notation $\{ (a,b) \mid a \in A,\ b \in B \}$

Cartesian

Products

The *product* $A \times B$ of two sets A and B is the set

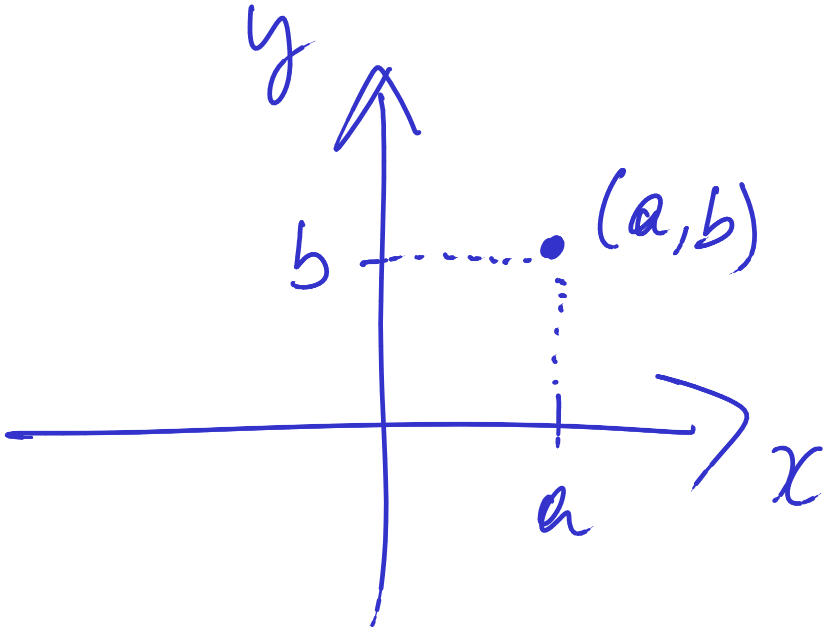
$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$



