Last time

System Fω

$$\frac{K_1 \text{ is a kind} \qquad K_2 \text{ is a kind}}{K_1 \Rightarrow K_2 \text{ is a kind}} \Rightarrow\text{-kind}$$

$$\frac{\Gamma, \alpha{::}K_1 \vdash A :: K_2}{\Gamma \vdash \lambda\alpha{::}K_1.A :: K_1 \Rightarrow K_2} \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash A :: K_1 \Rightarrow K_2 \quad \Gamma \vdash B :: K_1}{\Gamma \vdash A\ B :: K_2} \Rightarrow\text{-elim}$$

(and encoding data types: 1, 2, $\mathbb{N}$, $+$, lists, nested types and $\equiv$)

This time

$$\Gamma \vdash M : ?$$

What is type inference?

```
# fun f g x -> f (g x);;
- : ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b = <fun>
```

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Goal

succinctness of annotation-free code

$+$

safety and expressiveness of System Fω

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Goal

succinctness of annotation-free code

$+$

safety and expressiveness of System Fω

Bad news

the goal is unachievable

The ML calculus

Prenex quantifification

$\forall\alpha.\alpha \to \alpha$

$\forall\alpha.\forall\beta.\alpha \to (\beta \to \beta)$

$\forall\alpha.(\forall\beta.\beta \to \beta) \to \alpha$

$\forall\alpha.\alpha \to (\forall\beta.\beta \to \beta)$

Let-bound polymorphism

```
let id = fun x -> x
 in id id
```

```
let id x = x
 in id id
```

```
let f id = id id
 in f (fun x -> x)
```

```
(fun id -> id id)
  (fun x -> x)
```

Prenex quantifification

$\forall\alpha.\alpha \to \alpha$ ✓

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Let-bound polymorphism

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$\forall\alpha.\alpha \to (\forall\beta.\beta \to \beta)$ ✗

Let-bound polymorphism

```
let id = fun x -> x
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✓

```
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```

Prenex quantifification

$\forall\alpha.\alpha \to \alpha$ ✓

$\forall\alpha.\forall\beta.\alpha \to (\beta \to \beta)$ ✓

$\forall\alpha.(\forall\beta.\beta \to \beta) \to \alpha$ ✗

$\forall\alpha.\alpha \to (\forall\beta.\beta \to \beta)$ ✗

Let-bound polymorphism

```
let id = fun x -> x
 in id id
```

✓

```
let id x = x
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✓

```
let f id = id id
 in f (fun x -> x)
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```
(fun id -> id id)
  (fun x -> x)
```

Prenex quantifification

$\forall\alpha.\alpha \to \alpha$ ✓

$\forall\alpha.\forall\beta.\alpha \to (\beta \to \beta)$ ✓

$\forall\alpha.(\forall\beta.\beta \to \beta) \to \alpha$ ✗

$\forall\alpha.\alpha \to (\forall\beta.\beta \to \beta)$ ✗

Let-bound polymorphism

```
let id = fun x -> x
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```

✓

```
let id x = x
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```

✓

```
let f id = id id
 in f (fun x -> x)
```

✗

```
(fun id -> id id)
  (fun x -> x)
```

Prenex quantifification

$\forall\alpha.\alpha \to \alpha$ ✓

$\forall\alpha.\forall\beta.\alpha \to (\beta \to \beta)$ ✓

$\forall\alpha.(\forall\beta.\beta \to \beta) \to \alpha$ ✗

$\forall\alpha.\alpha \to (\forall\beta.\beta \to \beta)$ ✗

Let-bound polymorphism

