

## Last time

$\Gamma \vdash M : ?$

This time

$$\Gamma \vdash A$$

## A suggestive notation

$A \rightarrow B$

$\forall \alpha. A$

$\exists \alpha. A$

## A suggestive notation

$A \rightarrow B$

$\forall \alpha. A$

$\exists \alpha. A$

$A \times B$

$A + B$

## A suggestive notation

$A \rightarrow B$

$\forall \alpha. A$

$\exists \alpha. A$

$A \wedge B$

$A \vee B$

## A suggestive notation

$A \rightarrow B$

$\forall \alpha. A$

$\exists \alpha. A$

$A \wedge B$

$A \vee B$

**Types correspond to propositions**

## A suggestive notation

$A \rightarrow B$

$\forall \alpha . A$

$\exists \alpha . A$

$A \wedge B$

$A \vee B$

**Types correspond to propositions**

(Part 1 of the **Curry-Howard** correspondence)

# What logic?

$\lambda^\rightarrow$

$\mathcal{B}$        $A \rightarrow B$        $A \wedge B$        $A \vee B$

## What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$\mathcal{B}$

$A \rightarrow B$

$A \wedge B$

$A \vee B$

# What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$B$

$A \rightarrow B$

$A \wedge B$

$A \vee B$

## System F

$\forall\alpha.A$

$\exists\alpha.A$

## What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$$\mathcal{B} \quad A \rightarrow B \quad A \wedge B \quad A \vee B$$

**System F** corresponds to **second-order propositional logic**

$$\forall\alpha.A \quad \exists\alpha.A$$

## What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$$\mathcal{B} \quad A \rightarrow B \quad A \wedge B \quad A \vee B$$

**System F** corresponds to **second-order propositional logic**

$$\forall\alpha.A \quad \exists\alpha.A$$

**System  $F_\omega$**

$$\lambda\alpha.A \quad A\ B$$

## What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$$\mathcal{B} \quad A \rightarrow B \quad A \wedge B \quad A \vee B$$

**System F** corresponds to **second-order propositional logic**

$$\forall\alpha.A \quad \exists\alpha.A$$

**System F $\omega$**  corresponds to **higher-order propositional logic**

$$\lambda\alpha.A \quad A B$$

## What logic?

$\lambda^\rightarrow$  corresponds to **propositional logic**

$$\mathcal{B} \quad A \rightarrow B \quad A \wedge B \quad A \vee B$$

**System F** corresponds to **second-order propositional logic**

$$\forall\alpha.A \quad \exists\alpha.A$$

**System  $F_\omega$**  corresponds to **higher-order propositional logic**

$$\lambda\alpha.A \quad A B$$

What about **first-order logic**?

## Propositional vs predicate

**Propositional logic**

$$P \rightarrow Q$$

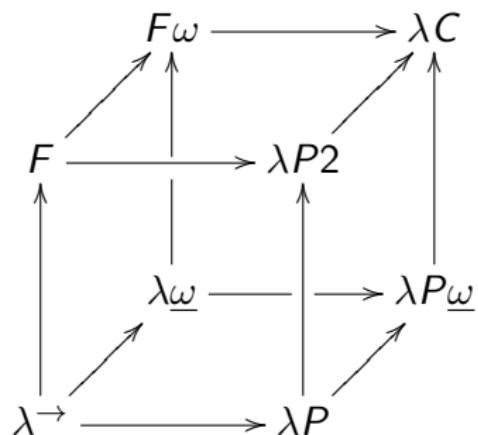
**Predicate logic (FOPL)**

$$P(x)$$

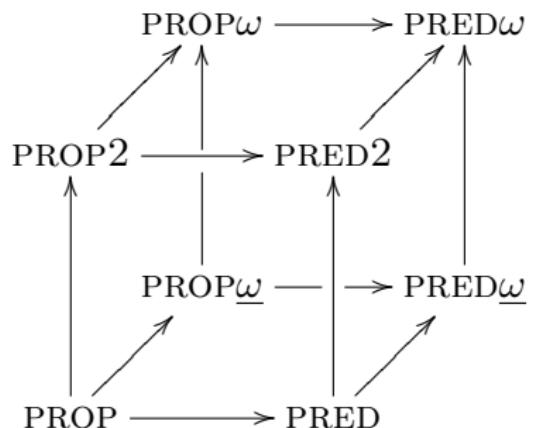
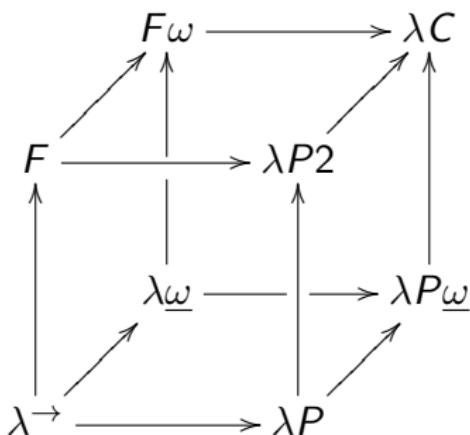
$$(\forall P. P \rightarrow P) \rightarrow (\exists Q. Q \rightarrow Q)$$

$$\forall x \in A. P(x)$$

# Lambda and logic cubes



# Lambda and logic cubes



## More suggestive notation

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B}$$

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**Terms correspond to proofs**