plane $= \mathbb{R} \times \mathbb{R}$

In general, $A \times B \neq B \times A$

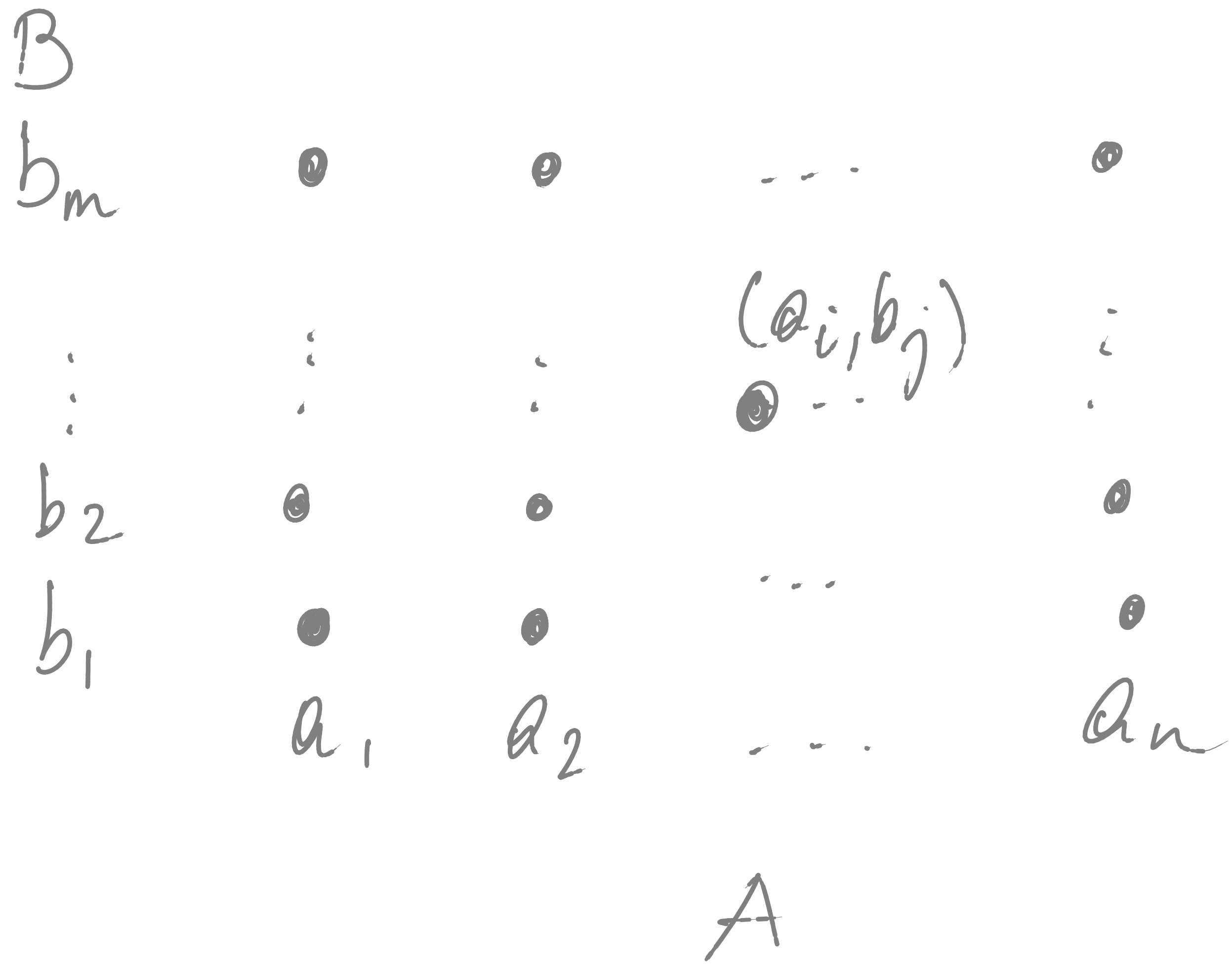
Products model the product type of ML:

$(a, b) : \alpha * \beta$ where $a : \alpha$ and $b : \beta$

Proposition 88 *For all finite sets* A *and* B*,*

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA:



$\mathcal{P}(U) = \{ S \mid S \subseteq U \}$

$A, B \in \mathcal{P}(U)$

$A \cup B$ $\quad$ $\vee$

$A \cap B$ $\quad$ $\wedge$

A^C $\quad$ $\neg$

$\bigcup$ $\quad$ $\exists$

$\bigcap$ $\quad$ $\forall$

is a set of subsets of $\mathcal{U}$ $\quad \mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

Big unions

Definition 89 *Let* $\mathbb{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathbb{U}))$*, we let the* big union *(relative to* $\mathbb{U}$*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathbb{U} \mid \exists A \in \mathcal{F}. x \in A \} \in \mathcal{P}(\mathbb{U}) \quad .$$

imagine

$\mathcal{F} = \{ A, B, C, \ldots \}$

Idea $\bigcup \mathcal{F} = A \cup B \cup C \cup \ldots$

$\mathcal{F} = \{ A_1, A_2, \ldots A_i \ldots \}$

$\bigcup \mathcal{F} = \bigcup_i A_i$

Cf. α list list

flatten : α list list $\to$ α list.

