I. Course Intro and Sorting Networks

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Easter 2016

UNIVERSITY OF CAMBRIDGE

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Load Balancing on Graphs

(Tentative) List of Topics

IA Algorithms → IB Complexity Theory → II Advanced Algorithms

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- I. Sorting Networks (Sorting, Counting, Load Balancing)
- II. Matrix Multiplication
- III. Linear Programming
- IV. Approximation Algorithms: Covering Problems
- V. Approximation Algorithms via Exact Algorithms
- VI. Approximation Algorithms: Travelling Salesman Problem
- VII. Approximation Algorithms: Randomisation and Rounding
- VIII. Approximation Algorithms: MAX-CUT Problem (if time permits)

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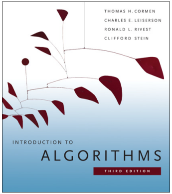

- closely follow CLRS3 and use the same numberring
- however, slides will be self-contained (mostly)

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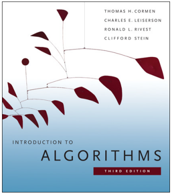

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\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
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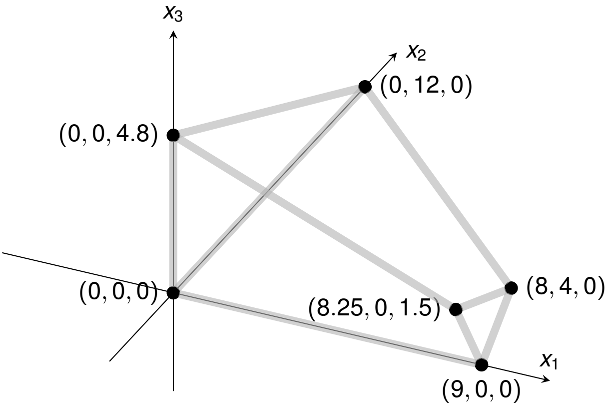

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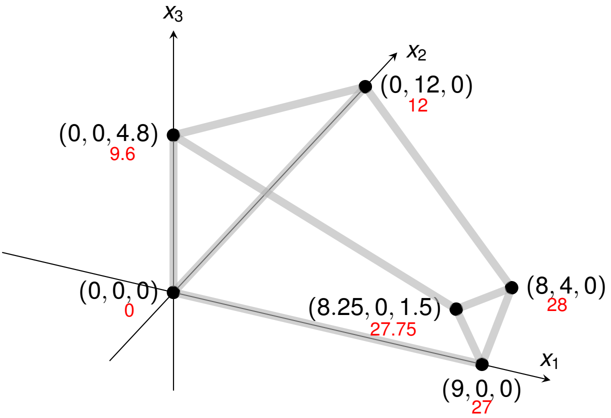

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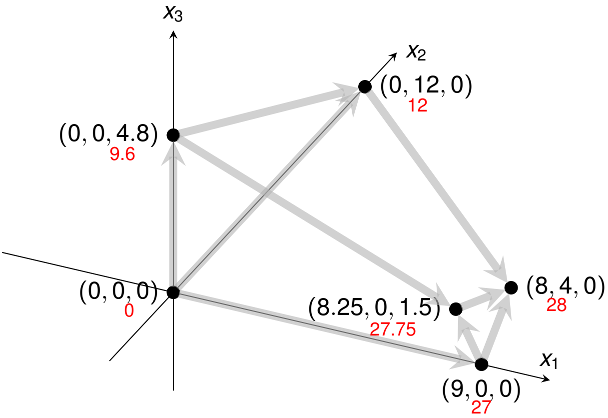

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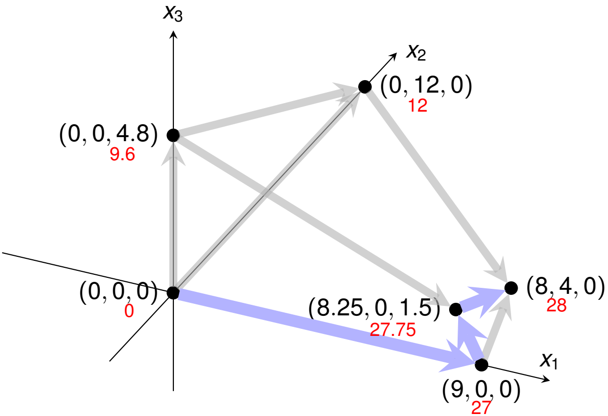

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The Original Article (1954)

SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON
The Rand Corporation, Santa Monica, California

(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

Travelling Salesman Problem: The 42 (49) Cities

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

Road Distances

TABLE I

Road Distances between Cities in Adjusted Units

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
2	8
3	39	45
4	37	47	9
5	50	49	21	15
6	61	62	21	20	17
7	58	60	16	17	18	6
8	59	60	15	20	26	17	10
9	62	66	20	25	31	22	15	5
10	81	81	40	44	50	41	35	24	20
11	103	107	62	67	72	63	57	46	41	23
12	108	117	66	71	77	68	61	51	46	26	11
13	145	149	104	108	114	106	99	88	84	63	49	40
14	181	185	140	144	150	142	135	124	120	99	85	76	35
15	187	191	146	150	156	142	137	130	125	105	90	81	41	10
16	161	170	120	124	130	115	110	104	105	90	72	64	34	31	27
17	142	146	101	104	111	97	91	85	86	75	51	59	29	53	48	21
18	174	178	133	138	143	129	123	117	118	107	83	84	54	46	35	26	31
19	185	186	142	143	140	130	126	124	128	118	93	101	72	69	58	58	43	26
20	164	165	120	123	124	106	106	105	110	104	86	97	71	93	82	62	42	45	22
21	137	139	94	96	94	80	78	77	84	77	56	64	65	90	87	58	36	68	50	30
22	117	122	77	80	83	68	62	60	61	50	34	42	49	82	77	60	30	62	70	49	21
23	114	118	73	78	84	69	63	57	59	48	28	36	43	77	72	45	27	59	69	55	27	5
24	85	89	44	48	53	41	34	28	29	22	23	35	69	105	102	74	56	88	99	81	54	32	29
25	77	80	36	40	46	34	27	19	21	14	29	40	77	114	111	84	64	96	107	87	60	40	37	8
26	87	89	44	46	46	30	28	29	32	27	36	47	78	116	112	84	66	98	95	75	47	36	39	12	11
27	91	93	48	50	48	34	32	33	36	30	34	45	77	115	110	83	63	97	91	72	44	32	36	9	15	3
28	105	106	62	63	64	47	46	49	54	48	46	59	85	119	115	88	66	98	79	59	31	36	42	28	33	21	20
29	111	113	69	71	66	51	53	56	61	57	59	71	96	130	126	98	75	98	85	62	38	47	53	39	42	29	30	12
30	91	92	50	51	46	30	34	38	43	49	60	71	103	141	136	109	90	115	99	81	53	61	62	36	34	24	28	20	20
31	83	85	42	43	38	22	26	32	36	51	63	75	106	142	140	112	93	126	108	88	60	64	66	39	36	27	31	28	28	8
32	89	91	55	55	50	34	39	44	49	63	76	87	120	155	150	123	100	123	109	86	62	71	78	52	49	39	44	35	24	15	12
33	95	97	64	63	56	42	49	56	60	75	86	97	126	160	155	128	104	128	113	90	67	76	82	62	59	49	53	40	29	25	23	11
34	74	81	44	43	35	23	30	39	44	62	78	89	121	159	155	127	108	136	124	101	75	79	81	54	50	42	46	43	39	23	14	14	21
35	67	69	42	41	31	25	32	41	46	64	83	90	130	164	160	133	114	146	134	111	85	84	86	59	52	47	51	53	49	32	24	24	30	9
36	74	76	61	60	42	44	51	60	66	83	102	110	147	185	179	155	133	159	146	122	98	105	107	79	71	66	70	70	60	48	40	36	33	25	18
37	57	59	46	41	25	30	36	47	52	71	93	98	136	172	172	148	126	158	147	124	121	97	99	71	65	59	63	67	62	46	38	37	43	23	13	17
38	45	46	41	34	20	34	38	48	53	73	96	99	137	176	178	151	131	163	159	135	108	102	103	73	67	64	69	75	72	54	46	49	54	34	24	29	12
39	35	37	35	26	18	34	36	46	51	70	93	97	134	171	176	151	129	161	163	139	118	102	101	71	65	65	70	84	78	58	50	56	62	41	32	38	21	9
40	29	33	30	21	18	35	33	40	45	65	87	91	117	166	171	144	125	157	156	139	113	95	97	67	60	62	67	79	82	62	53	59	66	45	38	45	27	15	6
41	3	11	41	37	47	57	55	58	63	83	105	109	147	186	188	164	144	176	182	161	134	119	116	86	78	84	88	101	108	88	80	86	92	71	64	71	54	41	32	25
42	5	12	55	41	53	64	61	61	66	84	111	113	150	186	192	166	147	180	188	167	140	124	119	90	87	90	94	107	114	77	86	92	98	80	74	77	60	48	38	32	6

The (Unique) Optimal Tour (699 Units $\approx$ 12,345 miles)

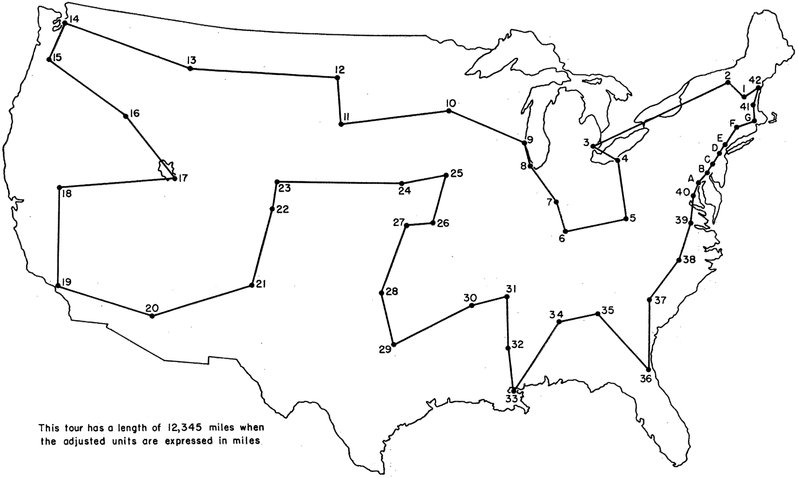

FIG. 16. The optimal tour of 49 cities.

Outline

Overview: Sorting Networks

(Serial) Sorting Algorithms

- we already know several (comparison-based) sorting algorithms: Insertion sort, Bubble sort, Merge sort, Quick sort, Heap sort
- execute one operation at a time
- can handle arbitrarily large inputs
- sequence of comparisons is not set in advance

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Simple concept, but surprisingly deep and complex theory!

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  - comparator is a device with, on given two inputs, x and y, returns two outputs $x' = \min(x, y)$ and $y' = \max(x, y)$

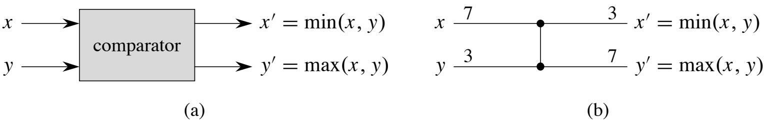

Figure 27.1 **(a)** A comparator with inputs x and y and outputs x' and y'. **(b)** The same comparator, drawn as a single vertical line. Inputs $x = 7$, $y = 3$ and outputs $x' = 3$, $y' = 7$ are shown.

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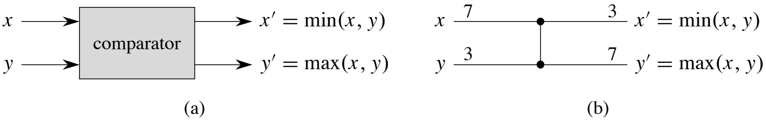
operates in $O(1)$



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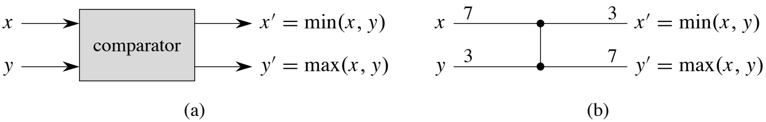

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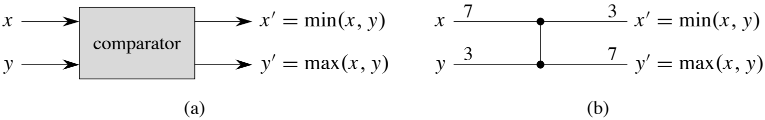

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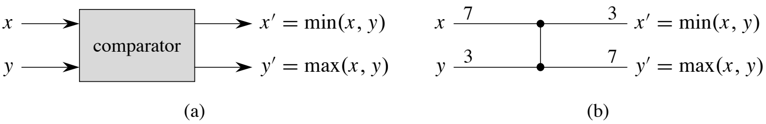
Convention: use the same name for both a wire and its value.



