

Exponentials

Given sets $X, Y \in \text{Set}$, we have

$Y^X \in \text{Set}$ set of all functions with
domain X & codomain Y

$Y^X = \text{Set}(X, Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued,} \\ \text{\& total} \}$

Aim to characterize Y^X
Category theoretically

Function application:

$$\text{app} \in \text{Set}(Y^X \times X, Y)$$

$$\text{app}(f, x) = fx \quad (f \in Y^X, x \in X)$$

So $\text{app} \subseteq (Y^X \times X) \times Y$ is
 $\{(f, x), y \mid (x, y) \in f\}$

Function application:

$$\text{app} \in \text{Set}(Y^X \times X, Y)$$

$$\text{app}(f, x) \triangleq fx \quad (f \in Y^X, x \in X)$$

Function *currying*:

$$f \in \text{Set}(Z \times X, Y)$$

$$\text{cur } f \in \text{Set}(Z, Y^X)$$

$$\text{cur } f \ z \ x \triangleq f(z, x) \quad (z \in Z, x \in X)$$

$$\text{so } \text{cur } f \ z = \{ (x, y) \mid ((z, x), y) \in f \}$$

Function application:

$$\text{app} \in \text{Set}(Y^X \times X, Y)$$

$$\text{app}(f, x) \triangleq fx \quad (f \in Y^X, x \in X)$$

Function **currying**:

$$\frac{f \in \text{Set}(Z \times X, Y)}{\text{cur } f \in \text{Set}(Z, Y^X)}$$

this notation just means "cur is a function from the set $\text{Set}(Z \times X, Y)$ to the set $\text{Set}(Z, Y^X)$ "

$$\text{cur } f \ z \ x \triangleq f(z, x) \quad (z \in Z, x \in X)$$

$$\text{so } \text{cur } f \ z = \{ (x, y) \mid ((z, x), y) \in f \}$$

Haskell Curry

Mathematician

Haskell Brooks Curry was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by ... [Wikipedia](#)



Born: September 12, 1900, [Millis, Massachusetts, United States](#)

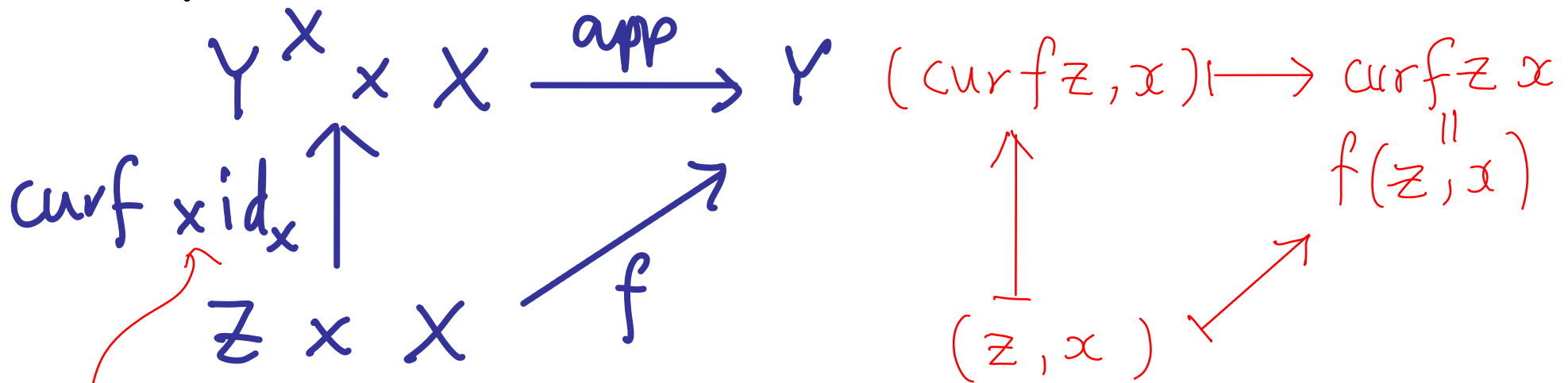
Died: September 1, 1982, [State College, Pennsylvania, United States](#)