```
let id = fun x -> x
 in id id
```

✓

```
let id x = x
 in id id
```

✓

```
let f id = id id
 in f (fun x -> x)
```

✗

```
(fun id -> id id)
  (fun x -> x)
```

✗

Types and schemes

$$\frac{}{\Gamma \vdash \mathcal{B} \text{ is a type}}\ \mathcal{B}\text{-types}$$

$$\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha \text{ is a type}}\ \alpha\text{-types}$$

$$\frac{\Gamma \vdash A \text{ is a type} \qquad \Gamma \vdash B \text{ is a type}}{\Gamma \vdash A \to B \text{ is a type}}\ \to\text{-types}$$

$$\frac{\Gamma, \overline{\alpha} \vdash A \text{ is a type}}{\Gamma \vdash \forall \overline{\alpha}.A \text{ is a scheme}}\ \text{scheme}$$

Environments

$$\frac{}{\cdot \text{ is an environment}}\ \Gamma\text{-}\cdot$$

$$\frac{\Gamma \text{ is an environment} \qquad \Gamma \vdash S \text{ is a scheme}}{\Gamma, x : S \text{ is an environment}}\ \Gamma\text{-:}$$

$$\frac{\Gamma \text{ is an environment}}{\Gamma, \alpha \text{ is an environment}}\ \Gamma\text{-::}$$

Typing rules for $\to$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to\text{-intro}$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \to\text{-elim}$$

Typing rules for schemes

$$\frac{\Gamma \vdash M : A \qquad \overline{\alpha} \cap fv(\Gamma) = \emptyset \qquad \Gamma, x : \forall \overline{\alpha}.A \vdash N : B}{\Gamma \vdash \text{let } x = M \text{ in } N : B} \text{ scheme-intro}$$

$$\frac{\begin{array}{c} x : \forall \overline{\alpha}.A \in \Gamma \\ \Gamma \vdash B \text{ is a type} \quad (\text{for } B \in \overline{B}) \end{array}}{\Gamma \vdash x : A[\overline{\alpha} := \overline{B}]} \text{ scheme-elim}$$

Milner's algorithm

Substitutions

$$[a_1 \mapsto A_1,\ a_2 \mapsto A_2,\ \ldots\ a_n \mapsto A_n]$$

For example, let

σ be $\{a \mapsto \mathcal{B},\ b \mapsto (\mathcal{B} \to \mathcal{B})\}$

A be $a \to b \to a$

Then

σA is $\mathcal{B} \to (\mathcal{B} \to \mathcal{B}) \to \mathcal{B}$.

If

$\sigma A = B$ (for some σ)

then we say

B is a *substitution instance* of A.

Constraints

$$a = b$$

$$a \to b = \mathcal{B} \to b$$

$$\mathcal{B} = \mathcal{B}$$

$$\mathcal{B} = \mathcal{B} \to \mathcal{B}$$

Unification

$$\text{unify} : \text{ConstraintSet} \rightarrow \text{Substitution}$$

$$
\begin{aligned}
\text{unify}(\emptyset) &= [] \\
\text{unify}(\{A = A\} \cup C) &= \text{unify}(C) \\
\text{unify}(\{a = A\} \cup C) &= \text{unify}([a \mapsto A]C) \circ [a \mapsto A] \\
&\quad \text{when } a \notin \mathit{ftv}(A) \\
\text{unify}(\{A = a\} \cup C) &= \text{unify}([a \mapsto A]C) \circ [a \mapsto A] \\
&\quad \text{when } a \notin \mathit{ftv}(A) \\
\text{unify}(\{A \rightarrow B = A' \rightarrow B'\} \cup C) &= \text{unify}(\{A = A', B = B'\} \cup C) \\
\text{unify}(\{A = B\} \cup C) &= \mathit{FAIL}
\end{aligned}
$$

Algorithm J

$$\texttt{J : Environment} \times \texttt{Expression} \rightarrow \texttt{Type}$$

$\texttt{J } (\Gamma, \lambda \texttt{x.M}) = b \rightarrow A$
where $\texttt{A} = \texttt{J } (\Gamma, \texttt{x}:b, \texttt{ M})$
and b *is fresh*

$\texttt{J } (\Gamma, \texttt{ M N}) = b$
where $A = \texttt{J } (\Gamma, \texttt{ M})$
and $B = \texttt{J } (\Gamma, \texttt{ N})$
and $\texttt{unify'} (\{A = B \rightarrow b\})$ *succeeds*
and b *is fresh*

$\texttt{J } (\Gamma, \texttt{ x}) = \texttt{A}[\overline{\alpha} := \overline{b}]$
where $\Gamma(\texttt{x}) = \forall \overline{\alpha}.A$
and $\overline{b}$ *are fresh*

$\texttt{J } (\Gamma, \texttt{ let x = M in N}) = B$
where $A = \texttt{J } (\Gamma, \texttt{ M})$
and $B = \texttt{J } (\Gamma, \texttt{x}:\forall \overline{\alpha}.A, \texttt{ N})$
and $\overline{\alpha} = \texttt{ftv}(A) \setminus \texttt{ftv}(\Gamma)$

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
       apply id) =
```