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Comparison Networks

A sorting network is a comparison network which works correctly (that is, it sorts every input)

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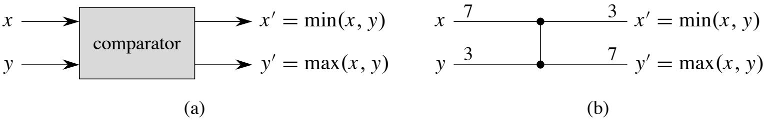

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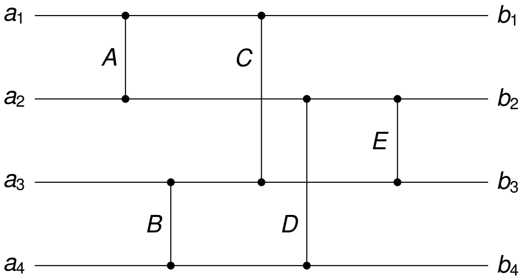
Example of a Comparison Network (Figure 27.2)



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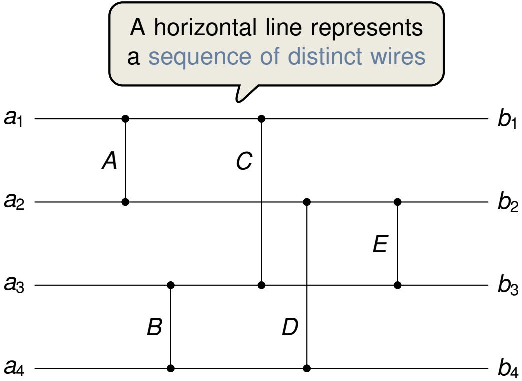

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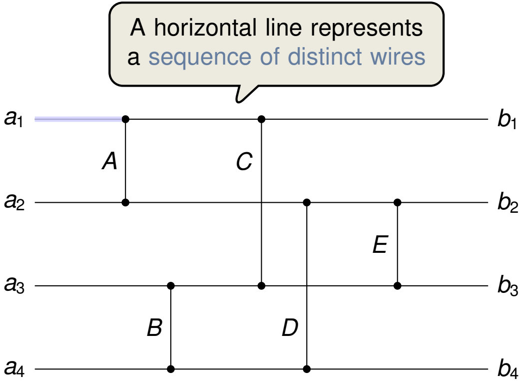

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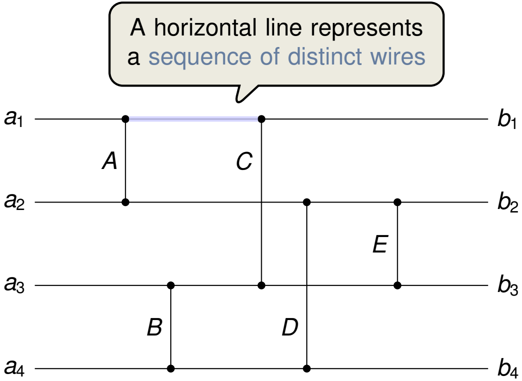

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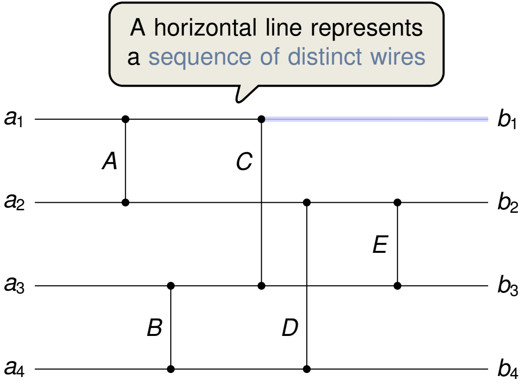

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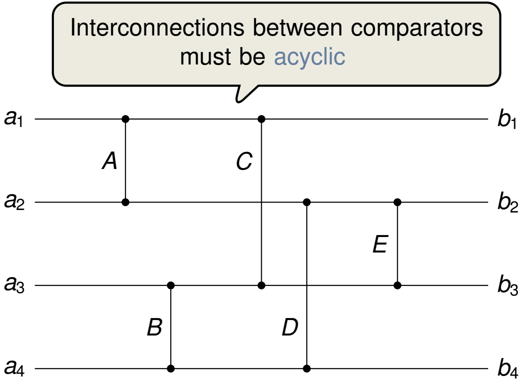

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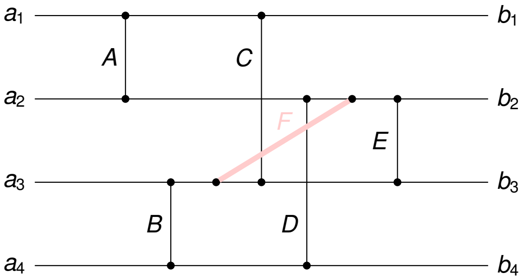
Interconnections between comparators must be acyclic



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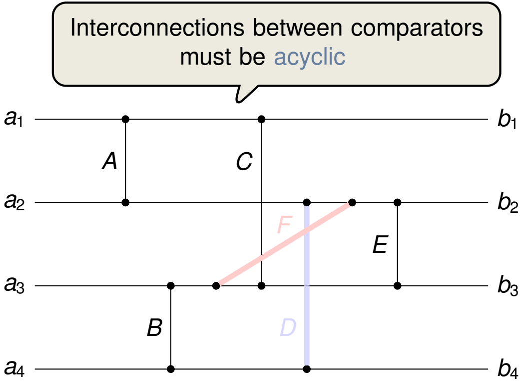

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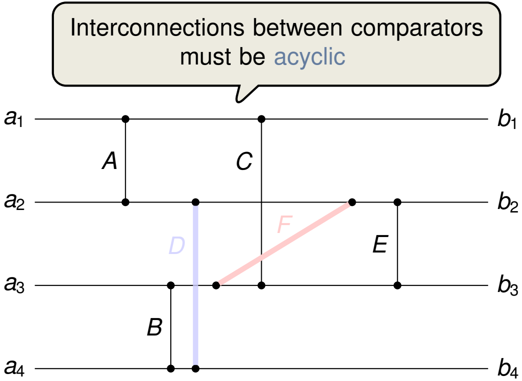

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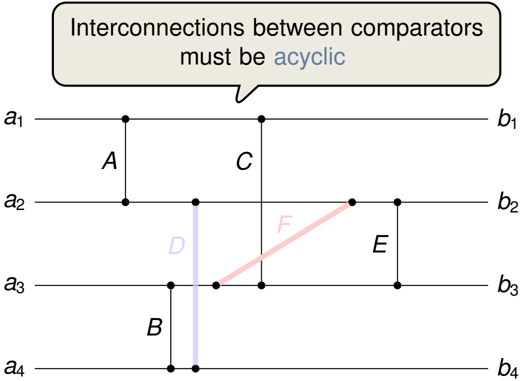

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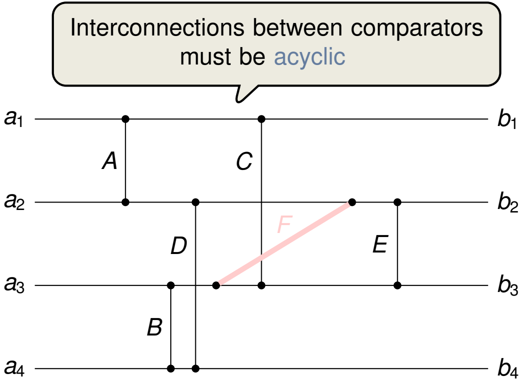

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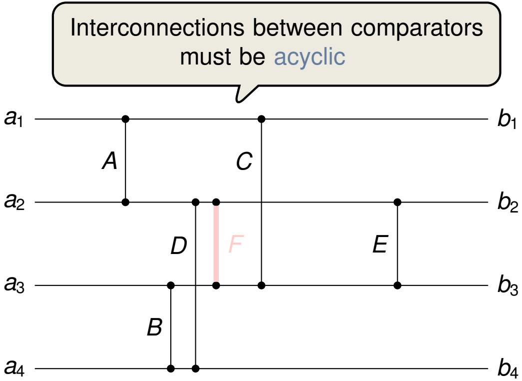
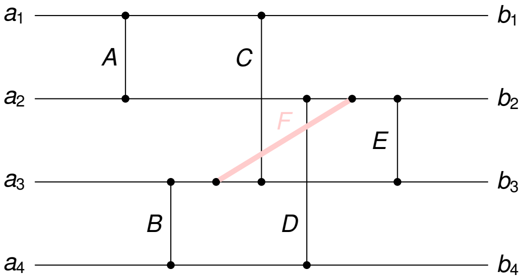


Example of a Comparison Network (Figure 27.2)

Interconnections between comparators must be acyclic ✓



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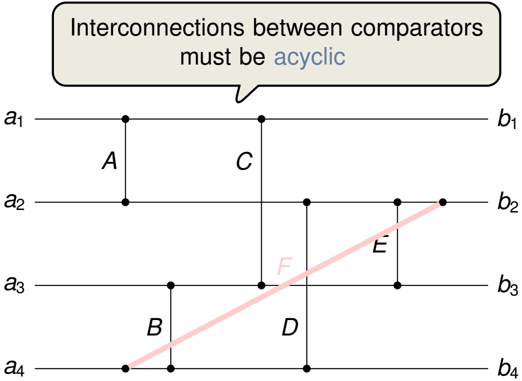

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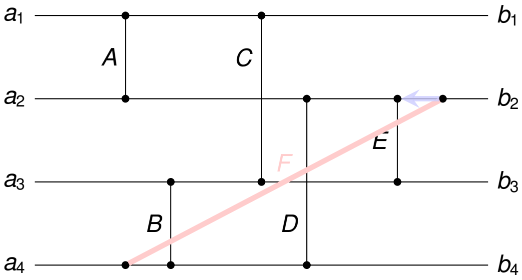

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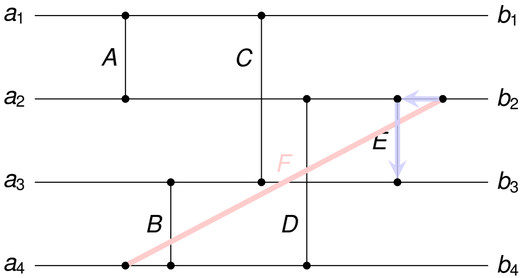

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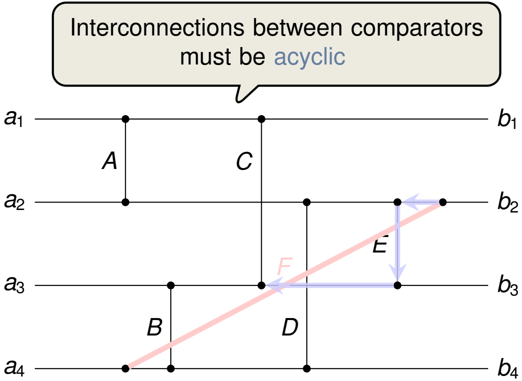

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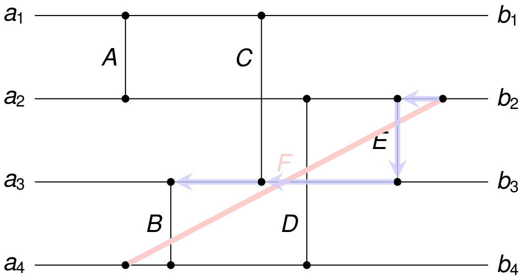

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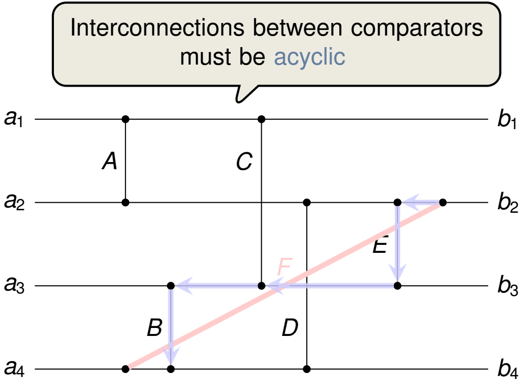

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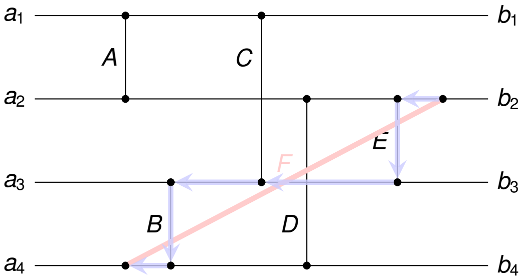

Example of a Comparison Network (Figure 27.2)

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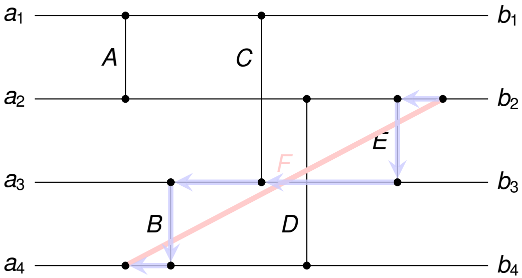
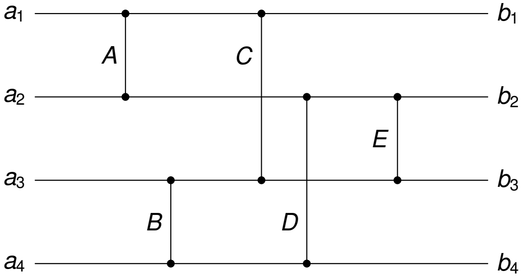


Tracing back a path must never cycle back on itself and go through the same comparator twice.

