Exponentials

Given sets $X, Y \in \mathbf{Set}$, we have

$Y^X \in \mathbf{Set}$ set of all functions with domain X & codomain Y

$$Y^X = \mathbf{Set}(X, Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued, \& total} \}$$

Aim to characterize Y^X category theoretically

Function application:

$$\mathrm{app} \in \mathbf{Set}(Y^X \times X, Y)$$

$$\mathrm{app}(f, x) = f x \qquad (f \in Y^X, x \in X)$$

So $\mathrm{app} \subseteq (Y^X \times X) \times Y$ is

$$\{((f, x), y) \mid (x, y) \in f\}$$

Function application:

$$app \in Set(Y^X \times X, Y)$$

$$app(f, x) \triangleq f\,x \qquad (f \in Y^X, x \in X)$$

Function currying:

$$\frac{f \in Set(Z \times X, Y)}{cur\,f \in Set(Z, Y^X)}$$

$$cur\,f\,z\,x \triangleq f(z, x) \qquad (z \in Z, x \in X)$$

so $cur\,f\,z = \{(x, y) \mid ((z, x), y) \in f\}$

Function application:

$$\mathrm{app} \in \mathrm{Set}(Y^X \times X, Y)$$

$$\mathrm{app}(f, x) \triangleq f\,x \qquad (f \in Y^X, x \in X)$$

Function currying:

$$\frac{f \in \mathrm{Set}(Z \times X, Y)}{\mathrm{cur}\,f \in \mathrm{Set}(Z, Y^X)}$$

this notation just means "cur is a function from the set $\mathrm{Set}(Z \times X, Y)$ to the set $\mathrm{Set}(Z, Y^X)$"

$$\mathrm{cur}\,f\,z\,x \triangleq f(z, x) \qquad (z \in Z, x \in X)$$

so $\mathrm{cur}\,f\,z = \{(x, y) \mid ((z, x), y) \in f\}$

Haskell Curry

Mathematician

Haskell Brooks Curry was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by ... Wikipedia

Born: September 12, 1900, Millis, Massachusetts, United States

Died: September 1, 1982, State College, Pennsylvania, United States

Parents: Samuel Silas Curry

Books: A Theory of Formal Deducibility, Foundations of Mathematical Logic

Education: University of Göttingen (1930), Harvard University

Given $f \in \mathbf{Set}(Z \times X, Y)$, get commutative diagram

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{app}"] & Y \\
Z \times X \arrow[u, "\mathrm{cur} f \times \mathrm{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{cur} f\, z, x) \arrow[r, mapsto] & \mathrm{cur} f\, z\, x \\
 & f(z, x) \arrow[u, phantom, "\shortparallel"] \\
(z, x) \arrow[uu, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}
$$

see Ex. Sheet 2, qu. 1(b)

Given $f \in \mathrm{Set}(Z \times X, Y)$, get commutative diagram

$$
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\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{cur} f\, z, x) \arrow[r, mapsto] & \mathrm{cur} f\, z\, x \\
& = f(z,x) \\
(z, x) \arrow[uu, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}
$$

Furthermore, if $g \in \mathrm{Set}(Z, Y^X)$ also satisfies

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{app}"] & Y \\
Z \times X \arrow[u, "g \times \mathrm{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

then $g = \mathrm{cur} f$,

because of function extensionality...

Function extensionality

Two functions $f, g \in Y^X$ are equal if (and only if)

$$(\forall x \in X)\ f x = g x$$

because this implies

$$\{(x, fx) \mid x \in X\} = \{(x, gx) \mid x \in X\}$$

ie $\{(x, y) \mid (x, y) \in f\} = \{(x, y) \mid (x, y) \in g\}$

i.e. $f = g$

Exponentials

in any cartesian category $\mathbb{C}$

An exponential for $\mathbb{C}$-objects X & Y is specified by

object Y^X + morphism $app : Y^X \times X \to Y$

with the universal property:

for all $f \in \mathbb{C}(Z \times X, Y)$ there is a unique morphism $g \in \mathbb{C}(Z, Y^X)$ such that

$$\begin{tikzcd}
Y^X \times X \arrow[r, "app"] & Y \\
Z \times X \arrow[u, "g \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes

Exponentials

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$$\begin{tikzcd} Y^X \times X \arrow[r, "app"] & Y \\ Z \times X \arrow[u, "g \times id_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes

Notation: we'll write $cur\,f$ for this unique g

Exponentials

in any cartesian category $\mathbb{C}$

The universal property of $\mathrm{app} : Y^X \times X \to Y$ says that there is a bijection:

$$\mathbb{C}(Z, Y^X) \cong \mathbb{C}(Z \times X, Y)$$

$$g \longmapsto \mathrm{app} \circ (g \times id_X)$$

$$\mathrm{cur}\, f \longleftarrow f$$

$$\mathrm{app} \circ (\mathrm{cur}\, f \times id_X) = f$$

$$\mathrm{cur}(\mathrm{app} \circ (g \times id_X)) = g$$

ex: Why does the universal property imply this?

Exponentials

in any cartesian category $\mathbb{C}$

An **exponential** for $\mathbb{C}$-objects X & Y is specified by

object Y^X + morphism $app : Y^X \times X \to Y$

such that

(Y^X, app) is terminal in the category with
- objects (Z, f) where $f \in \mathbb{C}(Z \times X, Y)$
- morphisms $g : (Z, f) \to (Z', f')$ are $g \in \mathbb{C}(Z, Z')$ such that $f' \circ (g \times id_X) = f$
- composition & identities as in $\mathbb{C}$

Exponentials

in any cartesian category $\mathbb{C}$

An exponential for $\mathbb{C}$-objects X & Y

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- composition & identities as in $\mathbb{C}$

so if they exist, exponentials are unique up to (unique) isomorphism

Ccc's

<u>Definition</u> A cartesian closed category (ccc) is a category $\mathbb{C}$ with

- a terminal object
- binary products

$\Big\}$ hence, all finite products i.e. $\mathbb{C}$ is cartesian

- an exponential for every pair of objects

Examples of ccc's

- **Set** is a ccc – as we've seen.
- **Pre** is a ccc: the exponential of $(P, \leq)$ and $(Q, \leq)$ is $(P \to Q, \leq)$ where

$$P \to Q = \{ f \in Q^P \mid (\forall p, p' \in P)\ p \leq p' \Rightarrow fp \leq fp' \}$$

this is just $\mathbf{Pre}((P, \leq), (Q, \leq))$

Examples of ccc's

- **Set** is a ccc – as we've seen.
- **Pre** is a ccc: the exponential of $(P, \leq)$ and $(Q, \leq)$ is $(P \to Q, \leq)$

$$\boxed{\begin{array}{l} P \to Q = \{ f \in Q^P \mid (\forall p, p' \in P)\ p \leq p' \Rightarrow fp \leq fp' \} \\ \text{with pre-order} \\ \qquad f \leq f' \triangleq (\forall p \in P)\ fp \leq f'p \end{array}}$$

with application $app \in \mathbf{Pre}((P \to Q, \leq) \times (P, \leq), (Q, \leq))$

$$app(f, p) \triangleq fp$$

Have to check this is monotone & has correct universal property...