Binary products

In a category $\mathbb{C}$, a **product** for objects $X, Y \in \mathbb{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

for all $X \xleftarrow{f} Z \xrightarrow{g} Y$

there is a unique $h : Z \to P$

such that

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f"'] \arrow[d, "h"] \arrow[dr, bend left, "g"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commutes

Binary products

In a category $\mathbb{C}$, a **product** for objects $X, Y \in \mathbb{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ satisfying

- (P, π_1, π_2) is terminal in the category with
- objects (Z, f, g) where $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbb{C}$
- morphisms $h : (Z, f, g) \to (Z', f', g')$ are $h \in \mathbb{C}(Z, Z')$ such that $f' \circ h = f$ & $g' \circ h = g$
- composition & identities as in $\mathbb{C}$

Binary products

In a category $\mathbb{C}$, a product for objects $X, Y \in \mathbb{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ satisfying

$\boxed{\begin{array}{l} (P, \pi_1, \pi_2) \text{ is terminal in the category with} \\ \text{- objects } (Z, f, g) \text{ where } X \xleftarrow{f} Z \xrightarrow{g} Y \text{ in } \mathbb{C} \\ \text{- morphisms } h : (Z, f, g) \to (Z', f', g') \text{ are} \\ \quad h \in \mathbb{C}(Z, Z') \text{ such that } f' \circ h = f \;\&\; g' \circ h = g \\ \text{- composition \& identities as in } \mathbb{C} \end{array}}$

so if they exist, products are unique up to (unique) isomorphism.

Binary products – notation

Usual notation for product of X & Y is

$$\boxed{X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y}$$

and, given $X \xleftarrow{f} Z \xrightarrow{g} Y$, the unique $h : Z \to X \times Y$ with $\begin{cases} \pi_1 \circ h = f \\ \pi_2 \circ h = g \end{cases}$

will be written

$$\boxed{\langle f, g \rangle : Z \to X \times Y}$$

Examples of products

In Set : $X \times Y = \{(x,y) \mid x \in X \wedge y \in Y\}$

$$\pi_1(x,y) = x$$

$$\pi_2(x,y) = y$$

because . . .

Examples of products

In **Pre** : $(P, \leq) \times (Q, \leq)$

$= (P \times Q, \leq)$

product in **Set**

$(p_1, q_1) \leq (p_2, q_2) \triangleq p_1 \leq p_2 \wedge q_1 \leq q_2$

in P in Q

$\left.\begin{array}{l} \pi_1(p, q) = p \\ \pi_2(p, q) = q \end{array}\right\}$ are monotone functions

Examples of products

In **Mon**: $(M, \cdot, e) \times (N, \cdot, e)$

$= (M \times N, \cdot, (e, e))$

product in Set

$$\boxed{(m_1, n_1) \cdot (m_2, n_2) \triangleq (m_1 \cdot m_2, n_1 \cdot n_2)}$$

in M in N

unit for this is (e, e)

$$\left.\begin{aligned} \pi_1(m, n) &= m \\ \pi_2(m, n) &= n \end{aligned}\right\} \text{give monoid homomorphisms}$$

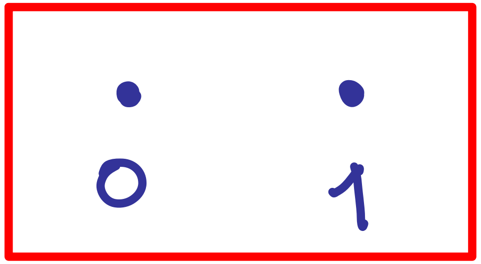
Examples of products

In a pre-ordered set $(P, \leq)$, regarded as a category, the product of $p, q \in P$ is a greatest lower bound (glb, or meet) $p \wedge q$:

$$p \wedge q \leq p \quad \& \quad p \wedge q \leq q$$

$$(\forall r \in P)\ r \leq p \ \& \ r \leq q \Rightarrow r \leq p \wedge q$$

Non-example

The poset 

, ~~that~~ is

$(\{0,1\}, \leq)$ where $0 \leq 0$ & $1 \leq 1$

does not possess a product (= meet)
for 0 & 1.

Duality

Binary product in $\mathbb{C}^{op}$ is called binary coproduct in $\mathbb{C}$.

E.g. coproduct of $X, Y \in \mathbf{Set}$ is

$$X \xrightarrow{i_1} X \overset{+}{\cup} Y \xleftarrow{i_2} Y$$

$$X \overset{+}{\cup} Y \triangleq \{(x,0) \mid x \in X\} \cup \{(y,1) \mid y \in Y\}$$

$i_1(x) \triangleq (x, 0)$

$i_2(y) \triangleq (y, 1)$