preserves the structure

$\sqsubseteq, \perp, \bigsqcup$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\left(\mathit{fix}(f)\right) = \mathit{fix}(f)$, and hence is the least fixed point of f.

$f : D \to D$

by induction

$\perp \sqsubseteq f(\perp) \sqsubseteq ff\perp \sqsubseteq \dots$

$f^n(\perp) \sqsubseteq f^{n+1}(\perp) \sqsubseteq \dots$

$\perp$ is least

f monotone

$f\perp \sqsubseteq ff\perp$

we have a countable chain

Consider $\bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

apply f to the chain to obtain the new chain

$f(\perp) \sqsubseteq ff\perp \sqsubseteq \dots \sqsubseteq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp) \sqsubseteq \dots$

The green chain is the blue chain without the first element hence they have the same lub; i.e.

$$\bigsqcup_n f^n(\bot) = \bigsqcup_m f^{m+1}(\bot)$$

$$f\left(\bigsqcup_n f^n(\bot)\right) \underbrace{=}_{\text{by cont}} \bigsqcup_n f(f^n \bot) = \bigsqcup_n f^{n+1}(\bot)$$

$$\| $$

$$\bigsqcup_n f^n(\bot)$$

Hence, $\underline{fix}(f) =^{\text{def}} \bigsqcup_n f^n(\bot)$
is s.t.

$$f(\underline{fix} f) = \underline{fix}(f)$$

We show $\mathrm{fix}(f) =_{\mathrm{def}} \bigsqcup_n f^n(\bot)$ is least amongst all prefixed points. So let x be a prefixed point; i.e. $f(x) \sqsubseteq x$. We want to show $\mathrm{fix}(f) \sqsubseteq x$

$$\begin{tikzcd}
 & f\bot \sqsubseteq fx \sqsubseteq x \arrow[dr, Rightarrow] & & ff\bot \sqsubseteq fx \sqsubseteq x \arrow[dr, Rightarrow] & \\
\overline{\bot \sqsubseteq x} \arrow[ur, Rightarrow] & , & f(\bot) \sqsubseteq x \arrow[ur, Rightarrow] & , & ff(\bot) \sqsubseteq x \quad , \dots
\end{tikzcd}$$

$$\frac{\forall n \quad f^n(\bot) \sqsubseteq x \quad \text{by induction}}{\bigsqcup_n f^n(\bot) = \mathrm{fix}(f) \sqsubseteq x}$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$
\begin{aligned}
&[\![\textbf{while } B \textbf{ do } C]\!] \\
&= \mathit{fix}(f_{[\![B]\!],[\![C]\!]}) \\
&= \textstyle\bigsqcup_{n\geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot) \\
&= \quad \lambda s \in \mathit{State}. \\
&\qquad \begin{cases}
[\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\
& \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\
\\
\text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0
\end{cases}
\end{aligned}
$$

Modelling ML types. bool type, nat type.

PRODUCTS α, β type $\Longrightarrow$ $\alpha * \beta$ type

FUNCTIONS $\Longrightarrow$ $\alpha \to \beta$ type

Topic 3

Constructions on Domains

ENUMERATED

DATATYPES

RECURSIVE

UI

INDUCTIVE

(lists, trees, ..)

outside the scope of the course

Discrete cpo's and flat domains

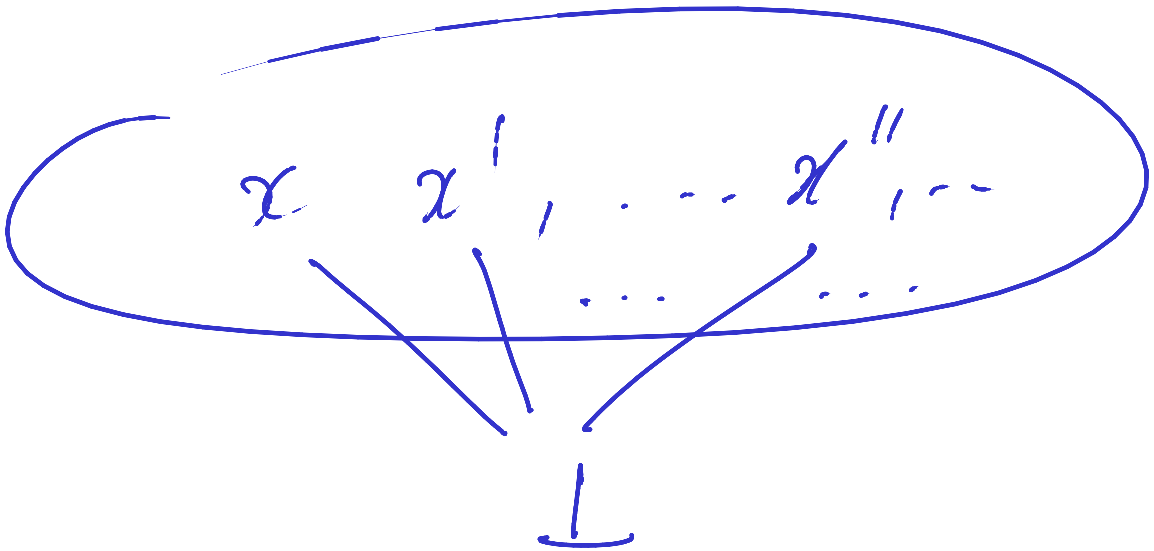
For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