## More suggestive notation

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**Terms correspond to proofs**

(Part 2 of the **Curry-Howard** correspondence)

## Inference rules for $\rightarrow$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \text{ tvar}$$

$$\frac{A \in \Gamma}{\Gamma \vdash A}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A. M : A \rightarrow B} \rightarrow\text{-intro}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B} \rightarrow\text{-elim}$$

$$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

## Inference rules for $\times$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times\text{-intro}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \text{fst } M : A} \times\text{-elim-1}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \text{snd } M : B} \times\text{-elim-2}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-elim-1}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-elim-2}$$

# Classical vs intuitionistic logic

## Classical logic

Emphasis on **truth**

Truth values:  $\top, \perp$

$A \vee \neg A$  always holds

## Intuitionistic logic

Emphasis on **proof**

Proofs inhabit propositions

$A \vee \neg A$  doesn't hold in general

## Brouwer-Heyting-Kolmogorov (BHK) interpretation

A proof of  $A \rightarrow B$ :

a function that builds a proof of  $B$  from a proof of  $A$ .

A proof of  $A \wedge B$ :

a pair of a proof of  $A$  and a proof of  $B$ .

$\neg A$

means  $A \rightarrow \perp$

$\perp$

has no proof

## Continuing the correspondence

**Types** *correspond to propositions*

**Programs** *correspond to proofs*

# Continuing the correspondence

**Types** *correspond to propositions*

**Programs** *correspond to proofs*

**Evaluation** *corresponds to proof simplification*

## Continuing the correspondence

**Types** *correspond to propositions*

**Programs** *correspond to proofs*

**Evaluation** *corresponds to proof simplification*

(The three-part **Curry-Howard** correspondence)

# Who should care?

## Language designers

e.g. *linear logic*: restrictions on structural rules

corresponds to a language with resource management guarantees

## Logicians

since results about programming languages transfer “for free”

e.g. strong normalization implies consistency

## Authors (and users) of proof assistants

e.g. Coq and other tools based on type theory

## Programmers?

# Logical equivalences

$$\forall \beta. (\forall \alpha. (P\alpha \rightarrow \beta)) \rightarrow \beta \quad \leftrightarrow \quad \exists \alpha. P\alpha$$

$$\forall \beta. (P \rightarrow \beta) \wedge (Q \rightarrow \beta) \rightarrow \beta \quad \leftrightarrow \quad P \vee Q$$

Proof: we must show

$$\forall \beta. (\forall \alpha. (P\alpha \rightarrow \beta)) \rightarrow \beta \vdash \exists \alpha. P\alpha$$

$$\exists \alpha. P\alpha \vdash \forall \beta. (\forall \alpha. (P\alpha \rightarrow \beta)) \rightarrow \beta$$

etc.

## A proof

Let  $\Gamma = \forall\beta.(\forall\alpha.P\alpha \rightarrow \beta) \rightarrow \beta$ . Then

$$\frac{\frac{\frac{\Gamma \vdash \forall\beta.(\forall\alpha.P\alpha \rightarrow \beta) \rightarrow \beta}{\Gamma \vdash (\forall\alpha.P\alpha \rightarrow \exists\alpha.P\alpha) \rightarrow \exists\alpha.P\alpha} \forall\text{-elim}}{\Gamma \vdash \exists\alpha.P\alpha} \exists\text{-intro}}{\frac{\frac{\Gamma, \alpha \vdash P\alpha \rightarrow \exists\alpha.P\alpha}{\Gamma, \alpha \vdash \exists\alpha.P\alpha} \rightarrow\text{-intro}}{\frac{\Gamma \vdash \forall\alpha.P\alpha \rightarrow \exists\alpha.P\alpha}{\Gamma \vdash \forall\alpha.P\alpha \rightarrow \exists\alpha.P\alpha} \forall\text{-intro}} \forall\text{-intro}} \rightarrow\text{-elim}}$$

# A program from a proof

Let  $\Gamma = \forall\beta.(\forall\alpha.P\alpha \rightarrow \beta) \rightarrow \beta$ . Then

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Right subtree:

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$$\frac{\begin{array}{c} \frac{\Gamma, \alpha, P\alpha \vdash \exists\alpha.P\alpha}{\Gamma, \alpha \vdash P\alpha \rightarrow \exists\alpha.P\alpha} \exists\text{-intro} \\ \frac{\Gamma, \alpha \vdash P\alpha \rightarrow \exists\alpha.P\alpha}{\Gamma \vdash \forall\alpha.P\alpha \rightarrow \exists\alpha.P\alpha} \rightarrow\text{-intro} \end{array}}{\Gamma \vdash \forall\alpha.P\alpha \rightarrow \exists\alpha.P\alpha} \forall\text{-intro}$$
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Finally:

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## Is it useful?

$$\forall \beta. (P \rightarrow \beta) \wedge (Q \rightarrow \beta) \rightarrow \beta \quad \leftrightarrow \quad P \vee Q$$

These type equivalences can be useful in constructing programs.

The data type encodings we saw last week can be derived this way.

## Closing thoughts

The correspondence suggests a way of thinking about programming  
— and a way of systematically constructing (some) programs

However, propositional logic is quite weak  
(and our types are often uninformative)

We'll have richer types available later (GADTs, monads),  
at which point we'll revisit the question of usefulness

Next time

## **Abstraction**