Example of a Comparison Network (Figure 27.2)



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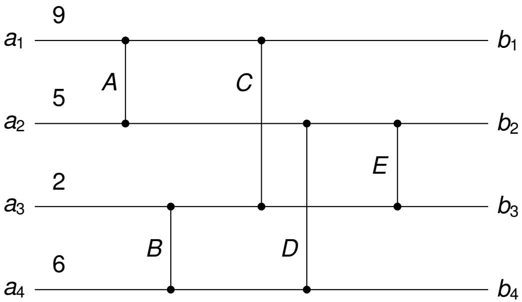

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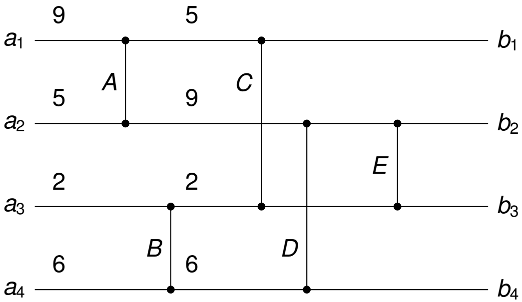

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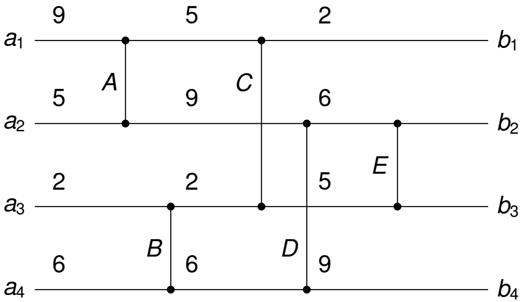

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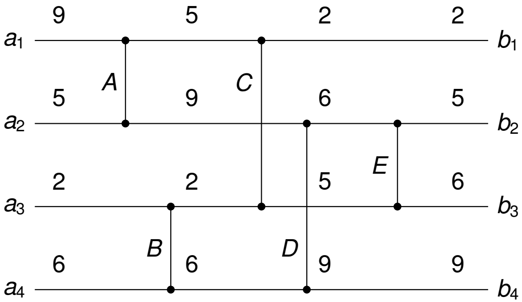
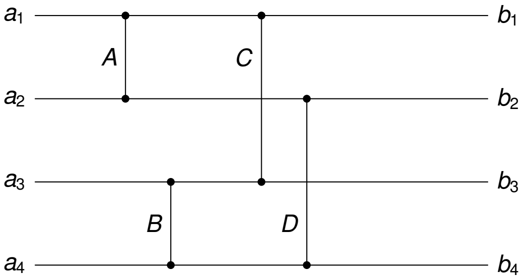


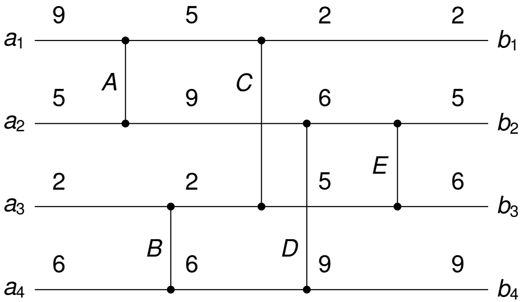
This network is in fact a sorting network (Exercise)

Example of a Comparison Network (Figure 27.2)



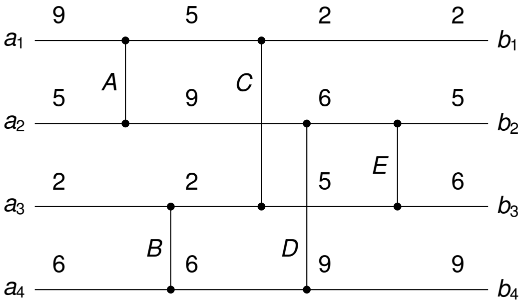
This network would not be a sorting network (Why??)

Example of a Comparison Network (Figure 27.2)



Depth of a wire:

Example of a Comparison Network (Figure 27.2)



Depth of a wire:

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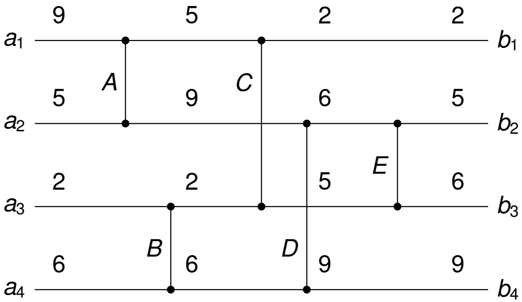

Depth of a wire:

- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

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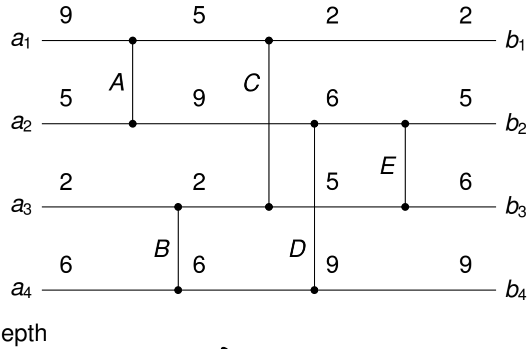

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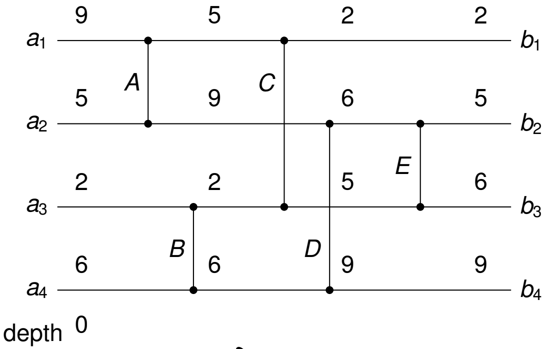

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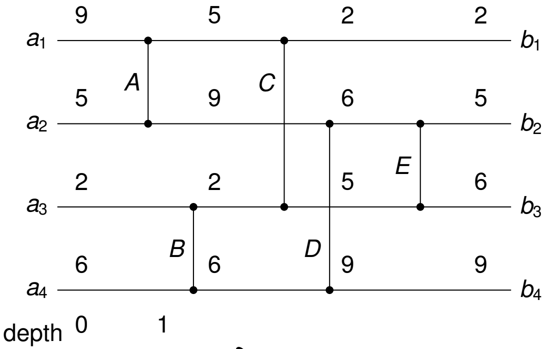

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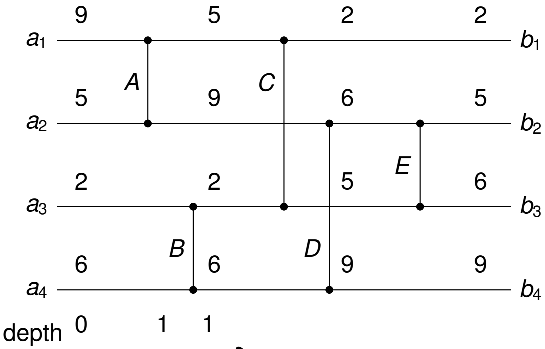

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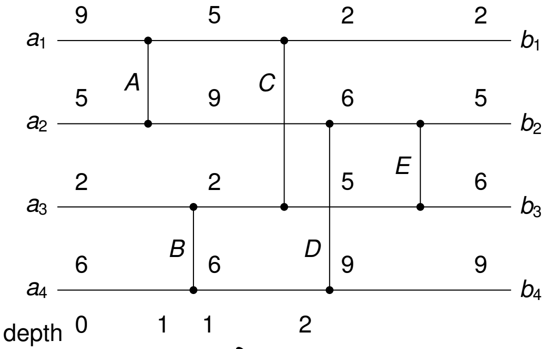

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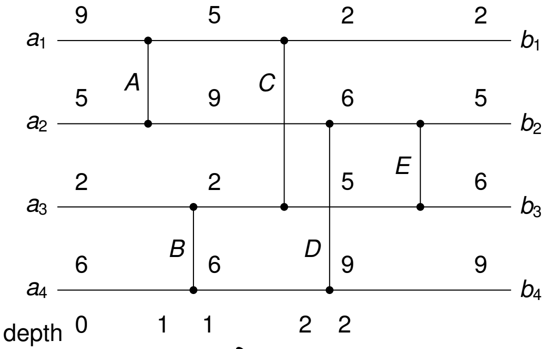

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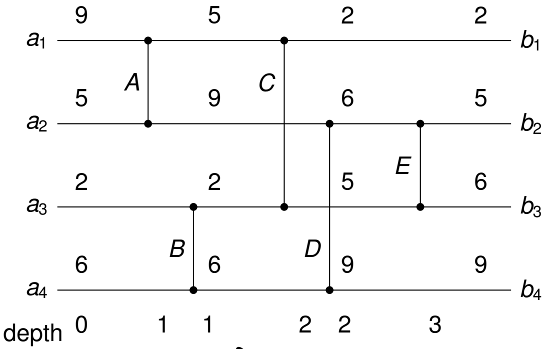

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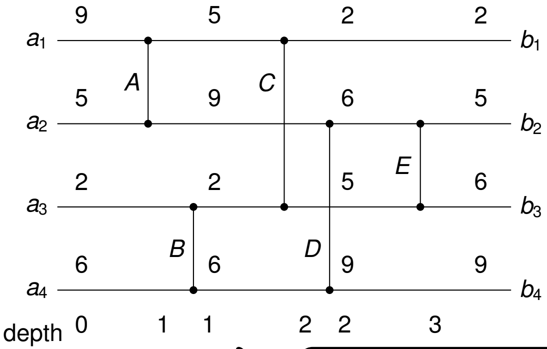

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Maximum depth of an output wire equals total running time

Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

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Lemma 27.1

If a comparison network transforms the input $a = \langle a_1, a_2, \ldots, a_n \rangle$ into the output $b = \langle b_1, b_2, \ldots, b_n \rangle$, then for any monotonically increasing function f, the network transforms $f(a) = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle$ into $f(b) = \langle f(b_1), f(b_2), \ldots, f(b_n) \rangle$.

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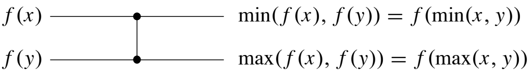

$$\begin{tikzcd} a \arrow[r, "\text{Network}"] \arrow[d, "f"'] & b \arrow[d, "f"] \\ f(a) \arrow[r, "\text{Network}"] & f(b) \end{tikzcd}$$

Figure 27.4 The operation of the comparator in the proof of Lemma 27.1. The function f is monotonically increasing.

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Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

Proof of the Zero-One Principle

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$$f(x) = \begin{cases} 0 & \text{if } x \leq a_i, \\ 1 & \text{if } x > a_i. \end{cases}$$

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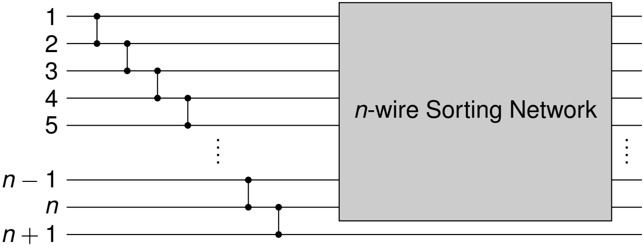
- Since the network places a_j before a_i, by the previous lemma $\Rightarrow f(a_j)$ is placed before $f(a_i)$
- But $f(a_j) = 1$ and $f(a_i) = 0$, which contradicts the assumption that the network sorts all sequences of 0's and 1's correctly $\square$

Some Basic (Recursive) Sorting Networks



???

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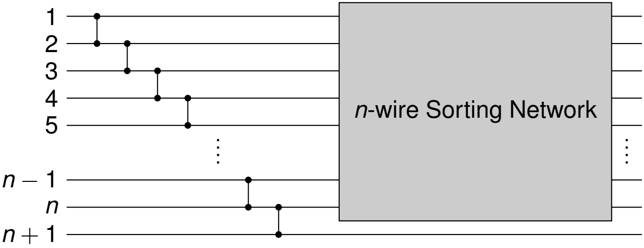

Bubble Sort

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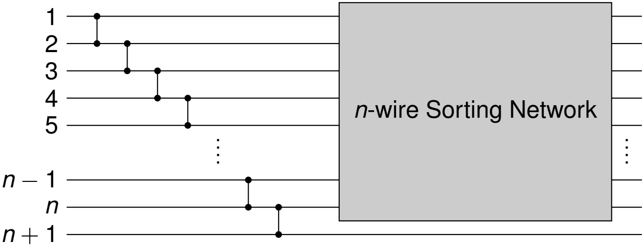

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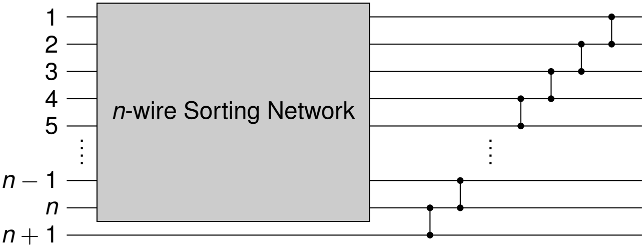

???

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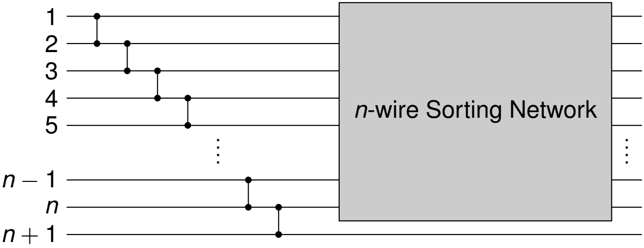

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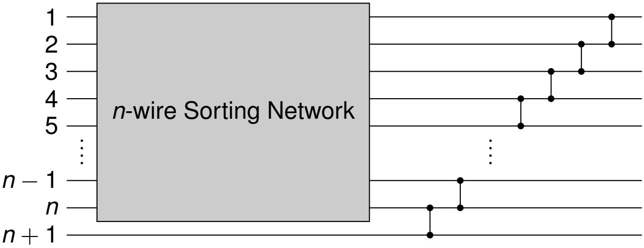
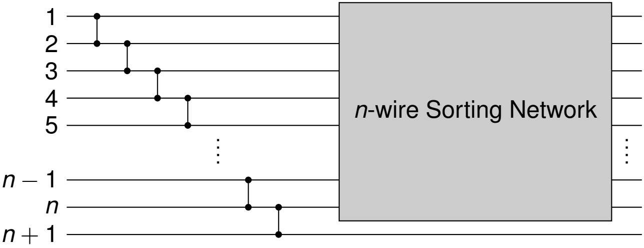


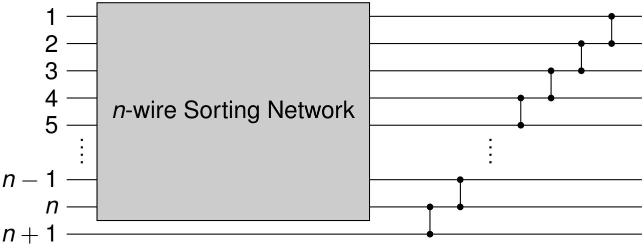
Insertion Sort

Some Basic (Recursive) Sorting Networks



Bubble Sort

These are Sorting Networks, but with depth $\Theta(n)$.