$\iff X \in \mathcal{P}(Y)$

$X \subseteq Y$

Big intersections

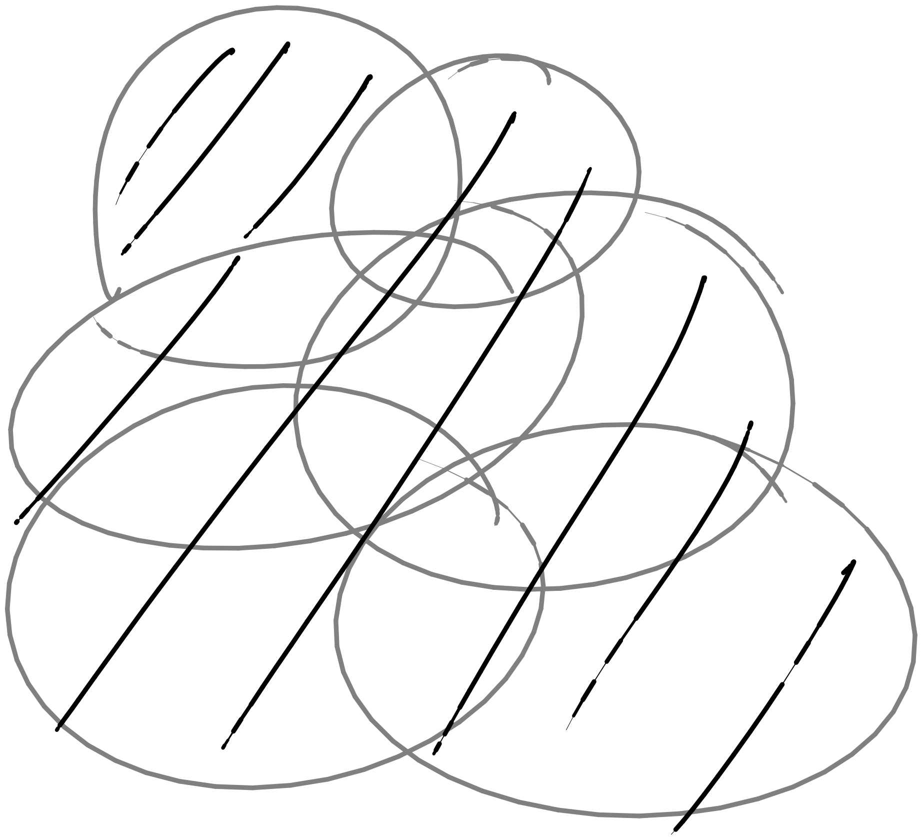
Definition 91 *Let* U *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(U)$, *we let the* big intersection *(relative to* U*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in U \mid \forall A \in \mathcal{F}. x \in A \} \quad .$$

imagine $\mathcal{F} = \{ A_i \}_{i \in I}$

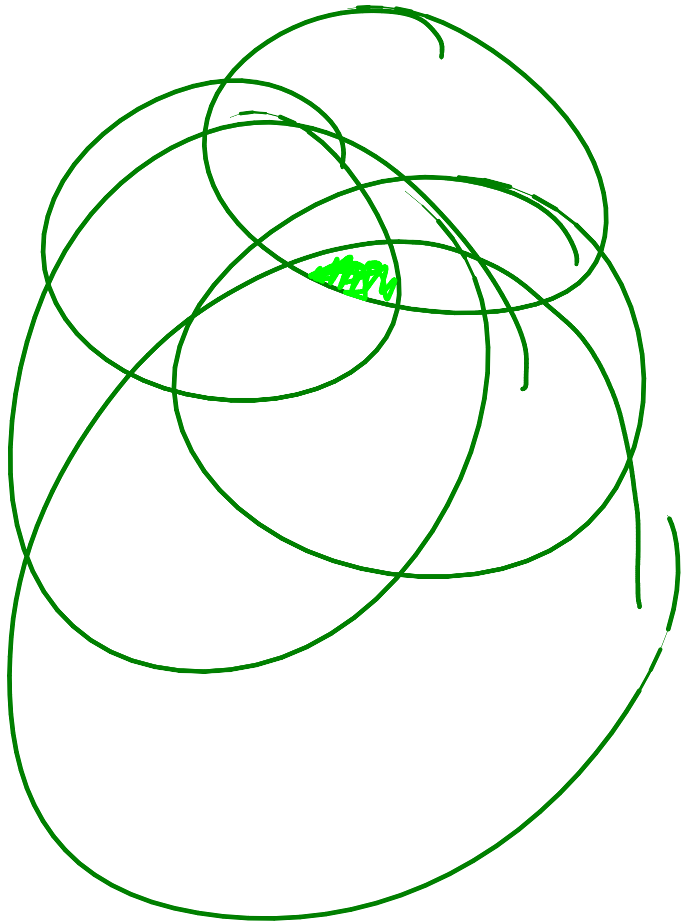
$\bigcap \mathcal{F} = \cdots \cap A_i \cap \cdots$

