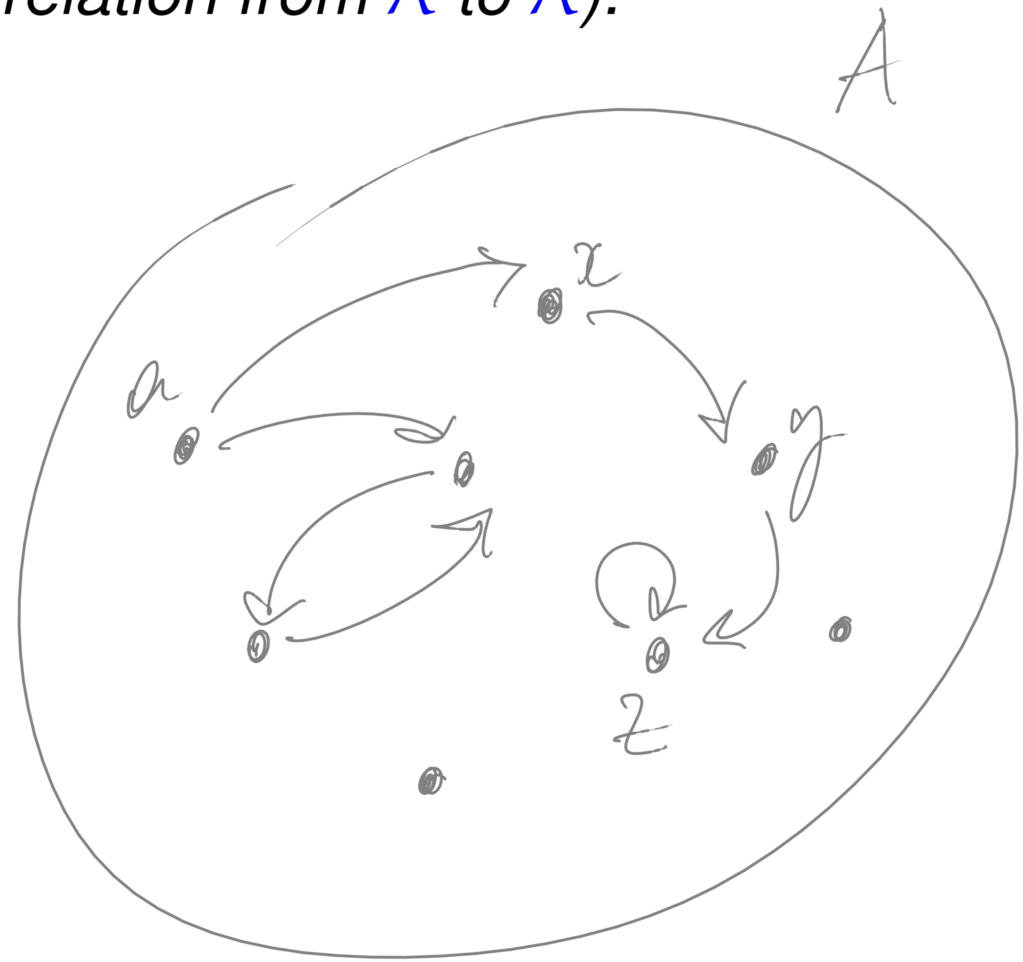
Directed graphs

Definition 107 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*

$R \subseteq A \times A$



xRy

yRz

aRx

$\Leftrightarrow (a,x) \in R$

zRz

$= \underline{Rel}(A,A) = \mathcal{P}(A \times A)$

$$\frac{A \xrightarrow{R} B \xrightarrow{S} C}{A \xrightarrow{S \circ R} C}$$

Corollary 109 *For every set* A*, the structure*

$$(\ \mathrm{Rel}(A)\ ,\ \mathrm{id}_A\ , \circ\ )$$

is a monoid.

Definition 110 *For* $R \in \mathrm{Rel}(A)$ *and* $n \in \mathbb{N}$*, we let*

$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \in \mathrm{Rel}(A)$$

be defined as id_A *for* $n = 0$*, and as* $R \circ R^{\circ m}$ *for* $n = m + 1$*.*

Paths

Proposition 112 *Let (A, R) be a directed graph. For all $n \in \mathbb{N}$ and $s, t \in A$, $s \; R^{\circ n} \; t$ iff there exists a path of length n in R with source s and target t.*

