UNIVERSITY OF CAMBRIDGE
Computer Laboratory

Computer Systems Modelling

Computer Science Tripos, Part II
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Problem sheets

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Examples (i)

1. In what contexts is simulation an appropriate technique for performance evaluation? When it is inappropriate?

2. How would you generate random variates for the exponential distribution? Write pseudo code for a simulation component which models the arrival process of customers at a bank, given that the mean arrival rate is 40 customers per hour.

3. How would you generate pseudo random variables from the mixture of exponential distributions with probability density function $f(x)$ for $x > 0$ given by

$$f(x) = \frac{1}{3}\lambda_1 \mathrm{e}^{-\lambda_1 x} + \frac{2}{3}\lambda_2 \mathrm{e}^{-\lambda_2 x}$$

   where $\lambda_1 = 5$ and $\lambda_2 = 10$?

4. Let $X_1, X_2, \ldots$ be a sequence of observations and define the sample mean, $\overline{X}_n$, of the first n observations by $\overline{X}_n = \sum_{i=1}^{n} X_i/n$. Show that for $n > 1$

$$\overline{X}_n = \overline{X}_{n-1} + \frac{X_n - \overline{X}_{n-1}}{n}\,.$$

   Now let $\overline{S}_n = \sum_{i=1}^{n}(X_i - \overline{X}_n)^2$ and show that for $n > 1$

$$\overline{S}_n = \overline{S}_{n-1} + (X_n - \overline{X}_{n-1})(X_n - \overline{X}_n)\,.$$

   Use these results to write a pseudo-code procedure to update the sample mean and sample variance of the first $n-1$ observations with the addition of the latest observation, X_n, at time n. Comment on how your pseudo-code improves on naive methods which you should state to compute the sample mean and variance of n observations. Now implement your algorithm using pseudo-random exponential random variables with some fixed parameter λ and demonstrate that your computations of the sample mean and sample variance give accurate results with large sample sizes.

5. Suppose that a simulation is constructed to estimate the mean response time, in milliseconds, of an interactive computer system. A number of repetitions are performed with the following results (measured in milliseconds): 4.1, 3.6, 3.1, 4.5, 3.8, 2.9, 3.4, 3.3, 2.8, 4.5, 4.9, 5.3, 1.9, 3.7, 3.2, 4.1, 5.1.

   Calculate the sample mean and variance of these results and hence derive an approximate 95% confidence interval for the mean response time. You may wish to make use the result that if $Z \sim N(0,1)$ then $\Phi(1.96) = \mathbb{P}(Z \leq 1.96) = 0.975$.

6. An experiment is performed to estimate the performance of a $M/M/1$ system with FIFO queueing. The response times of each of 1000 successive customers are recorded and the sample mean ($\overline{X}$) and sample variance (S^2) of these numbers are calculated.

A student believes that $100(1-\alpha)\%$ confidence bounds for the mean can be derived using the expression

$$\overline{X} \pm \frac{z_{\alpha/2} S}{\sqrt{1000}}$$

where $\mathbb{P}(Z > z_{\alpha/2}) = \alpha/2$ and Z is a standard Normal random variable.

What mistake has this student made? Would their technique be valid if they were estimating the service time of this queue?

7. Using the inverse transform method show that

$$X = \left\lfloor \frac{\log(U)}{\log(1-p)} \right\rfloor + 1$$

has a geometric distribution with parameter p when U has the $U(0,1)$ distribution.

Examples (ii)

1. Describe a procedure for the generation of the first T time units of a Poisson process of fixed rate λ. How could you modify the procedure to instead simulate a non-homogeneous Poisson process with an arrival rate, $\lambda(t)$, as a function of time t. You may assume that the arrival rate is bounded above by λ^*, say.

2. Describe the variance reduction technique based on antithetic variables and give an example of how it is used.

3. For the variance reduction technique based on control variates derive the optimal choice of $c = c^*$ and its associated variance given in the lectures.