$\mathcal{F}$



$\bigcup \mathcal{F}$

$\mathcal{F}$



$\bigcap \mathcal{F}$

Theorem 92 *Let* $\mathbb{Z} \in \mathcal{F}$, $\mathbb{Q} \in \mathcal{F}$, $\mathbb{N} \in \mathcal{F}$

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \,\middle|\, (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF: $\mathbb{N} \subseteq \bigcap \mathcal{F}$

$\iff \forall n \in \mathbb{N}.\ n \in \bigcap \mathcal{F}$

$\iff \forall n \in \mathbb{N}.\ \forall S \in \mathcal{F}.\ n \in S$

prove it by induction.

Union axiom

$$\boxed{\text{Every collection of sets has a union.}}$$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F}. x \in X\right) \quad .$$

$A, B \rightsquigarrow \{A, B\} \rightsquigarrow \bigcup \{A, B\} = A \cup B$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad \{x \mid x \in A \vee x \in B\}$

Disjoint unions

Definition 93 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B = \underbrace{(\{1\} \times A)}_{= \{(1,a) \mid a \in A\}} \cup \overbrace{(\{2\} \times B)}^{= \{(2,b) \mid b \in B\}} \quad .$$

Thus,

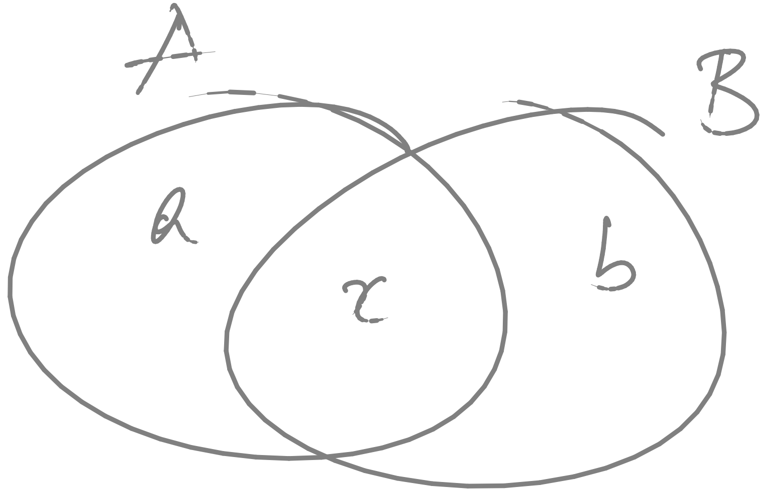
$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1, a)\big) \vee \big(\exists b \in B.\, x = (2, b)\big)\,.$$