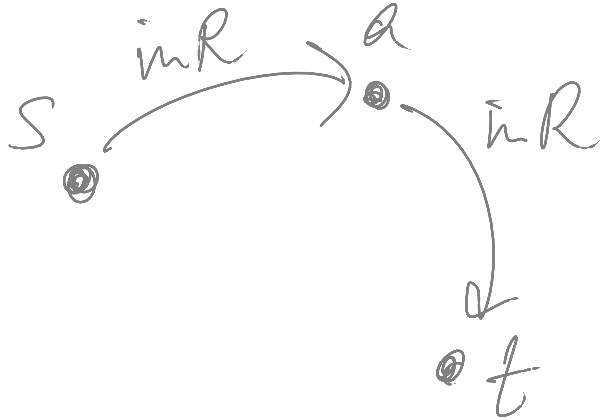
PROOF:

Example: $R^{\circ 0} = id_A$

$R^{\circ 1} = R \circ id_A = R$

$R^{\circ 2} = R \circ R^{\circ 1} = R \circ R$

$s\,(R \circ R)\,t \overset{def}{\iff} \exists a \in A.\; aRt \wedge sRa.$



$R^{\circ 3} = R \circ R^{\circ 2}$

Exercise: prove it by induction.

$x\, R^{\circ *}\, y \overset{\text{def}}{\iff} \exists n \in \mathbb{N}.\ x\, R^{\circ n}\, y \iff \exists$ a path of length n from x to y

Definition 113 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 114 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s\ R^{\circ *}\ t$ *iff there exists a path with source* s *and target* t *in* R.

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 & = & I_n \\ M_{k+1} & = & I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

Definition 115 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \wedge y \sqsubseteq z) \implies x \sqsubseteq z$$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\ \mathbb{Z}\ ,\ |\ )$.

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satisfy

$x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

Posets

Theorem 117 *For* $R \subseteq A \times A$, *let*

is the least relation containing R that is a preorder.

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF:

because $R \subseteq R^{\circ *}$ ✓

$R^{\circ *}$ a preorder:

There a path from x to x always.

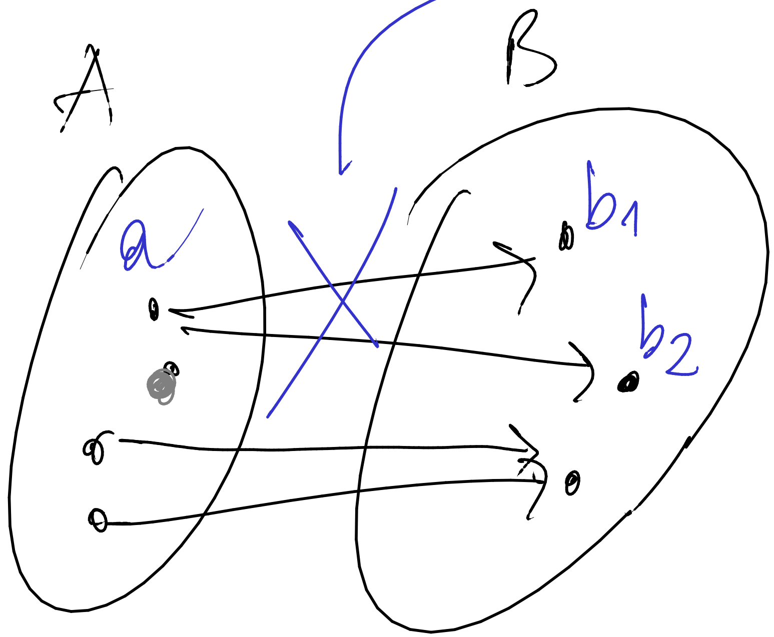
If there is a path from x to y and there is a path from y to z, so there is a path from x to z, namely by their concatenation.

Partial functions

"deterministic input/output relationship"

Definition 118 *A relation* $R : A \rightarrowtail B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A.\, \forall b_1, b_2 \in B.\ a\, R\, b_1 \wedge a\, R\, b_2 \implies b_1 = b_2 \quad .$$