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
       let id = λy.y in
        apply id) =
   J(·, λf.λx.f x) =
```

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
       apply id) =
  J(·, λf.λx.f x) =
    J(·,f:b₁, λx.f x) =
```

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id = }\lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id) =} \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x) = } b_1 \to b_2 \to b_3 \\
\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\ \lambda\texttt{x}.\texttt{f x) = } b_2 \to b_3 \\
\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f x) = } b_3
\end{array}
$$

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
       let id = λy.y in
        apply id) =
   J(·, λf.λx.f x) = b1 → b2 → b3
     J(·,f:b1, λx.f x) = b2 → b3
       J(·,f:b1,x:b2, f x) = b3
         J(·,f:b1,x:b2, f) =
```

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = b_1 \to b_2 \to b_3 \\
\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\ \lambda\texttt{x}.\texttt{f x}) = b_2 \to b_3 \\
\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f x}) = b_3 \\
\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f}) = b_1
\end{array}
$$

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
        apply id) =
    J(·, λf.λx.f x) = b1 → b2 → b3
      J(·,f:b1, λx.f x) = b2 → b3
        J(·,f:b1,x:b2, f x) = b3
          J(·,f:b1,x:b2, f) = b1
          J(·,f:b1,x:b2, x) =
```

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
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\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f x}) = b_3 \\
\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f}) = b_1 \\
\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{x}) = b_2
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f.}\lambda\texttt{x.f x in} \\
\quad\texttt{let id = }\lambda\texttt{y.y in} \\
\quad\quad\texttt{apply id) =} \\
\quad\texttt{J}(\cdot,\ \lambda\texttt{f.}\lambda\texttt{x.f x) = } b_1 \to b_2 \to b_3 \\
\quad\quad\texttt{J}(\cdot,\texttt{f:}b_1,\ \lambda\texttt{x.f x) = } b_2 \to b_3 \\
\quad\quad\quad\texttt{J}(\cdot,\texttt{f:}b_1,\texttt{x:}b_2,\ \texttt{f x) = } b_3 \\
\quad\quad\quad\quad\texttt{J}(\cdot,\texttt{f:}b_1,\texttt{x:}b_2,\ \texttt{f) = } b_1 \\
\quad\quad\quad\quad\texttt{J}(\cdot,\texttt{f:}b_1,\texttt{x:}b_2,\ \texttt{x) = } b_2 \\
\quad\quad\quad\quad\texttt{unify(\{}b_1\texttt{ = }b_2 \to b_3\texttt{\}) =}
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = b_1 \to b_2 \to b_3 \\
\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\ \lambda\texttt{x}.\texttt{f x}) = b_2 \to b_3 \\
\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f x}) = b_3 \\
\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{f}) = b_1 \\
\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_1,\texttt{x}:b_2,\ \texttt{x}) = b_2 \\
\quad\quad\quad\quad \texttt{unify}(\{b_1 = b_2 \to b_3\}) = \{b_1 \mapsto b_2 \to b_3\}
\end{array}
$$