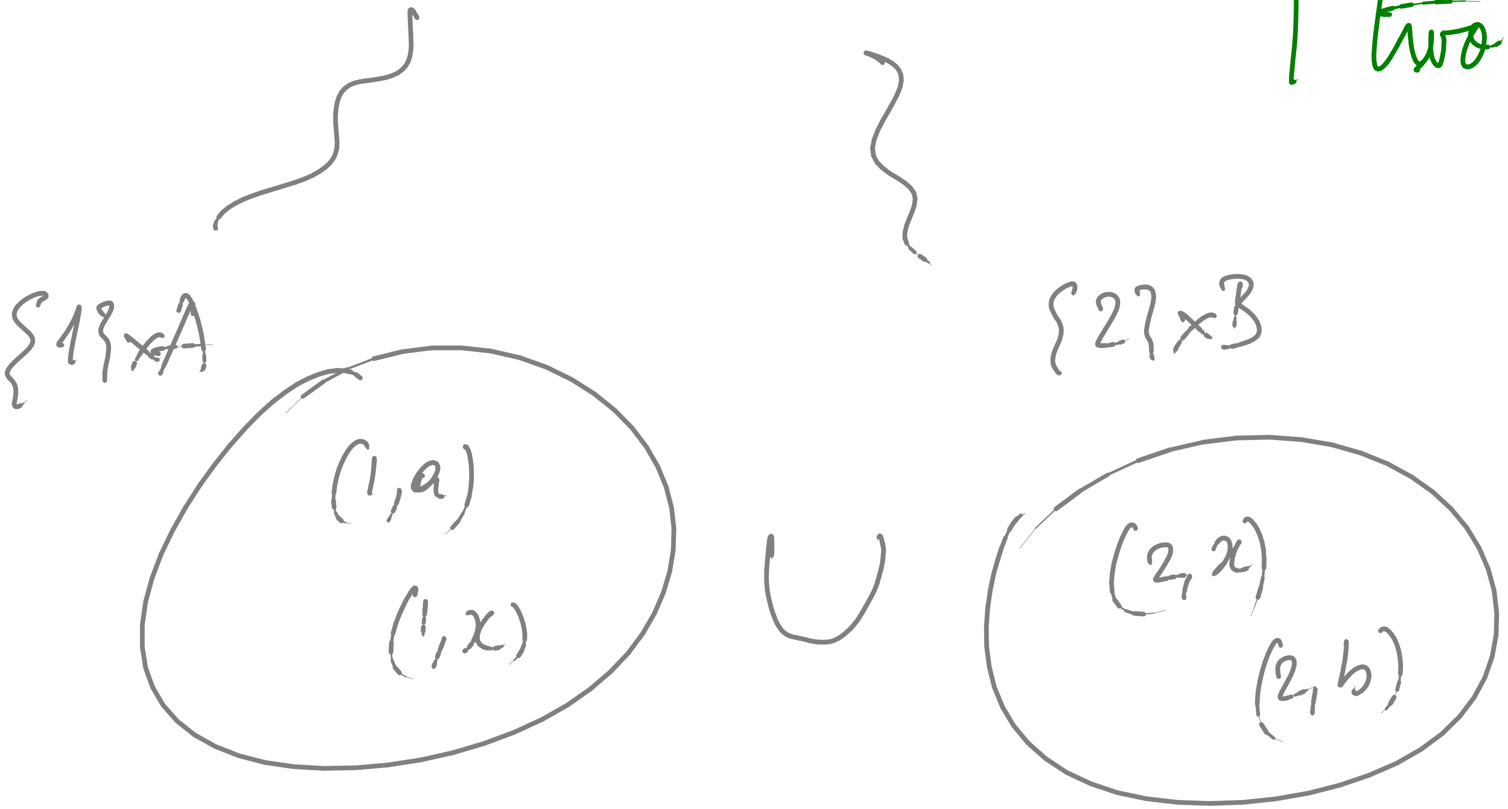
Cf. In ML datatype

(α, β) dunion

= one of α

| two of β





Proposition 95 *For all finite sets* A *and* B*,*

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$$\underbrace{a_1 \ldots a_n \quad b_1 \ldots b_m}_{n+m}$$

$$(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$$

Corollary 96 *For all finite sets* A *and* B*,*

$$\#(A \uplus B) = \#A + \#B \quad .$$

$$a \cdot b = b \cdot a$$

$$\#(A \times B) = (\#A) \cdot (\#B) = \#(B \times A)$$

but in general

$$A \times B \neq B \times A$$

but

$$\underbrace{A \times B \cong B \times A}_{\text{to be defined.}}$$