$X_\perp$ — flat domain

$\left\{ \begin{array}{l} X \\ \\ \end{array} \right.$



$\mathbb{B} = \{true, false\} \rightsquigarrow \mathbb{B}_\perp$ boolean domain

Discrete cpo's and flat domains

For any set X, the relation of equality

$\mathbb{N}_\perp$ natural numbers domain

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \;\&\; d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \;\&\; d_2 \sqsubseteq_2 d_2' \;.$$

pointwise order

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Show that $D_1 \times D_2$ is a cpo / domain for D_1, D_2 cpos / domains

Suppose $\bot_1$ is least in D_1 and $\bot_2$ is least in D_2

Then, $(\bot_1, \bot_2)$ is least in $D_1 \times D_2$

$$\dfrac{\dfrac{\checkmark}{\bot_1 \sqsubseteq d_1} \qquad \dfrac{\checkmark}{\bot_2 \sqsubseteq d_2}}{(\bot_1, \bot_2) \sqsubseteq (d_1, d_2)}$$

Let $(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$

be a chain in $D_1 \times D_2$

Let (d_1, d_2) be an upper bound for the above chain; i.e

$\forall n \; (x_n, y_n) \sqsubseteq (d_1, d_2)$ iff $\forall n \; x_n \sqsubseteq d_1$ & $y_n \sqsubseteq d_2$

$$\iff \bigsqcup_n x_n \sqsubseteq d_1 \;\&\; \bigsqcup_n y_n \sqsubseteq d_2$$

Using $\Leftarrow$ with $d_1 = \bigsqcup_n x_n$ and $d_2 = \bigsqcup_n y_n$, we get

$$\forall n \; (x_n, y_n) \sqsubseteq \left(\bigsqcup_n x_n, \bigsqcup_n y_n\right)$$

and $(\Rightarrow)$ shows it is least.

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d\,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$$f : (D \times E) \to F$$

f monotone iff $(x,y) \sqsubseteq (x',y')$ in $D \times E$

$$\Rightarrow f(x,y) \sqsubseteq f(x',y') \text{ in } F$$

Lemma iff $x \sqsubseteq x' \Rightarrow f(x,y) \sqsubseteq f(x',y) \quad \forall y$

and $y \sqsubseteq y' \Rightarrow f(x,y) \sqsubseteq f(x,y') \quad \forall x$

($\bullet \Rightarrow \bullet$) $(x,y) \sqsubseteq (x',y') \Rightarrow x \overset{1}{\sqsubseteq} x'$ & $y \overset{2}{\sqsubseteq} y'$

$$\left.\begin{array}{l} 1 \Rightarrow (x,y) \sqsubseteq (x',y) \Rightarrow f(x,y) \sqsubseteq f(x',y) \\ 2 \Rightarrow (x',y) \sqsubseteq (x',y') \Rightarrow f(x',y) \sqsubseteq f(x',y') \end{array}\right\} \Rightarrow \text{transitivity}$$

f is cont. iff$_{\text{def}}$ it is monotone and

$$f\left(\bigsqcup_n (x_n, y_n)\right) = \bigsqcup_n f(x_n, y_n)$$

iff f is monotone and

$\forall$ chain $\langle x_n \rangle$ and y,

$$f\left(\bigsqcup_n x_n, y\right) = \bigsqcup_n f(x_n, y)$$

and

$\forall$ chain $\langle y_n \rangle$ and x,

$$f\left(x, \bigsqcup_n y_n\right) = \bigsqcup_n f(x, y_n)$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (\underline{f \text{ continuous}})$$

N.B:

$$f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_m f(x_m, \bigsqcup_n y_n) = \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

$$= \bigsqcup_k f(x_k, y_k)$$