Insertion Sort

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Load Balancing on Graphs

Bitonic Sequences

Bitonic Sequence

A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

Sequences of one or two numbers are defined to be bitonic.

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- $\langle 9, 8, 3, 2, 4, 6 \rangle$?

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Bitonic Sequences

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A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

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- binary sequences: $0^i 1^j 0^k$, or, $1^i 0^j 1^k$, for $i, j, k \geq 0$.

Towards Bitonic Sorting Networks

Half-Cleaner

A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

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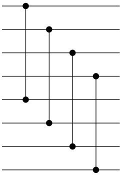

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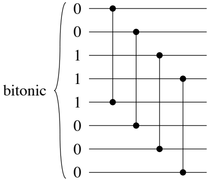

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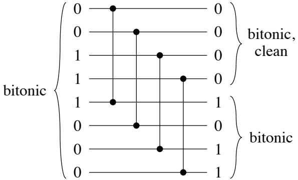

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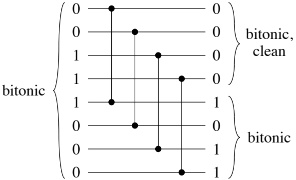

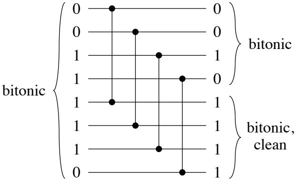

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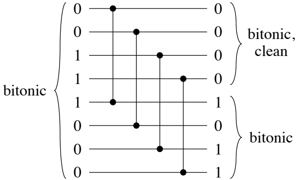
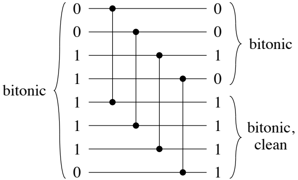
Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.





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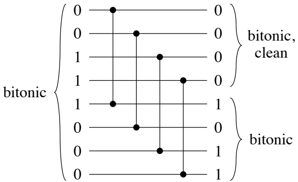

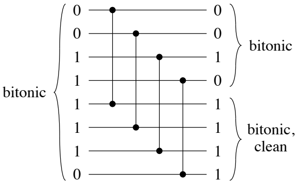

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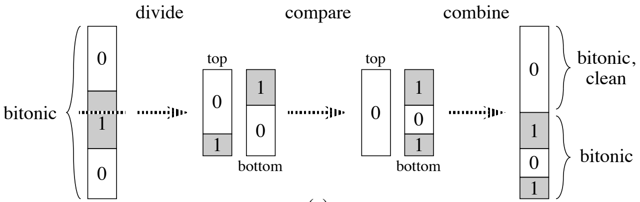

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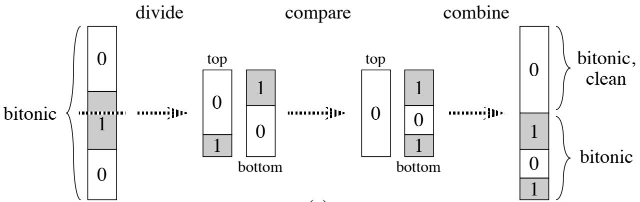

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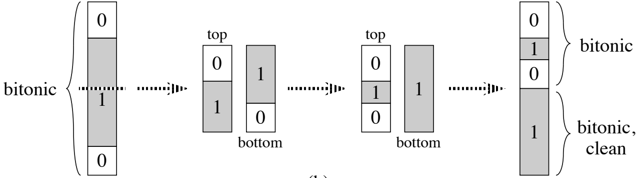

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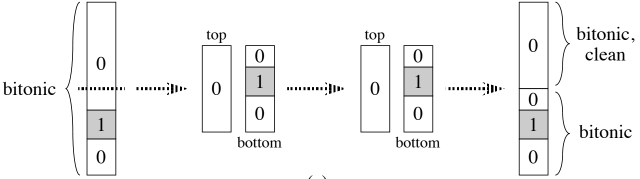

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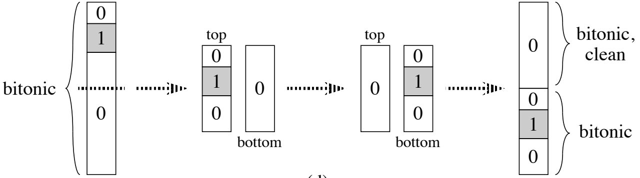

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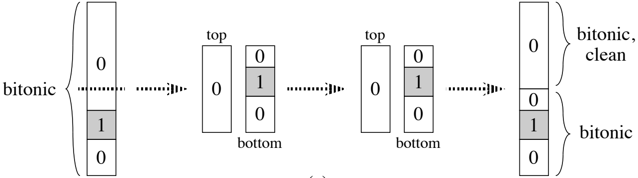

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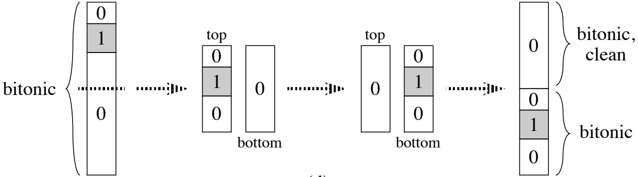

(d)

This suggests a recursive approach, since it now suffices to sort the top and bottom half separately.

The Bitonic Sorter

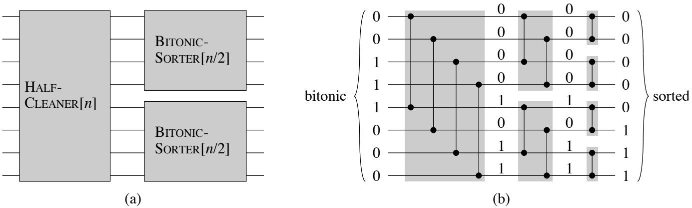

Figure 27.9 The comparison network BITONIC-SORTER$[n]$, shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER$[n]$ followed by two copies of BITONIC-SORTER$[n/2]$ that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

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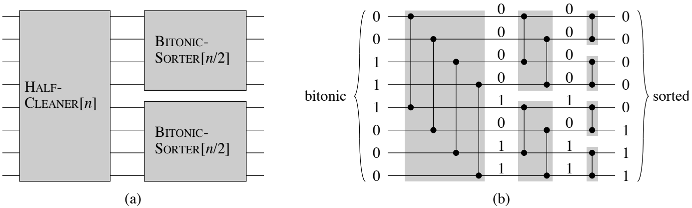

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Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

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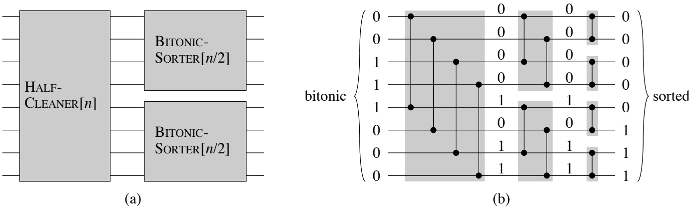

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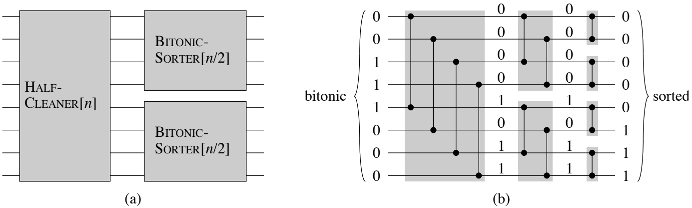

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BITONIC-SORTER[n] has depth $\log n$ and sorts any zero-one bitonic sequence.

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- will be based on a modification of BITONIC-SORTER[n]

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Hence in order to merge the sequences X and Y, it suffices to perform a bitonic sort on X concatenated with Y^R.

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$

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$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i$ for $i = 1, 2, \ldots, n/2$

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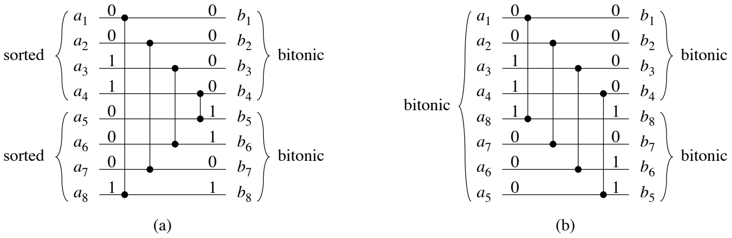

Figure 27.10 Comparing the first stage of MERGER[n] with HALF-CLEANER[n], for $n = 8$. **(a)** The first stage of MERGER[n] transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2}\rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n\rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n\rangle$. **(b)** The equivalent operation for HALF-CLEANER[n]. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1}\rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1}\rangle$.

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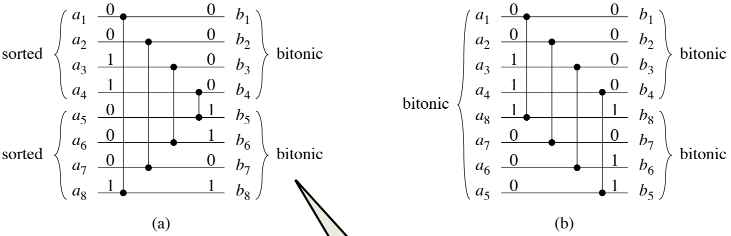

Lemma 27.3 still applies, since the reversal of a bitonic sequence is bitonic.

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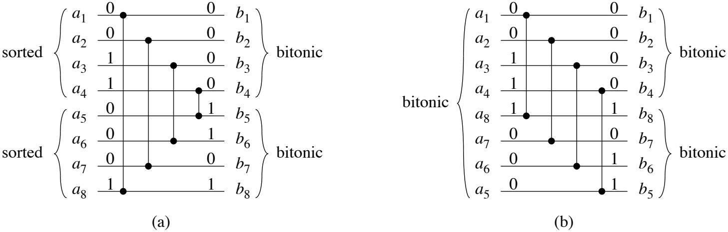

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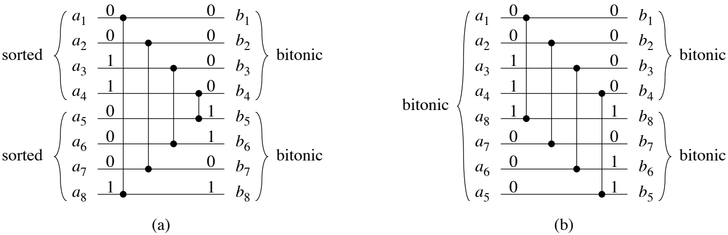
$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i$ for $i = 1, 2, \ldots, n/2$

- Remaining part is identical to BITONIC-SORTER[n]



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Construction of a Merging Network (2/2)

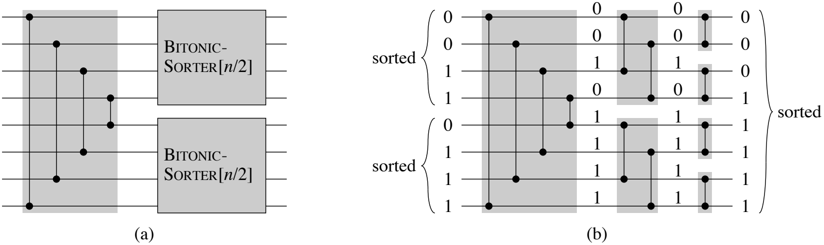
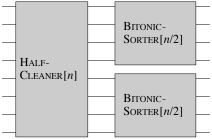


Figure 27.11 A network that merges two sorted input sequences into one sorted output sequence. The network MERGER[n] can be viewed as BITONIC-SORTER[n] with the first half-cleaner altered to compare inputs i and $n - i + 1$ for $i = 1, 2, \ldots, n/2$. Here, $n = 8$. **(a)** The network decomposed into the first stage followed by two parallel copies of BITONIC-SORTER[$n/2$]. **(b)** The same network with the recursion unrolled. Sample zero-one values are shown on the wires, and the stages are shaded.

Construction of a Sorting Network

Main Components

1. BITONIC-SORTER[n]
   - sorts any bitonic sequence
   - depth $\log n$

1.


Construction of a Sorting Network

Main Components

1. BITONIC-SORTER[n]
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2. MERGER[n]
   - merges two sorted input sequences
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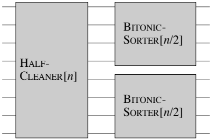

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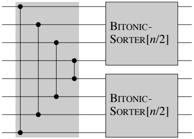

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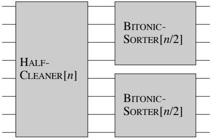

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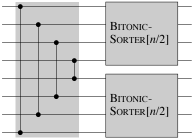
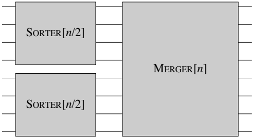


Batcher's Sorting Network

- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.