4. Suppose that X is a random variable having the Binomial distribution with parameters n and p and that $\lambda > 0$ is a constant. Write down the expression for $\mathbb{P}(X = k)$ where $k \in \{0, 1, 2, \ldots, n\}$. Now suppose that $n \to \infty$ and p is chosen so that $p = \lambda/n$. Show that under this limit $\mathbb{P}(X = k) \to e^{-\lambda}\lambda^k/k!$, that is, to a Poisson distribution with parameter λ.

5. Suppose that $N(t)$ is the random number of events in the time interval $[0, t]$ of a Poisson process with parameter λ. State the conditions that define the Poisson process $N(t)$ and show that for all $t > 0$ the random variable $N(t)$ has the Poisson distribution with parameter λt.

6. In what situations does queueing theory provide appropriate techniques for performance evaluation? When does it not?

7. Show for the M/M/1 queue that the probability that there are n or more customers in the system is given by ρ^n.

   Use this result to find a service rate μ such that, for given λ, n, α where $0 < \alpha < 1$, the probability of n or more customers in the system is given by α.

   Find a value of μ for an M/M/1 queue for which the arrival rate is 10 customers per second, and subject to the requirement that the probability of 3 or more customers in the system is 0.05

Examples (iii)

1. Using the steady-state distribution of the number of jobs in a $M/M/1$ queueing system given in lectures derive the first and second moments of this distribution and hence the variance of the number of jobs present. Describe what happens to the queueing system as the load ρ increases.

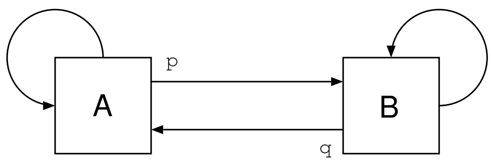
2. Given an $M/M/1/K$ queue with $\lambda = 10$, $\mu = 12$ and $K = 15$ over what proportion of time are customers rejected from the queue? What is the effective arrival rate? What is the utilization of the server?

3. For the $M/M/m/m$ loss system show that the loss probability, $E(\rho, m)$, that all servers are occupied is given by
$$E(\rho, m) = \frac{\rho^m/m!}{\sum_{i=0}^{m} \rho^i/i!}$$
where $\rho = \lambda/\mu$ is the ratio of the arrival rate to the service rate of a single server. Show also that $E(\rho, m)$ satisfies the recurrence equation
$$E(\rho, m) = \frac{\rho E(\rho, m-1)}{m + \rho E(\rho, m-1)}$$
with boundary condition $E(\rho, 0) = 1$. Why is this expression useful for the numerical computation of loss probabilities?

4. Repairing a computer takes 4 stages in sequence, namely removing the lid, finding the faulty part, replacing it, and reassembling the machine. Each step is independent and exponentially distributed with mean 3 minutes. What is the coefficient of variation of the repair time?

5. Road vehicles travel past a fixed location according to points of a Poisson process with constant rate λ. Suppose that a person wishing to cross the road at that location waits before crossing until they can see that no vehicle will arrive in the next T units of time. Find the expected time that the person waits before starting to cross the road. Note that if no cars will pass in the first T units of time then the waiting time is 0. Write a simulation program to validate your analytical result for the expected waiting time as a function of λ and T.

6. A closed queueing network (shown below) comprises two $\cdot/M/1$ nodes **A** and **B** between which n identical jobs circulate. The nodes have service rates μ_A and μ_B respectively. Upon completion at **A**, a job moves to **B** with probability p and otherwise it remains at **A**. Similarly, upon completion at **B** a job moves to **A** with probability q.



Write down the traffic equations for the arrival rates λ_A and λ_B at nodes **A** and **B**, respectively. Comment on what you find when solving the traffic equations and compute the steady state distribution for the 2-dimensional process describing the numbers of jobs present at each node.