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
       apply id) =
   J(·, λf.λx.f x) = (b2 → b3) → b2 → b3
      J(·,f:b2 → b3, λx.f x) = b2 → b3
         J(·,f:b2 → b3,x:b2, f x) = b3
            J(·,f:b2 → b3,x:b2, f) = b2 → b3
            J(·,f:b2 → b3,x:b2, x) = b2
            unify({b1 = b2 → b3}) = {b1 ↦ b2 → b3}
```

$$
\begin{aligned}
&\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f.}\lambda\texttt{x.f x in let id = }\lambda\texttt{y.y in apply id}) = \\
&\quad \texttt{J}(\cdot,\ \lambda\texttt{f.}\lambda\texttt{x.f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
&\quad\quad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\ \lambda\texttt{x.f x}) = b_2 \to b_3 \\
&\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{f x}) = b_3 \\
&\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{f}) = b_2 \to b_3 \\
&\quad\quad\quad\quad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{x}) = b_2 \\
&\quad\quad\quad\quad \texttt{unify}(\{b_1 = b_2 \to b_3\}) = \{b_1 \mapsto b_2 \to b_3\}
\end{aligned}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = } \lambda \texttt{f.}\lambda \texttt{x.f x in} \\
\qquad \texttt{let id = } \lambda \texttt{y.y in} \\
\qquad\quad \texttt{apply id) =} \\
\quad \texttt{J}(\cdot,\ \lambda \texttt{f.}\lambda \texttt{x.f x) = } (b_2 \to b_3) \to b_2 \to b_3 \\
\qquad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\ \lambda \texttt{x.f x) = } b_2 \to b_3 \\
\qquad\quad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{f x) = } b_3 \\
\qquad\qquad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{f) = } b_2 \to b_3 \\
\qquad\qquad \texttt{J}(\cdot,\texttt{f}:b_2 \to b_3,\texttt{x}:b_2,\ \texttt{x) = } b_2 \\
\quad \texttt{ftv}((b_2 \to b_3) \to b_2 \to b_3) = \{b_2,\ b_3\} \\
\quad \texttt{ftv}(\cdot) = \{\} \\
\quad \{b_2,\ b_3\} \setminus \{\} = \{b_2,\ b_3\}
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f.}\lambda\texttt{x.f x in} \\
\quad \texttt{let id = }\lambda\texttt{y.y in} \\
\quad\ \ \texttt{apply id) =} \\
\ \ \texttt{J}(\cdot,\ \lambda\texttt{f.}\lambda\texttt{x.f x) = } (b_2 \to b_3) \to b_2 \to b_3 \\
\ \ \texttt{J}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad \texttt{let id = }\lambda\texttt{y.y in apply id) =}
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda \texttt{f}.\lambda \texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda \texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda \texttt{f}.\lambda \texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}: \forall \alpha_2 \alpha_3 . (\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda \texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}: \forall \alpha_2 \alpha_3 . (\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda \texttt{y}.\texttt{y}) =
\end{array}
$$

Algorithm J in action

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\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{y}:b_4,\ \texttt{y}) \\
\quad\quad\quad\quad = b_4
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\ \ \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad \texttt{ftv}(b_4 \to b_4) = \{b_4\} \\
\quad\quad \texttt{ftv}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3) = \{\} \\
\quad\quad \{b_4\} \setminus \{\} = \{b_4\}
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4, \\
\quad\quad\quad \texttt{apply id}) = b_5
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id = }\lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id = }\lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4, \\
\quad\quad\quad \texttt{apply id}) = b_5 \\
\quad\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad\quad\quad \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4,\ \texttt{apply}) \\
\quad\quad\quad\quad = (b_6 \to b_7) \to b_6 \to b_7
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4, \\
\quad\quad\quad \texttt{apply id}) = b_5 \\
\quad\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad\quad\quad \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4,\ \texttt{apply}) \\
\quad\quad\quad\quad = (b_6 \to b_7) \to b_6 \to b_7 \\
\quad\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad\quad\quad \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4,\ \texttt{id}) \\
\quad\quad\quad\quad = b_8 \to b_8
\end{array}
$$

Algorithm J in action

$$\texttt{unify}\ (\{(b_6 \rightarrow b_7) \rightarrow b_6 \rightarrow b_7\ \texttt{=}\ (b_8 \rightarrow b_8)\ \rightarrow\ b_5\})$$

Algorithm J in action

