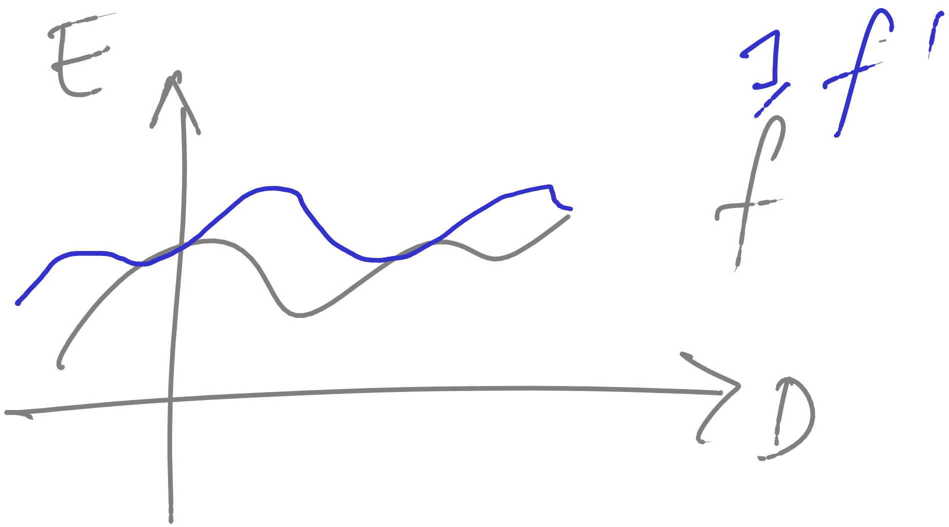
Function cpo's and domains

~ Model → Types in ML

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.



- $f \sqsubseteq f$ ✓
- $f \sqsubseteq g \sqsubseteq h \Rightarrow f \sqsubseteq h$
- $f \sqsubseteq g \sqsubseteq f \Rightarrow f = g$

$\bot_{D\to E}(d) = \bot_E$

$\Uparrow$

$\bot_{D\to E}(d) \sqsubseteq \bot_E$

$\Uparrow$

$\bot_{D\to E}(d) \sqsubseteq e$ $\forall d\ \forall e$

$\Uparrow$

NB E has least element, say $\bot_E$. Then $((D \to E), \sqsubseteq)$ has least element.

$\bot_{D\to E} \sqsubseteq \lambda d.e$ $\quad \forall e \in E$

$= _{\text{def}}\ \lambda d.\bot_E$

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has lubs of ω chains.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Consider

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad \text{in } (D \to E)$$

Show that there is a lub

Let g be an upper bound; i.e $\forall n \; f_n \sqsubseteq g$

i.e $\forall n \; \forall d \; f_n(d) \sqsubseteq g(d)$

Hence $g(d)$ for fixed d is an upper bound of the chain

$$f_0(d) \sqsubseteq f_1(d) \sqsubseteq \cdots \sqsubseteq f_n(d) \sqsubseteq \cdots \quad \text{in } E$$

Therefore $\bigsqcup_n f_n(d) \sqsubseteq g(d)$ for arbitrary d

So we define $\bigsqcup_n f_n \in (D \to E)$ as $\lambda d. \bigsqcup_n (f_n d)$.

Need to check that the definition

$$\bigsqcup_n f_n = \lambda d.\ \bigsqcup_n (f_n d)$$

$$\frac{x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n}$$

yields a continuous function:

- $x \sqsubseteq y \Rightarrow (\bigsqcup_n f_n)(x) \overset{?}{\sqsubseteq} (\bigsqcup_n f_n)(y)$

$$(\bigsqcup_n f_n)(x) = \bigsqcup_n (f_n(x)) \qquad (\bigsqcup_n f_n)(y) = \bigsqcup_n (f_n(y))$$

f_n mon.

$$f_n(x) \sqsubseteq f_n(y) \ \forall n$$

- $(\bigsqcup_n f_n)(\bigsqcup_m d_m) \overset{?}{=} \bigsqcup_m \left((\bigsqcup_n f_n)(d_m) \right)$