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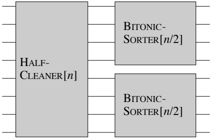

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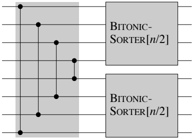

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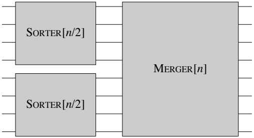
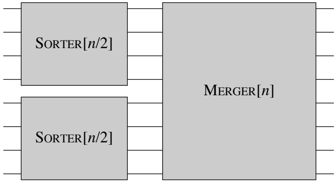
can be seen as a parallel version of merge sort



Unrolling the Recursion (Figure 27.12)



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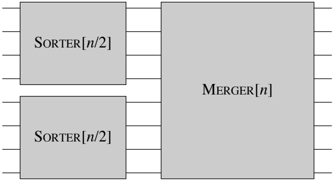

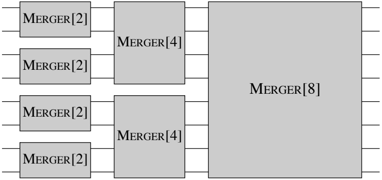

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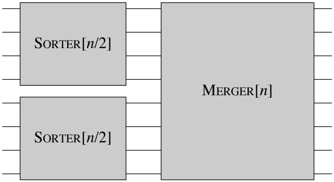

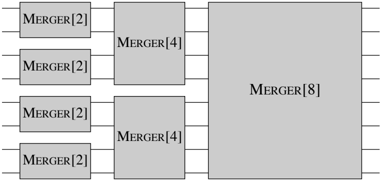

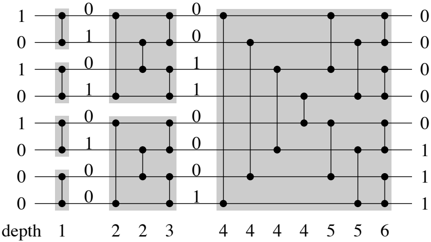

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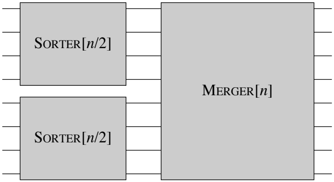

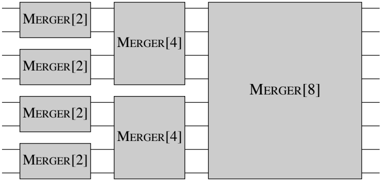

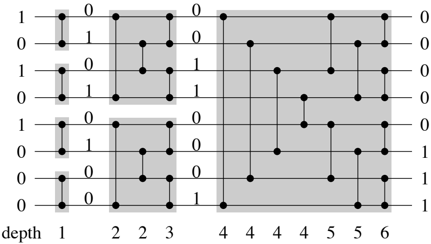

Recursion for $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + \log n & \text{if } n = 2^k. \end{cases}$$

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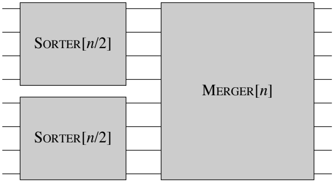

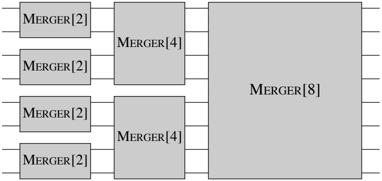

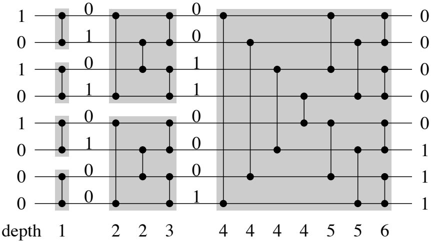

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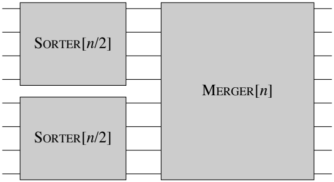

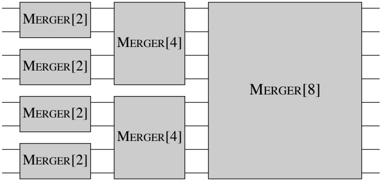

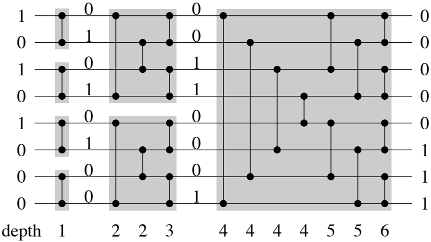

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$\textsc{Sorter}[n]$ has depth $\Theta(\log^2 n)$ and sorts any input.

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

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Quite elaborate construction, and involves huges constants.

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Perfect Halver

A perfect halver is a comparison network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\log_2 n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

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We will prove that such networks can be constructed in constant depth!

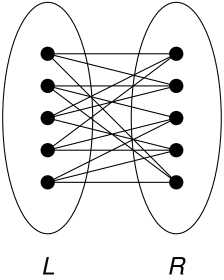
Expander Graphs

Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is union of d perfect matchings
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| > \min\{\mu \cdot |S|, n/2 - |S|\}$$



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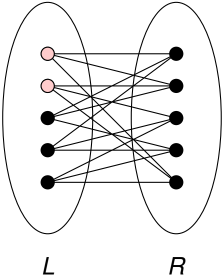

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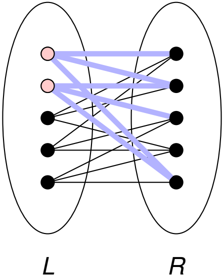

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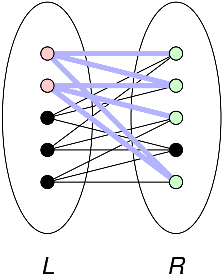

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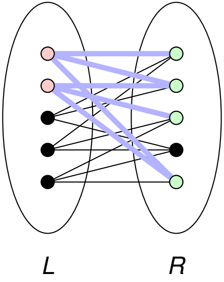
Specific definition tailored for sorting network - many other variants exist!



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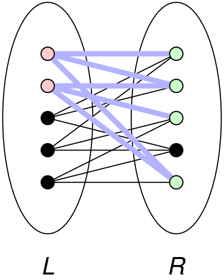
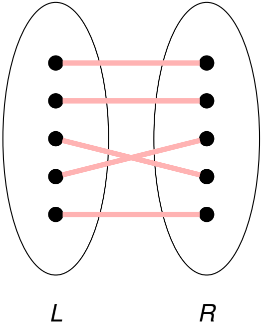


Expander Graphs:

- **probabilistic construction** "easy": take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers



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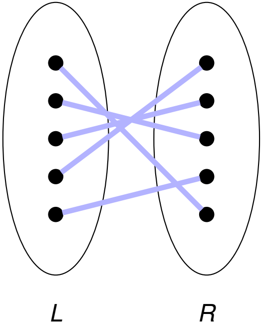

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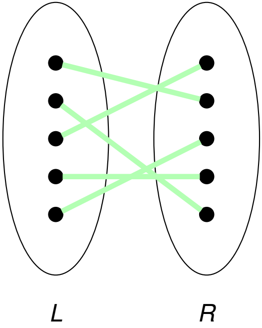

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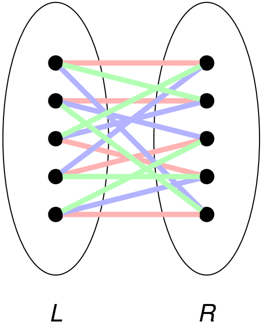

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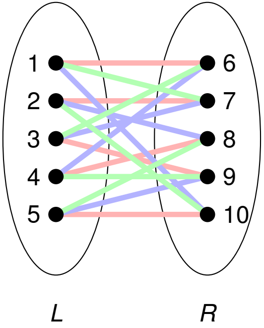

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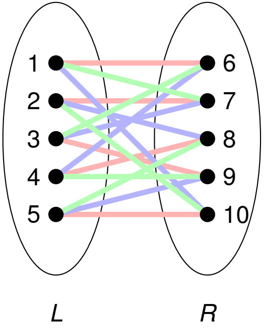

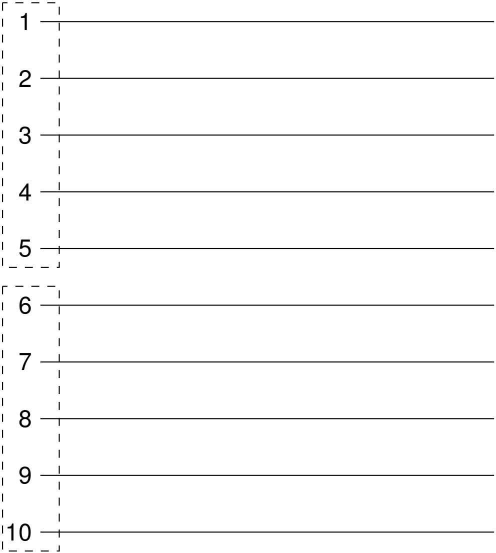

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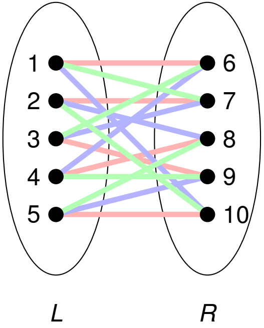

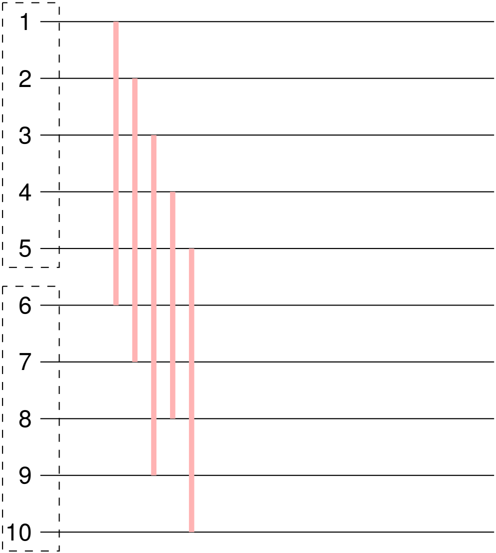

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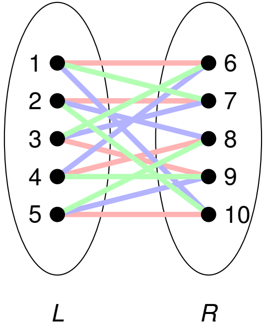

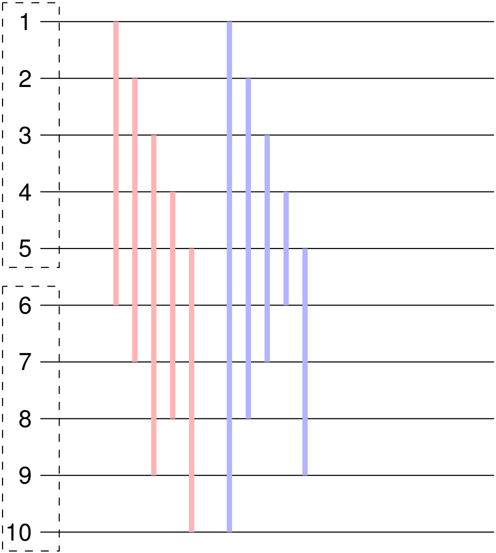

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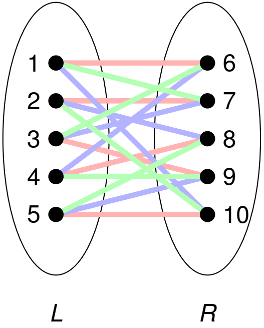
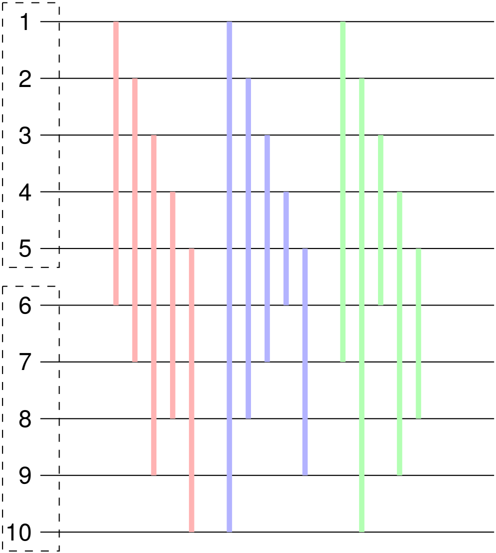
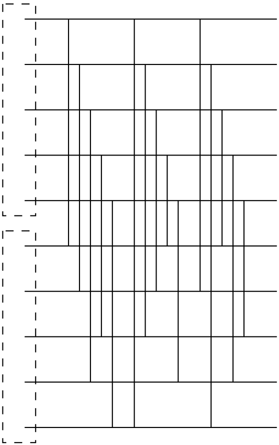




Existence of Approximate Halvers (not examinable)

Proof:



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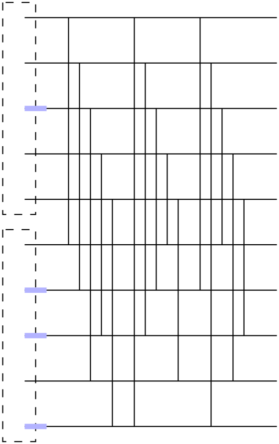