Parents: [Samuel Silas Curry](#)

Books: [A Theory of Formal Deducibility](#), [Foundations of Mathematical Logic](#)

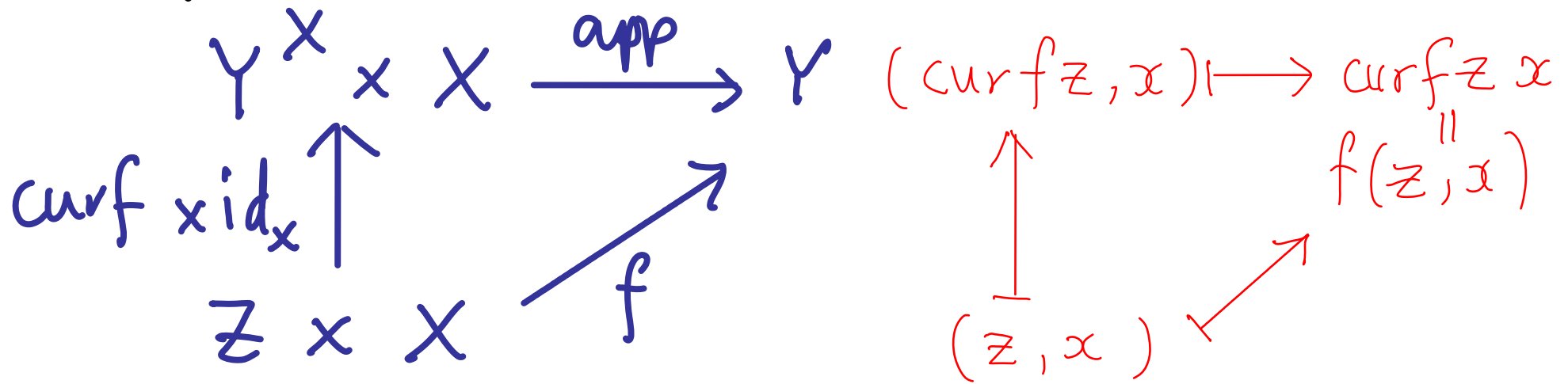
Education: [University of Göttingen \(1930\)](#), [Harvard University](#)

Given $f \in \text{Set}(Z \times X, Y)$, get commutative diagram

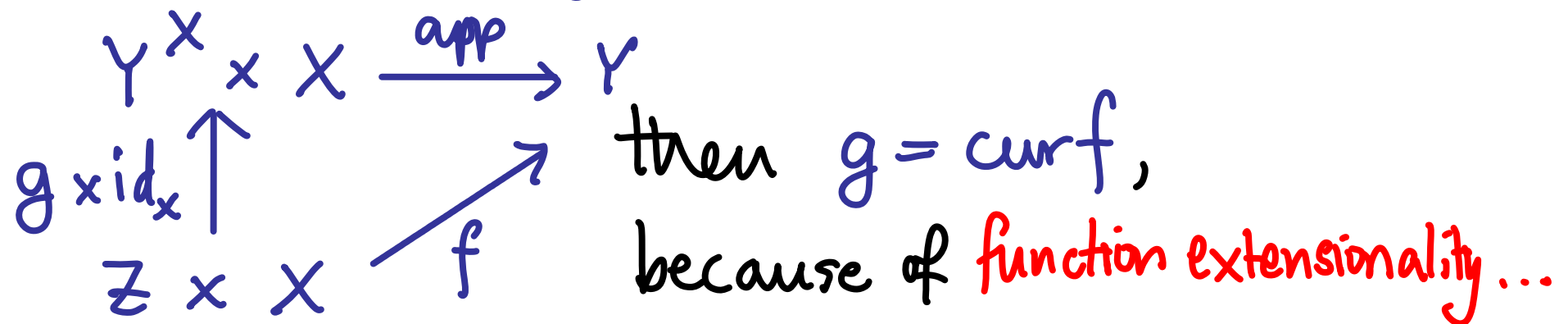


see Ex. Sheet 2, qu. 1(b)

