{\Gamma, \alpha \vdash P\alpha \to \exists\alpha.P\alpha}\ \to\text{-intro}}{\Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}\ \forall\text{-intro}
}{\Gamma \vdash \exists\alpha.P\alpha}\ \to\text{-elim}
$$

A program from a proof

Let $\Gamma = \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta$. Then

$$\dfrac{\dfrac{\overline{\Gamma \vdash \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta}}{\Gamma \vdash (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim} \qquad \dfrac{\dfrac{\overline{\Gamma, \alpha, P\alpha \vdash \exists\alpha.P\alpha}\ \exists\text{-intro}}{\Gamma, \alpha \vdash P\alpha \to \exists\alpha.P\alpha}\ {\to}\text{-intro}}{\Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}\ \forall\text{-intro}}{\Gamma \vdash \exists\alpha.P\alpha}\ {\to}\text{-elim}$$

Right subtree:

$$\dfrac{\dfrac{\overline{\Gamma, \alpha, P\alpha \vdash \exists\alpha.P\alpha}\ \exists\text{-intro}}{\Gamma, \alpha \vdash P\alpha \to \exists\alpha.P\alpha}\ {\to}\text{-intro}}{\Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}\ \forall\text{-intro}$$

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$$

$$
\dfrac{\Gamma \vdash (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha \qquad \Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}{\Gamma \vdash \exists\alpha.P\alpha}\ {\to}\text{-elim}
$$

Right subtree:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, \alpha, v : P\alpha \vdash \text{pack } \alpha, v \text{ as } \exists\alpha.P\alpha : \exists\alpha.P\alpha}\ \exists\text{-intro}}{\Gamma, \alpha \vdash P\alpha \to \exists\alpha.P\alpha}\ {\to}\text{-intro}}{\Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}\ \forall\text{-intro}
$$

A program from a proof

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Left subtree:

$$\dfrac{\dfrac{}{\Gamma \vdash \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta}}{\Gamma \vdash (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim}$$

A program from a proof

Let $\Gamma = H : \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta$. Then

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta}}{\Gamma \vdash (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \alpha, P\alpha \vdash \exists\alpha.P\alpha}\ \exists\text{-intro}}{\Gamma, \alpha \vdash P\alpha \to \exists\alpha.P\alpha}\ \to\text{-intro}}{\Gamma \vdash \forall\alpha.P\alpha \to \exists\alpha.P\alpha}\ \forall\text{-intro}}{\Gamma \vdash \exists\alpha.P\alpha}\ \to\text{-elim}$$

Right subtree:

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Right subtree:

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Left subtree:

$$\dfrac{\dfrac{}{\Gamma \vdash H : \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta}}{\Gamma \vdash H\,[\exists\alpha.V\alpha] : (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim}$$

A program from a proof

Let $\Gamma = H : \forall\beta.(\forall\alpha.P\alpha \to \beta) \to \beta$. Then

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$$\dfrac{\dots}{\Gamma \vdash H\ [\exists\alpha.V\alpha] : (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha}\ \forall\text{-elim}$$

Finally:

$$\dfrac{\begin{array}{c}\Gamma \vdash H\ [\exists\alpha.V\alpha] : (\forall\alpha.P\alpha \to \exists\alpha.P\alpha) \to \exists\alpha.P\alpha \\ \Gamma \vdash \Lambda\alpha.\lambda v : P\alpha.\text{pack } \alpha, v \text{ as } \exists\alpha.P\alpha : \forall\alpha.P\alpha \to \exists\alpha.P\alpha\end{array}}{\Gamma \vdash \exists\alpha.P\alpha}\ \to\text{-elim}$$

A program from a proof

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Is it useful?

$$\forall\beta.(P \to \beta) \wedge (Q \to \beta) \to \beta \qquad \leftrightarrow \qquad P \vee Q$$

These type equivalences can be useful in constructing programs.

The data type encodings we saw last week can be derived this way.

Closing thoughts

The correspondence suggests a way of thinking about programming
— and a way of systematically constructing (some) programs

However, propositional logic is quite weak
(and our types are often uninformative)

We'll have richer types available later (GADTs, monads),
at which point we'll revisit the question of usefulness

Next time

Abstraction