```
  unify ({(b6 → b7) → b6 → b7 = (b8 → b8) → b5})
= unify ({b6 → b7 = b8 → b8,
          b6 → b7 = b5})
```

$$\begin{aligned} & \texttt{unify}\ (\{(b_6 \to b_7) \to b_6 \to b_7 = (b_8 \to b_8) \to b_5\}) \\ = {} & \texttt{unify}\ (\{b_6 \to b_7 = b_8 \to b_8, \\ & \qquad b_6 \to b_7 = b_5\}) \end{aligned}$$

Algorithm J in action

$$
\begin{aligned}
& \texttt{unify}\ (\{(b_6 \to b_7) \to b_6 \to b_7 \ \texttt{=}\ (b_8 \to b_8) \to b_5\}) \\
= \ & \texttt{unify}\ (\{b_6 \to b_7 \ \texttt{=}\ b_8 \to b_8, \\
& \qquad\qquad b_6 \to b_7 \ \texttt{=}\ b_5\}) \\
= \ & \texttt{unify}\ (\{b_6 \ \texttt{=}\ b_8, \\
& \qquad\qquad b_7 \ \texttt{=}\ b_8, \\
& \qquad\qquad b_6 \to b_7 \ \texttt{=}\ b_5\})
\end{aligned}
$$

Algorithm J in action

$$
\begin{aligned}
& \texttt{unify}\ (\{(b_6 \to b_7) \to b_6 \to b_7 \ \texttt{=}\ (b_8 \to b_8) \to b_5\}) \\
\texttt{=}\ & \texttt{unify}\ (\{b_6 \to b_7 \ \texttt{=}\ b_8 \to b_8, \\
& \qquad\qquad b_6 \to b_7 \ \texttt{=}\ b_5\}) \\
\texttt{=}\ & \texttt{unify}\ (\{b_6 \ \texttt{=}\ b_8, \\
& \qquad\qquad b_7 \ \texttt{=}\ b_8, \\
& \qquad\qquad b_6 \to b_7 \ \texttt{=}\ b_5\}) \\
\texttt{=}\ & \{b_6 \mapsto b_8,\ b_7 \mapsto b_8,\ b_5 \mapsto b_6 \to b_7\}
\end{aligned}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply} = \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x in} \\
\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in} \\
\quad\quad \texttt{apply id}) = \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f}.\lambda\texttt{x}.\texttt{f x}) = (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id} = \lambda\texttt{y}.\texttt{y in apply id}) = \\
\quad\quad \texttt{J}(\cdot, \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y}.\texttt{y}) = b_4 \to b_4 \\
\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4, \\
\quad\quad\quad \texttt{apply id}) = b_5 \\
\quad\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad\quad\quad \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4,\ \texttt{apply}) \\
\quad\quad\quad\quad = (b_6 \to b_7) \to b_6 \to b_7 \\
\quad\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad\quad\quad \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4,\ \texttt{id}) \\
\quad\quad\quad\quad = b_8 \to b_8
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J(}\cdot\texttt{, let apply = }\lambda\texttt{f.}\lambda\texttt{x.f x in} \\
\quad\quad \texttt{let id = }\lambda\texttt{y.y in} \\
\quad\quad\quad \texttt{apply id) =} \\
\quad \texttt{J(}\cdot\texttt{, }\lambda\texttt{f.}\lambda\texttt{x.f x) = } (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J(}\cdot\texttt{,apply:}\forall\alpha_2\alpha_3\texttt{.(}\alpha_2 \to \alpha_3\texttt{)}\to \alpha_2 \to \alpha_3\texttt{,} \\
\quad\quad \texttt{let id = }\lambda\texttt{y.y in apply id) =} \\
\quad\quad \texttt{J(}\cdot\texttt{,apply:}\forall\alpha_2\alpha_3\texttt{.(}\alpha_2 \to \alpha_3\texttt{)}\to \alpha_2 \to \alpha_3\texttt{,} \\
\quad\quad\quad \lambda\texttt{y.y) = } b_4 \to b_4 \\
\quad\quad \texttt{J(}\cdot\texttt{, apply:}\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3\texttt{,id:}\forall\alpha_4.\alpha_4 \to \alpha_4\texttt{,} \\
\quad\quad\quad \texttt{apply id) = } b_8 \to b_8 \\
\quad\quad\quad \texttt{J(}\cdot\texttt{, apply:}\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3\texttt{,} \\
\quad\quad\quad\quad\quad \texttt{id:}\forall\alpha_4.\alpha_4 \to \alpha_4\texttt{, apply)} \\
\quad\quad\quad\quad \texttt{= (}b_8 \to b_8\texttt{)}\to b_8 \to b_8 \\
\quad\quad\quad \texttt{J(}\cdot\texttt{, apply:}\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3\texttt{,} \\
\quad\quad\quad\quad\quad \texttt{id:}\forall\alpha_4.\alpha_4 \to \alpha_4\texttt{, id)} \\
\quad\quad\quad\quad \texttt{= } b_8 \to b_8
\end{array}
$$

Algorithm J in action

$$
\begin{array}{l}
\texttt{J}(\cdot,\ \texttt{let apply = }\lambda\texttt{f.}\lambda\texttt{x.f x in} \\
\quad\quad \texttt{let id = }\lambda\texttt{y.y in} \\
\quad\quad\quad \texttt{apply id) =} \\
\quad \texttt{J}(\cdot,\ \lambda\texttt{f.}\lambda\texttt{x.f x) = } (b_2 \to b_3) \to b_2 \to b_3 \\
\quad \texttt{J}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad \texttt{let id = }\lambda\texttt{y.y in apply id) =} \\
\quad\quad \texttt{J}(\cdot,\texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \\
\quad\quad\quad \lambda\texttt{y.y) = } b_4 \to b_4 \\
\quad\quad \texttt{J}(\cdot,\ \texttt{apply}:\forall\alpha_2\alpha_3.(\alpha_2 \to \alpha_3) \to \alpha_2 \to \alpha_3, \texttt{id}:\forall\alpha_4.\alpha_4 \to \alpha_4, \\
\quad\quad\quad \texttt{apply id) = } b_8 \to b_8
\end{array}
$$

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
       apply id) =
   J(·, λf.λx.f x) = (b₂ → b₃) → b₂ → b₃
   J(·,apply:∀α₂α₃.(α₂ → α₃)→ α₂ → α₃,
     let id = λy.y in apply id) = b₈ → b₈
     J(·, apply:∀α₂α₃.(α₂ → α₃) → α₂ → α₃,id:∀α₄.α₄ → α₄,
       apply id) = b₈ → b₈
```