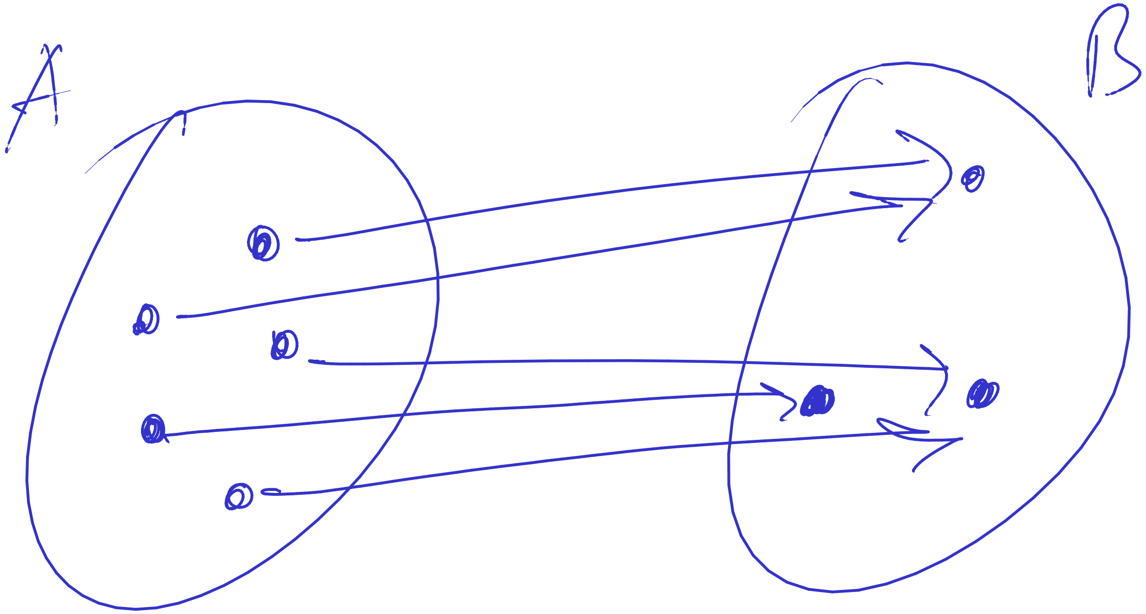
is a partial function

Obs: • id_A is a partial function.

• The composition of partial functions is a partial function.

Functions (or maps)

Definition 122 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*



the set of all the functions from A to B.

Theorem 123 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a\, f\, b \quad .$$

Proposition 124 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) = \#B^{\#A} \quad .$$

PROOF IDEA:

$A = \{a_1, \ldots, a_n\}$ $\quad\quad$ $B = \{b_1, \ldots, b_m\}$

To describe a function, I need

$$a_1 \longmapsto \{b_1, \ldots, b_m\} \quad m$$

$$\times$$

$$a_2 \longmapsto \{b_1, \ldots, b_m\} \quad m$$

$$\times$$

$$\vdots$$

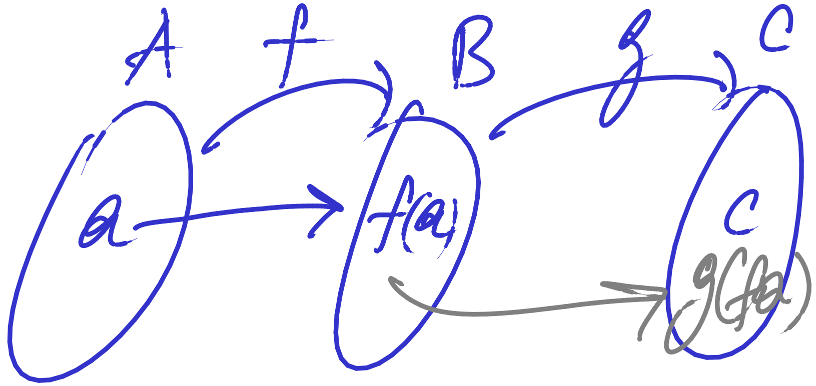
$$\times$$

$$a_n \longmapsto \{b_1, \ldots, b_m\} \quad m$$

$$\left.\begin{array}{c} m \\ \times \\ m \\ \times \\ \vdots \\ \times \\ m \end{array}\right\} = m^n$$

Notation: For a function $f : A \to B$ and $a \in A$, $f(a) \in B$ denotes the unique element related to a by f.

Theorem 125 *The identity partial function is a function, and the composition of functions yields a function.*



NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

$$\mathrm{id}_A(a) = a$$

and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

$$(g \circ f)(a) = g\big(f(a)\big) \quad .$$

Example: $R \mapsto \underline{mat}(R)$

$\underline{rel}(R) \leftarrow\!\shortmid M$

Bijections

Definition 126 *A function* $f : A \to B$ *is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* f*) such that*

1. g *is a* retraction *(or* left inverse*) for* f*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. g *is a* section *(or* right inverse*) for* f*:*

$$f \circ g = \mathrm{id}_B \quad .$$

s.t.

$g \circ f = id_A$

$\wedge$ $f \circ g = id_B$

$\iff \forall a \in A.\ g(fa) = a$

$\wedge\ \forall b \in B\ f(gb) = b$

Definition 130 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

2. $\mathbb{N} \cong \mathbb{N}^+$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.