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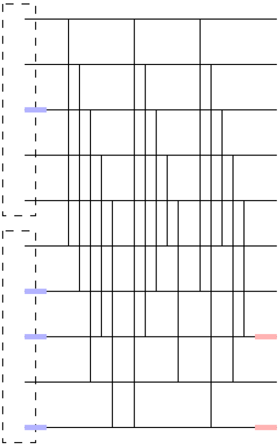

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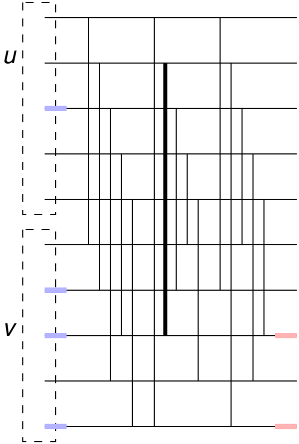

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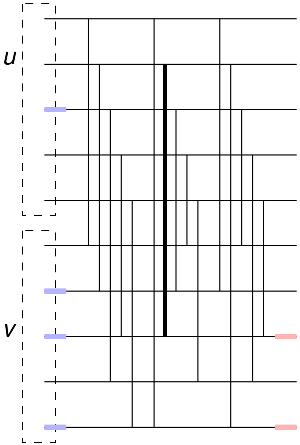

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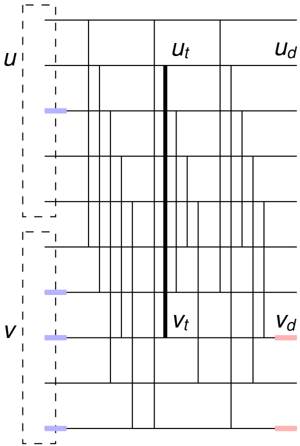

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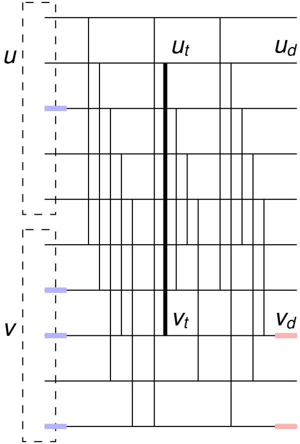
- Note that $v_d \in X$



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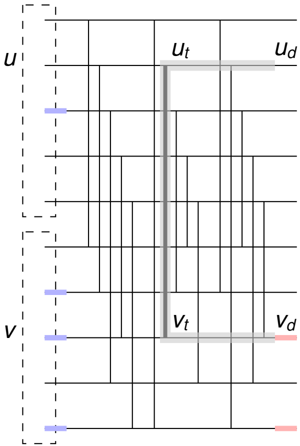

Existence of Approximate Halvers (not examinable)

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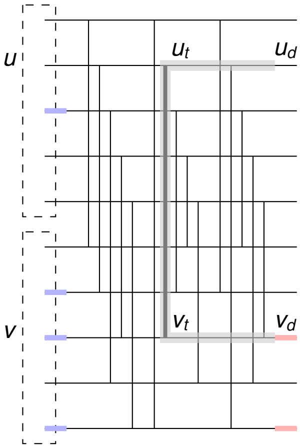

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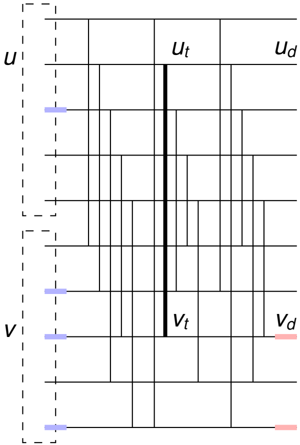

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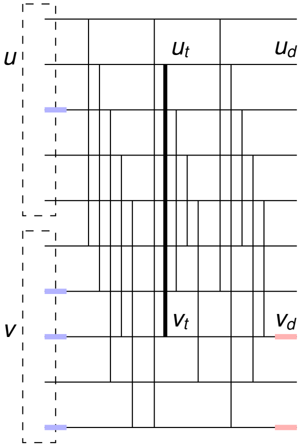

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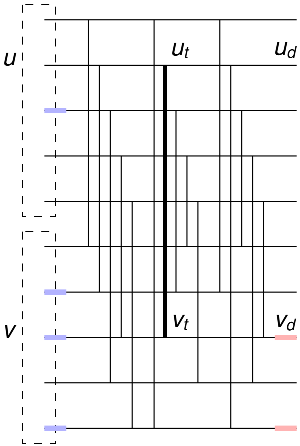

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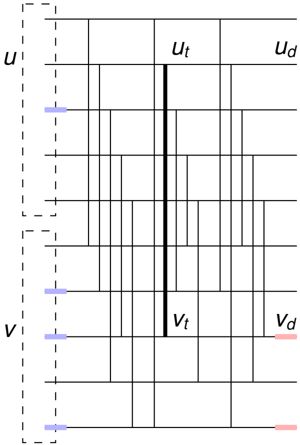

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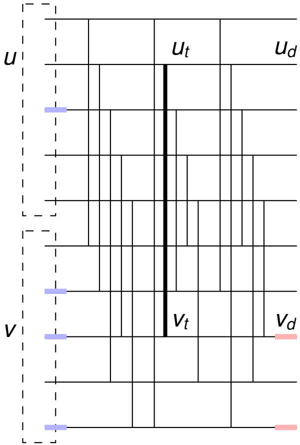

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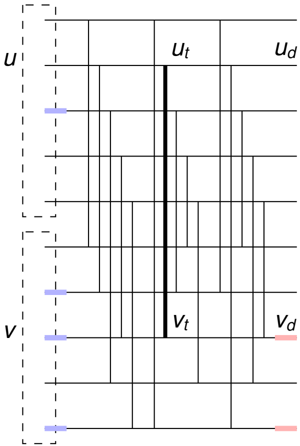
Here we used that $k \leq n/2$



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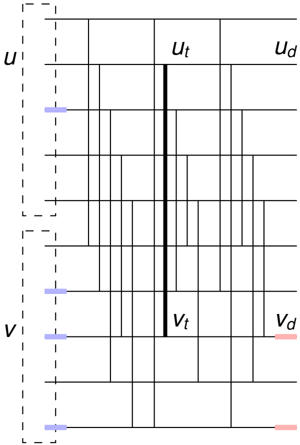
- Same argument $\Rightarrow$ at most $\epsilon \cdot k$, $\epsilon := 1/(\mu+1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. $\square$



- typical application of expander gaphs in parallel algorithms
- Much more work needed to construct the AKS sorting network

AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"

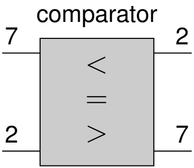
Richard J. Lipton (Georgia Tech)

*"The AKS sorting network is **galactic**: it needs that n be larger than 2^{78} or so to finally be smaller than Batcher's network for n items."*

Siblings of Sorting Network

Sorting Networks

- sorts any input of size *n*
- special case of Comparison Networks



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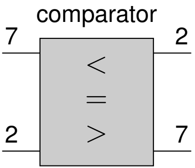
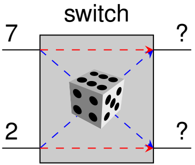


Switching (Shuffling) Networks

- creates a random permutation of n items
- special case of Permutation Networks



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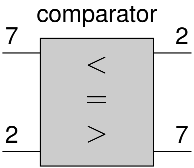

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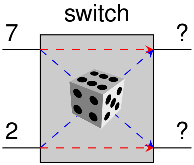
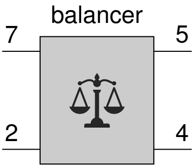


Counting Networks

- balances any stream of tokens over *n* wires
- special case of Balancing Networks



Outline

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

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Processors collectively assign successive values from a given range.

Values could represent addresses in memories or destinations on an interconnection network

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Balancing Networks

- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,. . .)

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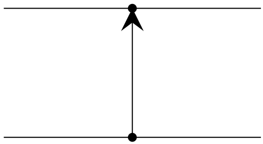

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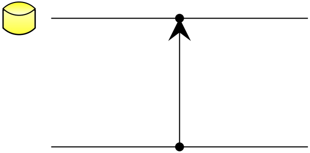

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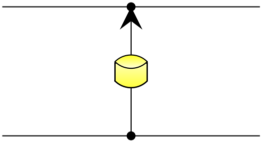

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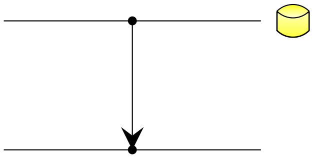

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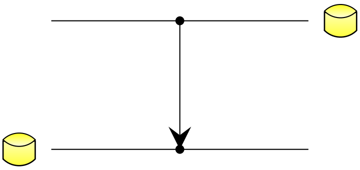

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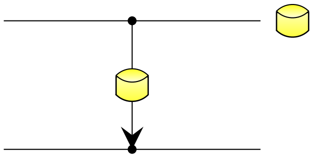

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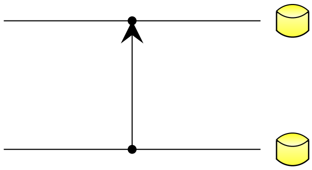

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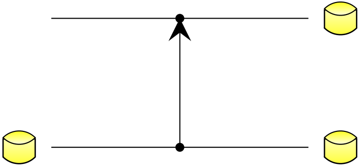

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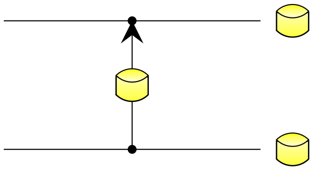

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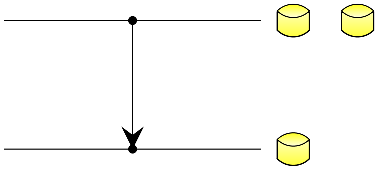

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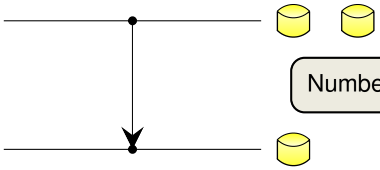

Number of tokens differs by at most one

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires

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3. In a quiescent state: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$
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Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.

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Key Lemma

Consider a MERGER[n]. Then if the inputs $x_1, \ldots, x_{n/2}$ and $x_{n/2+1}, \ldots, x_n$ have the step property, then so does the output $y_1, \ldots, y_n$.

Proof (by induction on n being a power of 2)

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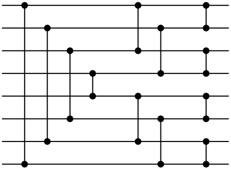

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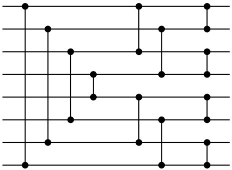

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer

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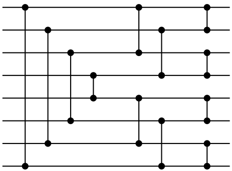
1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.



Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$:

Correctness of the Bitonic Counting Network

Facts

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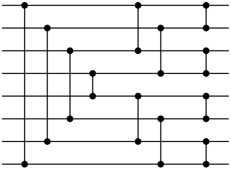

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks

Correctness of the Bitonic Counting Network

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Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

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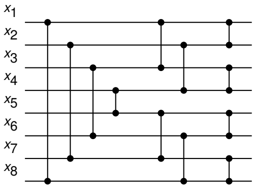

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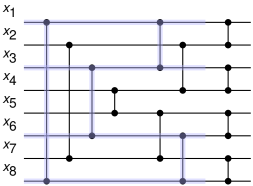

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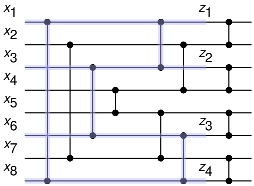

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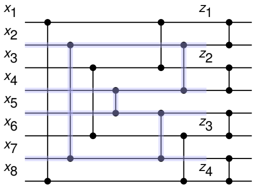

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Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

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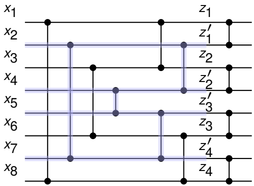

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ have the step property

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
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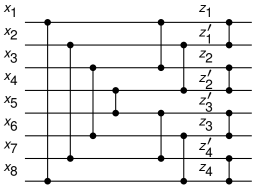

Proof (by induction on n being a power of 2)

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- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z_i'$

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Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

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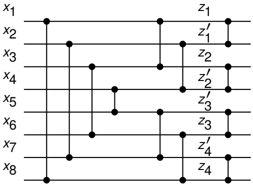

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- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$
- F1 $\Rightarrow$ $Z = \lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \rceil + \lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rfloor$ and $Z' = \lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \rfloor + \lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rceil$

Correctness of the Bitonic Counting Network

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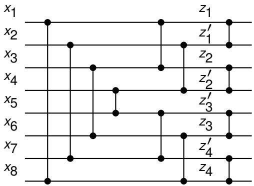

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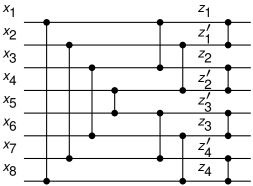

Proof (by induction on n being a power of 2)

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- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ have the step property
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- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

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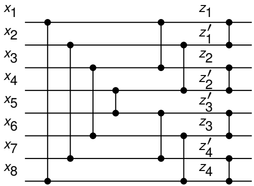

Proof (by induction on n being a power of 2)

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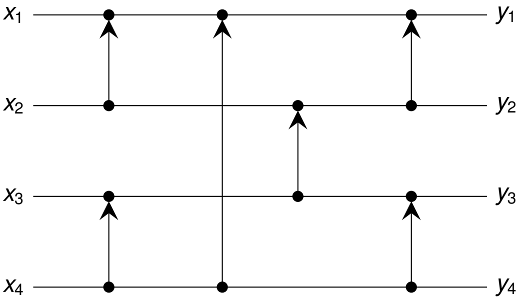
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$. Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action (Asychnronous Execution)



