Sets, cardinality, equality.

$$\#\{0\} = \#\{1\} \qquad \{0\} \neq \{1\}$$

Relations

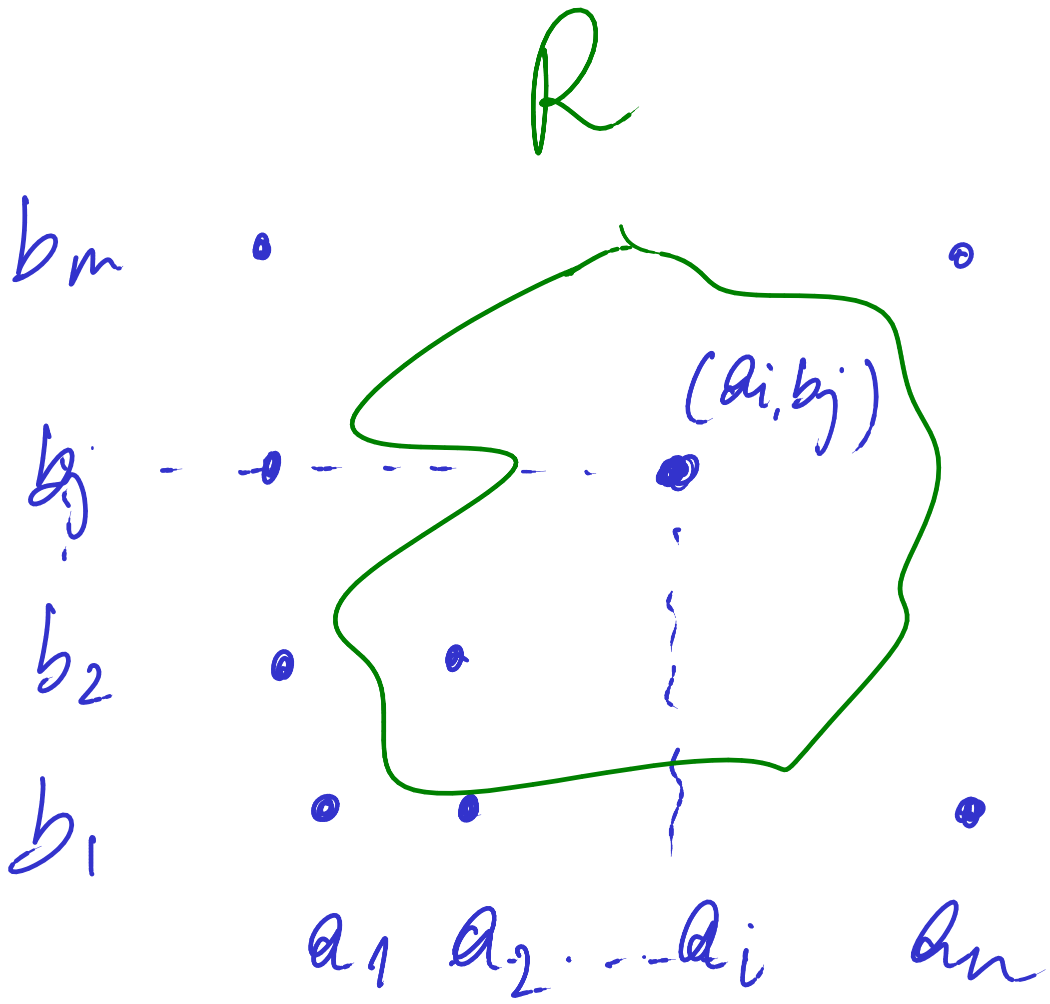
Definition 98 *A* (binary) relation R *from a set* A *to a set* B

$$R : A \nrightarrow B \qquad \text{or} \qquad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \qquad \text{or} \qquad R \in \mathcal{P}(A \times B) \quad .$$

Notation 99 *One typically writes* $a \mathrel{R} b$ *for* $(a, b) \in R$.



relations from A to B

$$\#\mathcal{P}(A \times B) = 2^{\#(A\times B)} = 2^{(\#A)\cdot(\#B)}$$

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

Examples:

- Empty relation.

  $\emptyset : A \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow B \qquad\qquad (a \; \emptyset \; b \iff \mathbf{false})$

- Full relation.