Given $f \in \text{Set}(Z \times X, Y)$, get commutative diagram



Furthermore, if $g \in \text{Set}(Z, Y^X)$ also satisfies



Function extensionality

Two functions $f, g \in Y^X$
are equal if (and only if)

$$\rightarrow (\forall x \in X) f x = g x$$

because this implies

$$\{ (x, f x) \mid x \in X \} = \{ (x, g x) \mid x \in X \}$$

ie $\{ (x, y) \mid (x, y) \in f \} = \{ (x, y) \mid (x, y) \in g \}$

i.e. $f = g$

Exponentials

in any cartesian category \mathbb{C}

An **exponential** for \mathbb{C} -objects X & Y

is specified by

object Y^X + morphism $\text{app} : Y^X \times X \rightarrow Y$

with the universal property:

for all $f \in \mathbb{C}(Z \times X, Y)$ there is

a unique morphism $g \in \mathbb{C}(Z, Y^X)$

such that

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\text{app}} & Y \\ g \times \text{id}_X \uparrow & & \nearrow f \\ Z \times X & & \end{array}$$

commutes

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Notation: we'll
write **cur f**
for **this** unique g

Exponentials

in any cartesian category \mathcal{C}
The universal property of $\text{app} : Y^X \times X \rightarrow Y$
says that there is a bijection:

$$\begin{array}{ccc} \mathcal{C}(Z, Y^X) & \cong & \mathcal{C}(Z \times X, Y) \\ g & \xrightarrow{\quad} & \text{app} \circ (g \times \text{id}_X) \\ \text{cur } f & \xleftarrow{\quad} & f \end{array}$$

$$\text{app} \circ (\text{cur } f \times \text{id}_X) = f$$

$$\text{cur}(\text{app} \circ (g \times \text{id}_X)) = g$$

ex: why does the universal property imply this?

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An **exponential** for \mathcal{C} -objects X & Y

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object Y^X + morphism $\text{app} : Y^X \times X \rightarrow Y$

such that

- (Y^X, app) is terminal in the category with
- objects (Z, f) where $f \in \mathcal{C}(Z \times X, Y)$
 - morphisms $g : (Z, f) \rightarrow (Z', f')$ are $g \in \mathcal{C}(Z, Z')$
such that $f' \circ (g \times \text{id}_X) = f$
 - composition & identities as in \mathcal{C}

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so if they exist, exponentials are unique up to (unique) isomorphism

Ccc's

Definition A cartesian closed category (ccc) is a category \mathcal{C} with

- a terminal object
 - binary products
 - an exponential for every pair of objects
- } hence, all finite products
ie. \mathcal{C} is cartesian

Examples of ccc's

- Set is a ccc — as we've seen.
- Pre is a ccc : the exponential of (P, \leq) and (Q, \leq) is $(P \rightarrow Q, \leq)$ where
$$P \rightarrow Q = \{ f \in Q^P \mid (\forall p, p' \in P) p \leq p' \Rightarrow fp \leq fp' \}$$

this is just $\text{Pre}((P, \leq), (Q, \leq))$

Examples of ccc's

• Set is a ccc — as we've seen.

• Pre is a ccc : the exponential of

(P, \leq) and (Q, \leq) is $(P \rightarrow Q, \leq)$ 

$$P \rightarrow Q = \{ f \in Q^P \mid (\forall p, p' \in P) p \leq p' \Rightarrow fp \leq fp' \}$$

with pre-order

$$f \leq f' \triangleq (\forall p \in P) fp \leq f'p$$

with application $\text{app} \in \text{Pre}((P \rightarrow Q, \leq) \times (P, \leq), (Q, \leq))$

$$\text{app}(f, p) \triangleq fp$$


Have to check this is monotone & has correct universal property...