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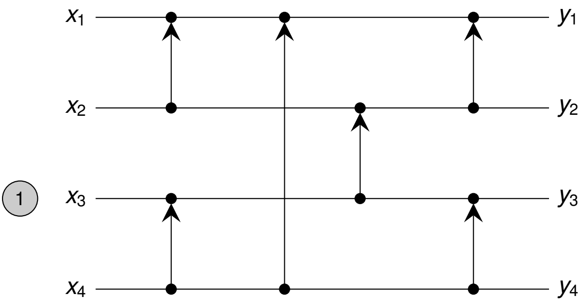

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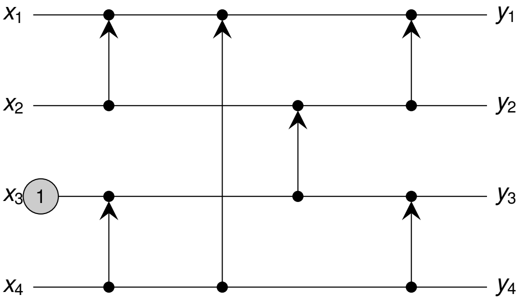

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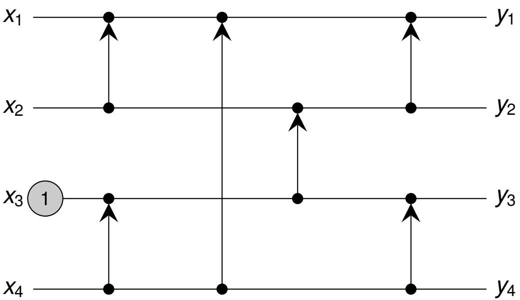

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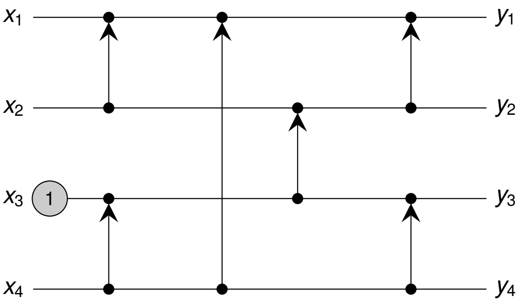

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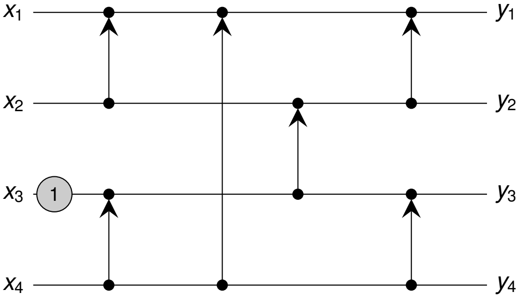

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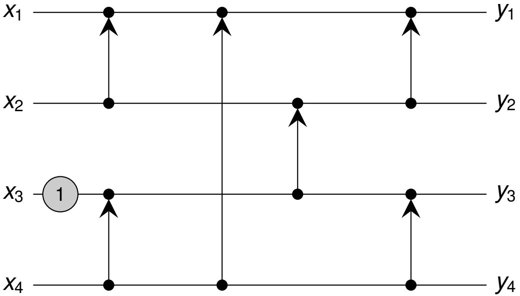

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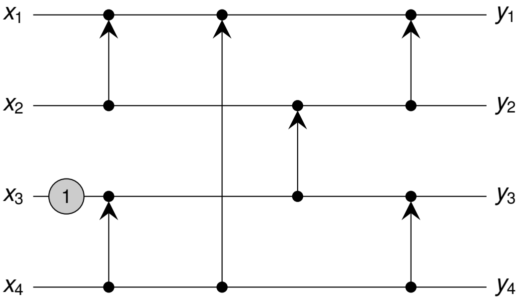

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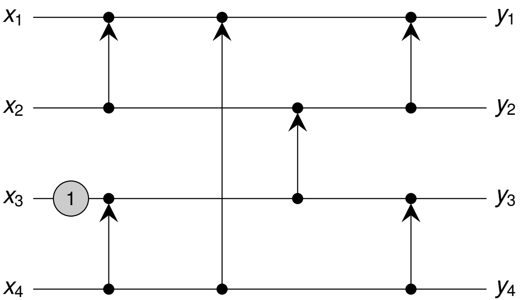

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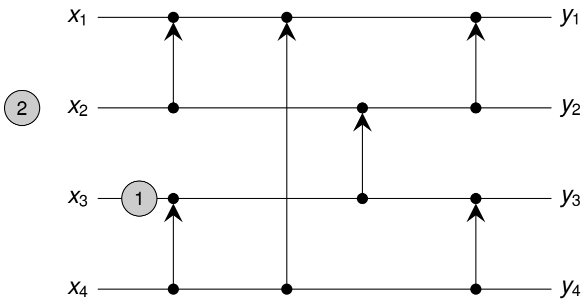

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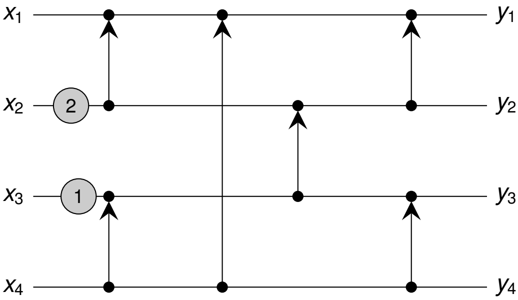
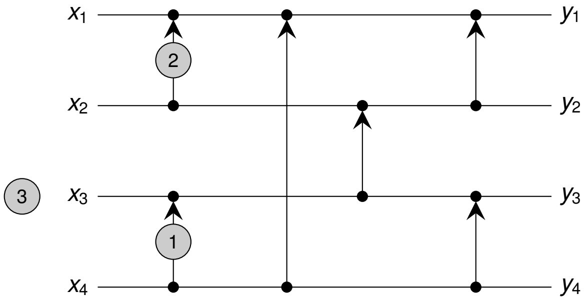


Bitonic Counting Network in Action (Asychnronous Execution)



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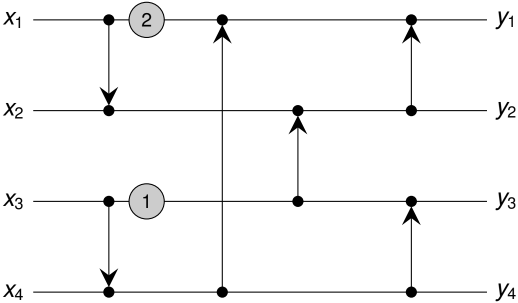

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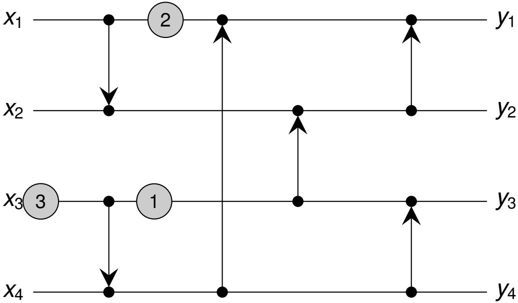

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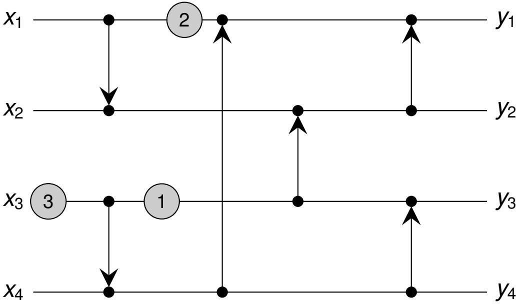

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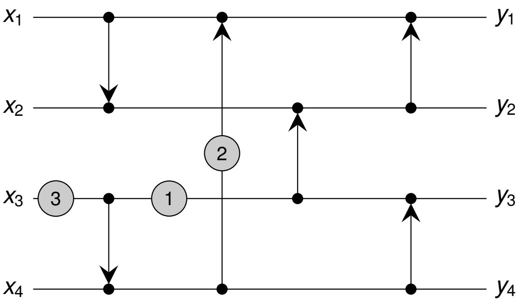

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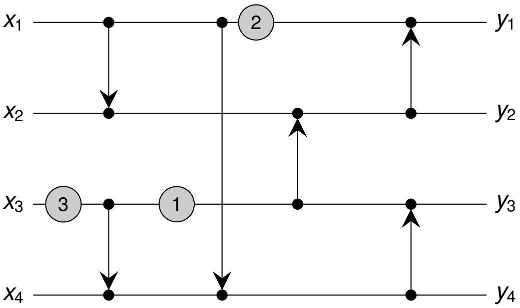

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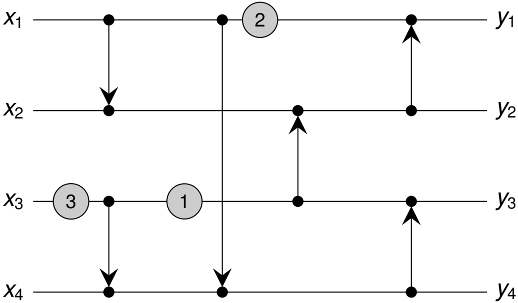

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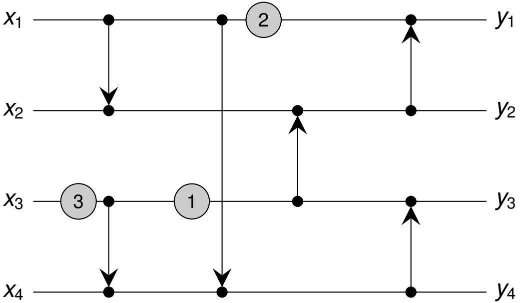

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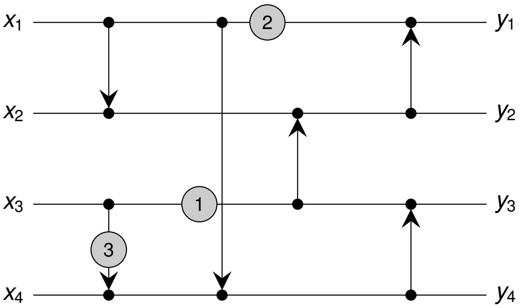

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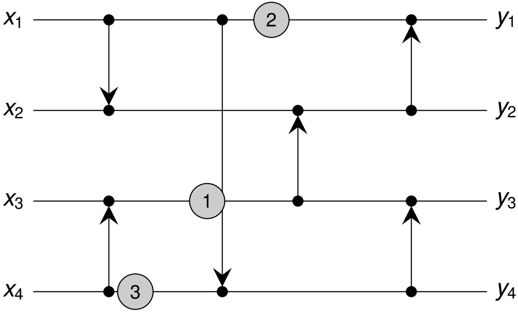

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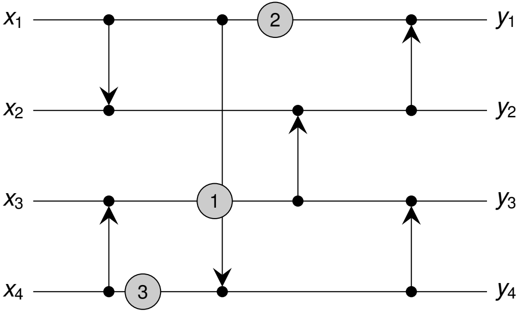

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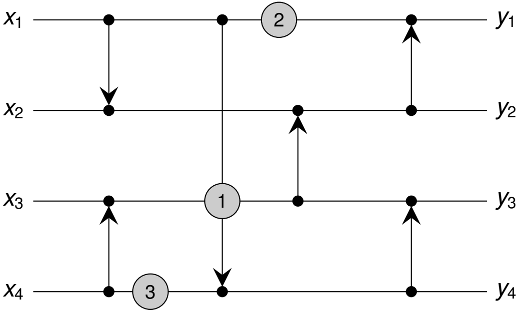

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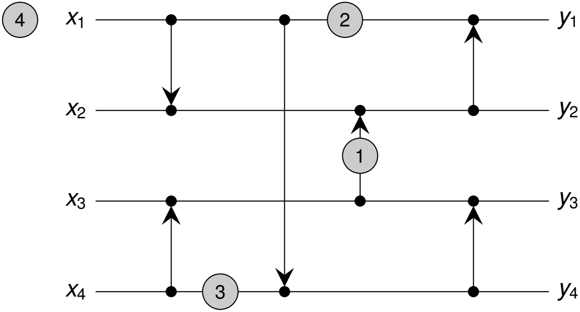

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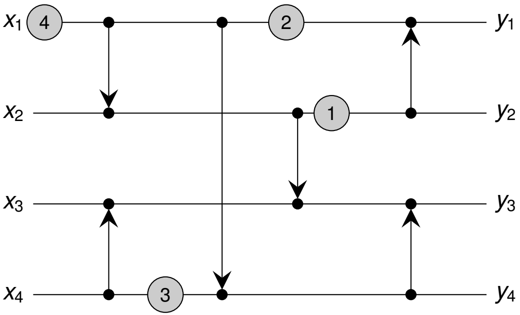

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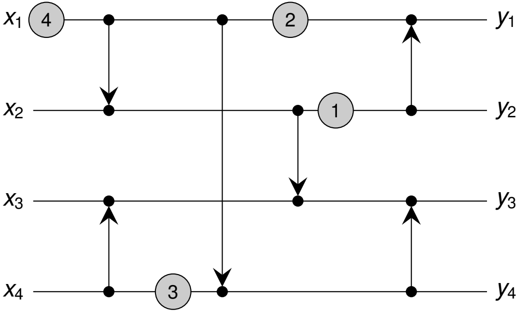