  $(A \times B) : A \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow B \qquad\qquad (a \; (A \times B) \; b \iff \mathbf{true})$

- Identity (or equality) relation.

  $\mathrm{id}_A = \{ \, (a, a) \mid a \in A \, \} : A \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow A \qquad\qquad (a \; \mathrm{id}_A \; a' \iff a = a')$

- Integer square root.

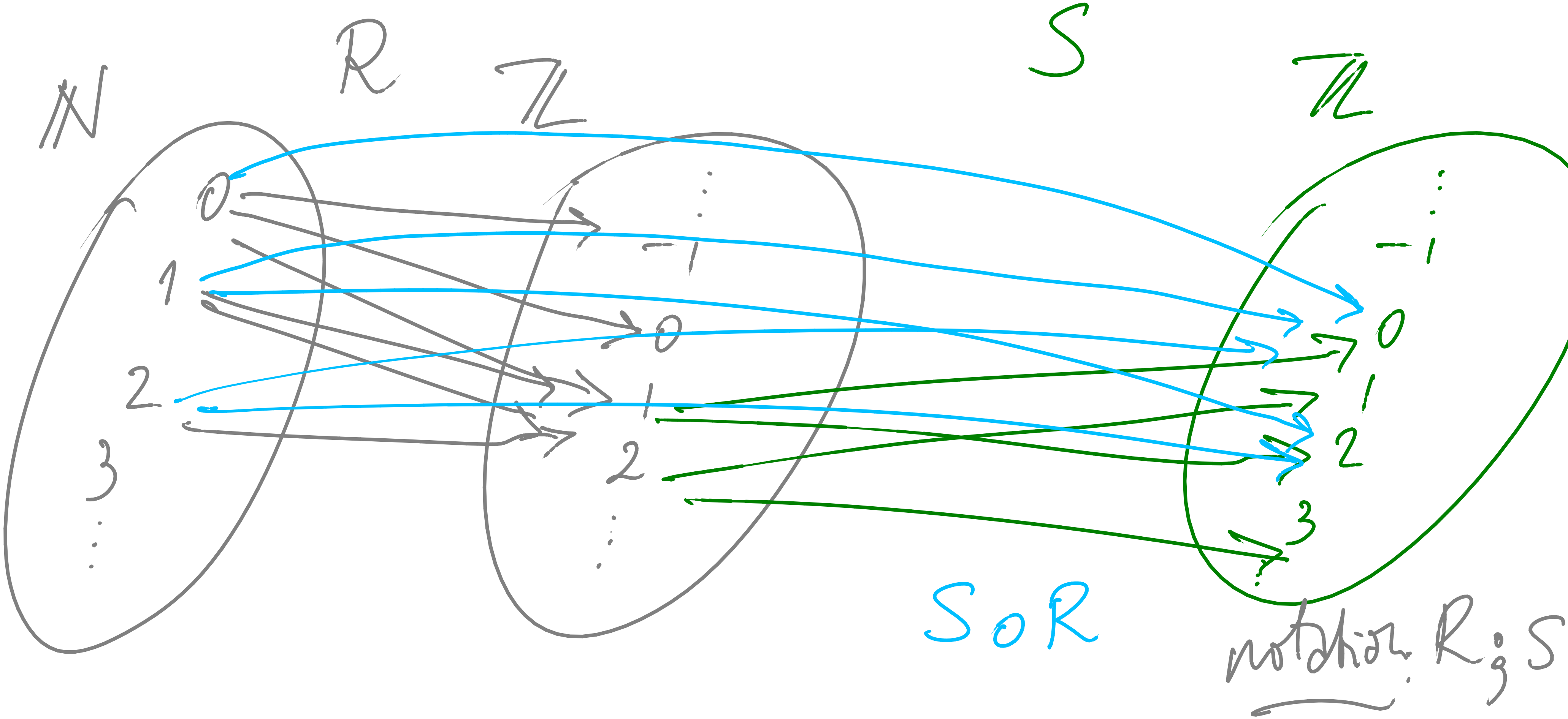
  $R_2 = \{ \, (m, n) \mid m = n^2 \, \} : \mathbb{N} \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow \mathbb{Z} \qquad\qquad (m \; R_2 \; n \iff m = n^2)$