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_m d_m\right) \overset{\text{by def}}{=} \bigsqcup_n \left(f_n\left(\bigsqcup_m d_m\right)\right)$$

$$\overset{f_n \text{ cont.}}{=} \bigsqcup_n \bigsqcup_m f_n(d_m)$$

$$\overset{\text{diag.}}{=} \bigsqcup_m \bigsqcup_n (f_n(d_m))$$

$$\overset{\text{by def}}{=} \bigsqcup_m \left(\bigsqcup_n f_n\right)(d_m)$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \;=\; \lambda d \in D.\ \bigsqcup_{n\geq 0} f_n(d)\ .$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \;=\; \lambda d \in D.\, \bigsqcup_{n\geq 0} f_n(d)\;.$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

∘ is monotone ⟺ is monotone in each argument

- $f \sqsubseteq f'$ in $(E \to F)$ g in $(D \to E)$
  $\Rightarrow g \circ f \sqsubseteq g \circ f'$

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : ((E \to F) \times (D \to E)) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f = \lambda d \in D . g(f(d))$$

is continuous.

- $g \sqsubseteq g'$ in $(D \to E)$ and f in $(E \to F)$
  $\Rightarrow g \circ f \sqsubseteq g' \circ f$.

$g(fd)$

$\|$

$(g \circ f)(d) \sqsubseteq (g' \circ f)(d) = g'(fd)$

• $\circ$ is cont $\Leftrightarrow$ cont in each argument.

$$g \circ \left(\bigsqcup_n f_n\right) \overset{?}{=} \bigsqcup_n (g \circ f_n)$$

&

$$\left(\bigsqcup_n g_n\right) \circ f \overset{?}{=} \bigsqcup_n (g_n \circ f)$$

$$\forall d.\ \left(\left(\bigsqcup_n g_n\right) \circ f\right)(d) \overset{?}{=} \left(\bigsqcup_n (g_n \circ f)\right)(d)$$

$$\left(\bigsqcup_n g_n\right)(f\,d) = \bigsqcup_n g_n(f\,d) = \bigsqcup_n (g_n \circ f)(d) = \left(\bigsqcup_n (g_n \circ f)\right)(d)$$

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

$f \sqsubseteq g$ in $(D \to D) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g)$ in D

f_n chain in $(D \to D) \implies \mathit{fix}(\bigsqcup_n f_n) = \bigsqcup_n \mathit{fix}(f_n)$

$$\frac{h(y) \sqsubseteq y}{\underline{fix}(h) \sqsubseteq y}\ (lfp2)$$

use that $f_n(\underline{fix}\ f_n) \sqsubseteq \underline{fix}(f_n)$

? $\sqsubseteq \bigsqcup_n \underline{fix}(f_n)$

$$\frac{(\bigsqcup_n f_n)(\bigsqcup_n \underline{fix}(f_n)) \sqsubseteq \bigsqcup_n \underline{fix}(f_n)}{\underline{fix}(\bigsqcup_n f_n) \sqsubseteq \bigsqcup_n \underline{fix}(f_n)}$$

$$\cfrac{\cfrac{\cfrac{\checkmark}{\forall n\ \ f_n \sqsubseteq \bigsqcup_n f_n}\ (lub\ 1)}{\forall n.\ \underline{fix}(f_n) \sqsubseteq \underline{fix}(\bigsqcup_n f_n)}\ \text{fix mon.}}{\bigsqcup_n \underline{fix}(f_n) \sqsubseteq \underline{fix}(\bigsqcup_n f_n)}$$

$$\frac{\underline{fix}(\bigsqcup_n f_n) \sqsubseteq \bigsqcup_n \underline{fix}(f_n) \qquad \bigsqcup_n \underline{fix}(f_n) \sqsubseteq \underline{fix}(\bigsqcup_n f_n)}{\underline{fix}(\bigsqcup_n f_n) = \bigsqcup_n \underline{fix}(f_n)}$$

Topic 4

Scott Induction

Suppose $\perp \in S \Rightarrow f(\perp) \in S \Rightarrow f^2(\perp) \in S$ ... $f^n(\perp) \in S$

Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq$ in S

$$fix(f) \in S \;,$$

$\Rightarrow \bigsqcup d_n \in S$

it suffices to prove

Then we have an argument for Scott Induction

$$\forall d \in D \; (d \in S \;\Rightarrow\; f(d) \in S) \; .$$

$$\frac{\forall d \quad d \in S \Rightarrow f(d) \in S}{fix(f) \in S} \quad (S \text{ ADMISSIBLE})$$

$\bigsqcup_n f^n(\perp)$ ===