Algorithm J in action

```
J(·, let apply = λf.λx.f x in
      let id = λy.y in
       apply id) = b8 → b8
```

Type inference in practice

Type inference and recursion

$$\frac{\Gamma, x : A \vdash M : A \qquad \overline{\alpha} \notin fv(\Gamma) \qquad \Gamma, x : \forall\overline{\alpha}.A \vdash N : B}{\Gamma \vdash \text{let rec x = M in N} : B}\ \text{let-rec}$$

Supporting imperative programming: the value restriction

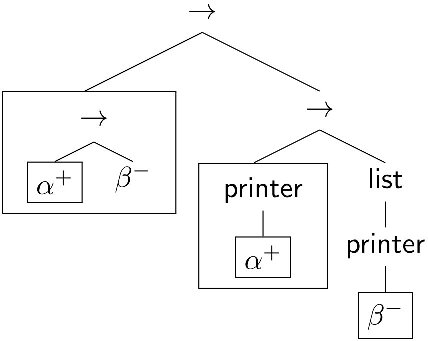
```
type 'a ref = { mutable contents : 'a }
val ref : 'a -> 'a ref
val ( ! ) : 'a ref -> 'a
val (:=) : 'a ref -> 'a -> unit

let r = ref None in
   r := Some "boom";
   match !r with
     None -> ()
   | Some f -> f ()
```

Relaxing the value restriction: variance

```
type 'a printer = 'a -> string

('a -> 'b) -> 'a printer -> 'b printer list
```



Relaxing the value restriction: the rules

Should we generalize?

- covariant type variables
- invariant type variables
- contravariant type variables
- bivariant type variables

Relaxing the value restriction: the rules

Should we generalize?

- covariant type variables ✓
- invariant type variables
- contravariant type variables
- bivariant type variables

Relaxing the value restriction: the rules

Should we generalize?

- covariant type variables ✓
- invariant type variables ✗
- contravariant type variables
- bivariant type variables

Relaxing the value restriction: the rules

Should we generalize?

- covariant type variables ✓
- invariant type variables ✗
- contravariant type variables ✗
- bivariant type variables

Relaxing the value restriction: the rules

Should we generalize?

- covariant type variables ✓
- invariant type variables ✗
- contravariant type variables ✗
- bivariant type variables ✓

Next time

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$