Internal diagrams

Example:

$$R = \{ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1) \} : \mathbb{N} \nrightarrow \mathbb{Z}$$

$$S = \{ (1,0), (1,2), (2,1), (2,3) \} : \mathbb{Z} \nrightarrow \mathbb{Z}$$



Relational extensionality

$$R = S : A \nrightarrow B$$

iff

$$\forall a \in A. \forall b \in B.\ a \, R \, b \iff a \, S \, b$$

Relational composition

$R : A \nrightarrow B$ $\qquad$ $S : B \nrightarrow C$

domain / source $\qquad$ codomain / target

$S \circ R : A \nrightarrow C$

$$a \,(S \circ R)\, c \overset{\text{def}}{\iff} \exists b \in B .\ b \,S\, c \wedge a \,R\, b$$

$$R : A \nrightarrow B$$

$\emptyset : D \nrightarrow A$ $\qquad$ $\emptyset : B \nrightarrow C$

$D \times A : D \nrightarrow A$ $\qquad$ $B \times C : B \nrightarrow C$

$id_A : A \nrightarrow A$ $\qquad$ $id_B : B \nrightarrow B$

Eg. $R \circ id_A : A \nrightarrow B$ $\qquad$ $R \circ id_A = R$

$\emptyset \circ R : A \nrightarrow C$ $\qquad$ $\emptyset \circ R = \emptyset$

$(B \times C) \circ R : A \nrightarrow C$ $\qquad$ $(B \times C) \circ R = ?$ exercise

$(T \circ S) \circ R$

Theorem 101 *Relational composition is associative and has the identity relation as neutral element.*

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[rr, bend right, "S \circ R"'] \arrow[rrr, bend left=60, "(T \circ S) \circ R"] & B \arrow[r, "S"] \arrow[rr, bend left, "T \circ S"] & C \arrow[r, "T"] & D \end{tikzcd}$$

- *Associativity.*

  For all $R : A \rightarrow\!\!\!\!\!+ B$, $S : B \rightarrow\!\!\!\!\!+ C$, *and* $T : C \rightarrow\!\!\!\!\!+ D$,

  $S \circ R$

  $$(T \circ S) \circ R = T \circ (S \circ R)$$

  $T \circ (S \circ R)$

  exercise.

  Notation:

  $T \circ S \circ R$

- *Neutral element.*

  For all $R : A \rightarrow\!\!\!\!\!+ B$,

  $$R \circ \mathrm{id}_A = R = \mathrm{id}_B \circ R \quad .$$

Definition 102

1. *For positive integers* m *and* n*, an* $(m \times n)$*-matrix* M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.

$(m \times n)$-matrix M

$(n \times k)$-matrix L

$$M = \begin{pmatrix} \cdots \\ \cdots M_{i,j} \cdots \\ \cdots \end{pmatrix}$$

$0 \leq i \leq m-1$

$0 \leq j \leq n-1$

$L * M$ $(m \times k)$-matrix

$$(L * M)_{i,j} \stackrel{\text{def}}{=} \bigoplus_k L_{k,j} \odot M_{i,k}$$

Theorem 103 *Matrix multiplication is associative and has the identity matrix as neutral element.*

$\overset{def}{=} \{0, \ldots, m-1\}$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

$M = (M_{i,j})_{i,j} \rightsquigarrow \underline{rel}(M) : \{0, \ldots, m-1\} \nrightarrow \{0, \ldots, n-1\}$

$i \ \underline{rel}(M)\ j \overset{def}{\iff} M_{ij} = true$

$\underline{mat}(R) \leftsquigarrow R : [m] \nrightarrow [n]$

$$\underline{mat}(R)_{i,j} = \begin{cases} true & \text{if } (i,j) \in R \\ false & \text{if } (i,j) \notin R \end{cases}$$

$\underline{mat}(\underline{rel}(M)) = M$ exercise

$\underline{rel}(\underline{mat}(R)) = R$ exercise.

matrices		relations
Identity matrix	$\longleftrightarrow$	$id_{[n]}$
$*$	$\longleftrightarrow$	$\circ$

$\underline{mat}(R \circ S) = (mat\ R) * (mat\ S)$

fact.