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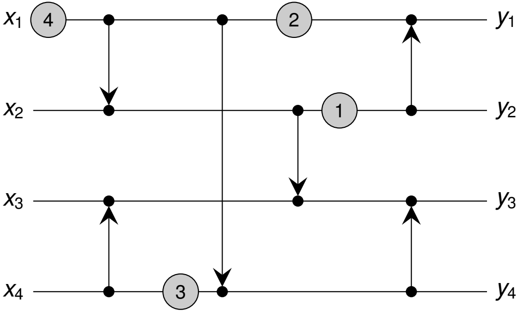

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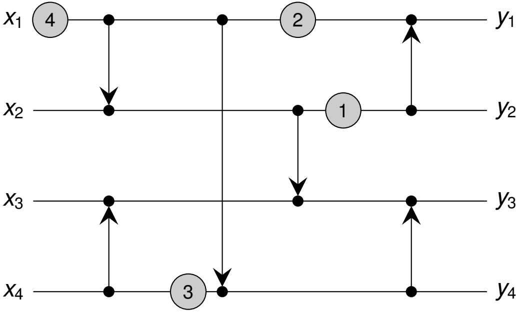

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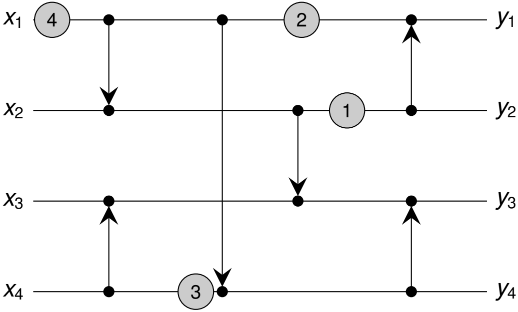

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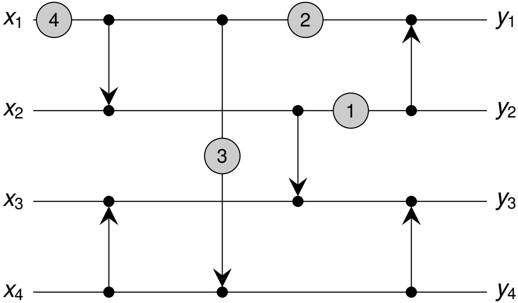

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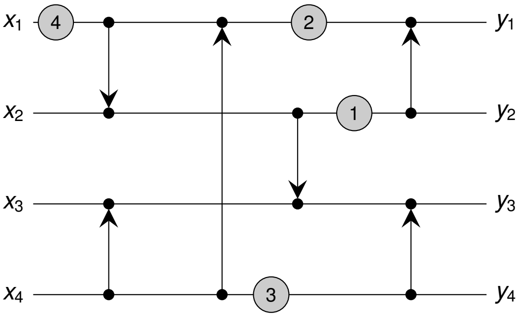

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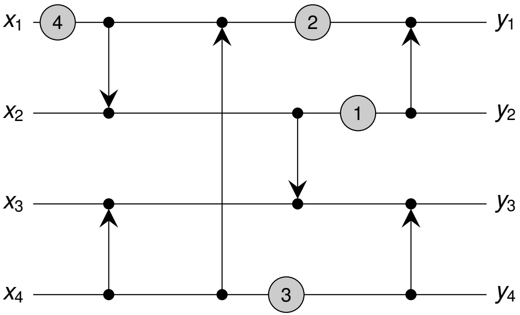

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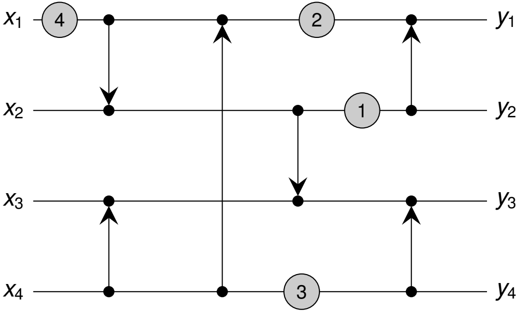

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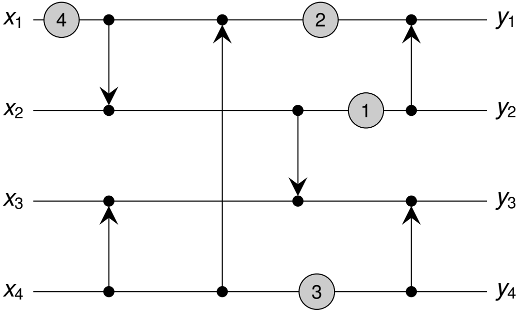

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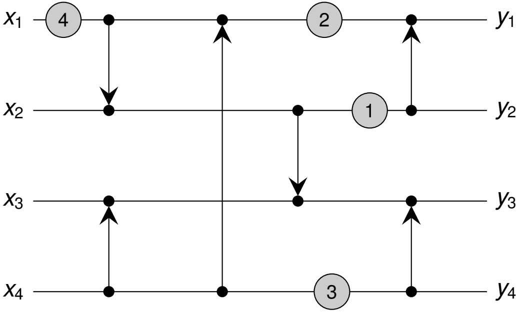

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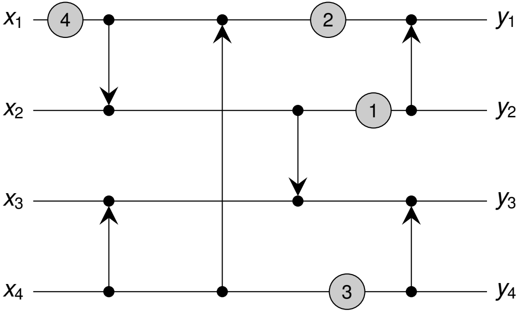

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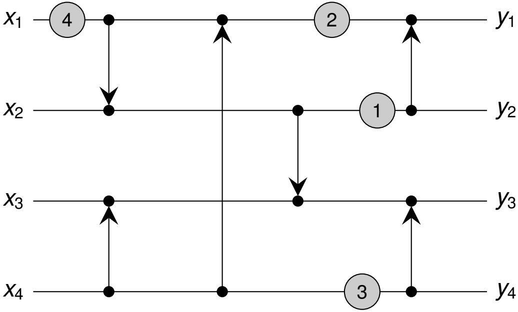

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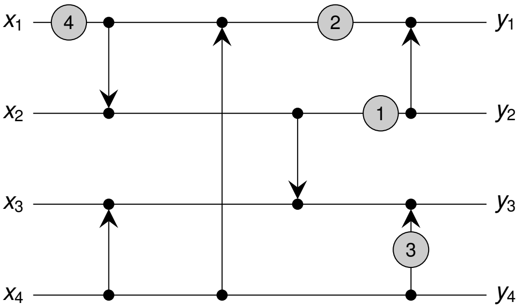

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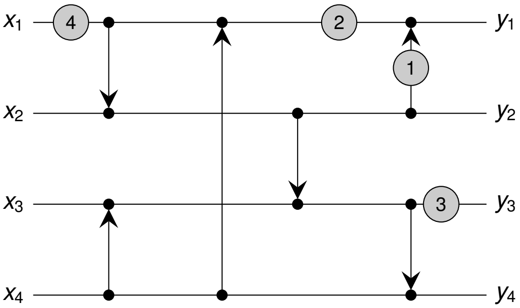

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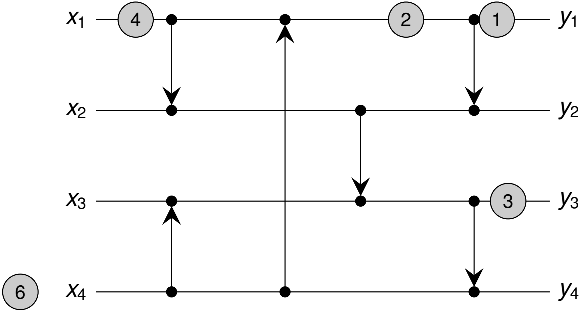

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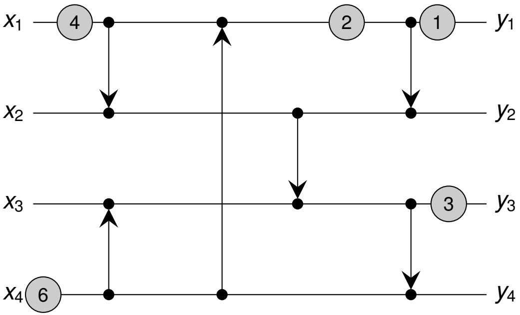

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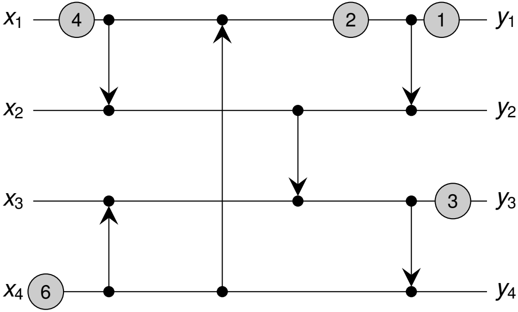

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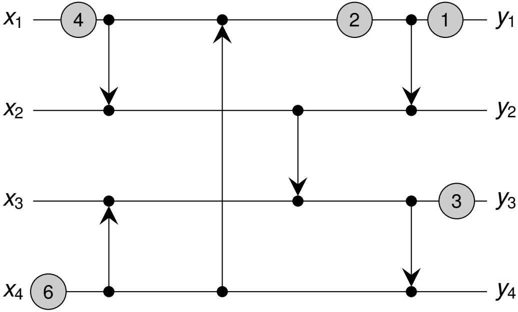

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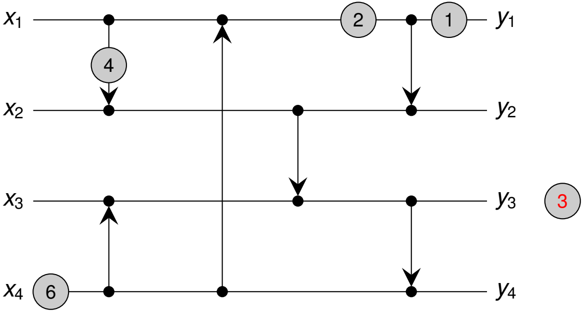

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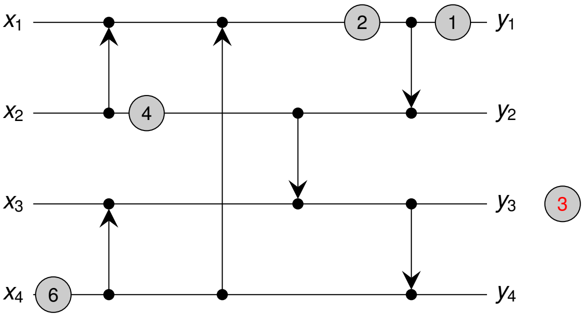

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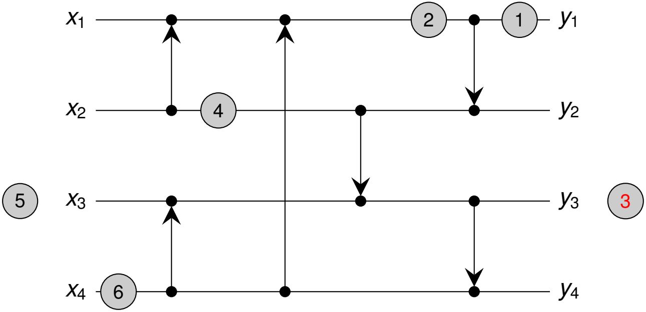

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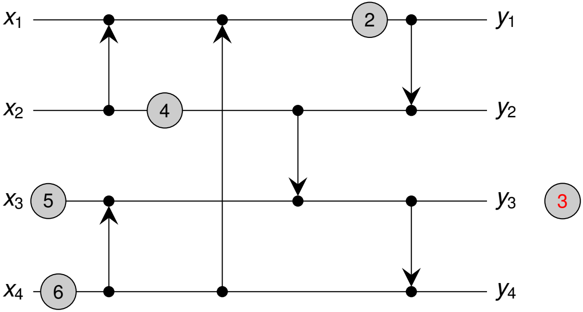

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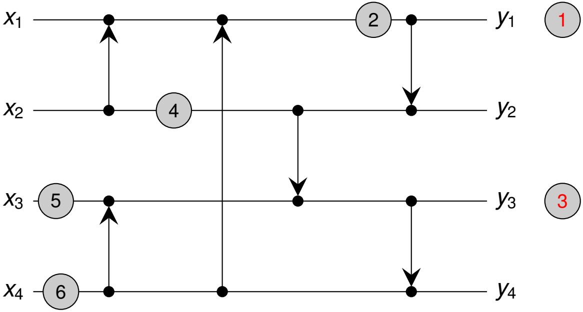

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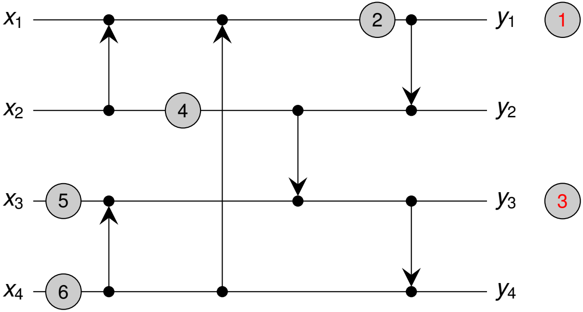

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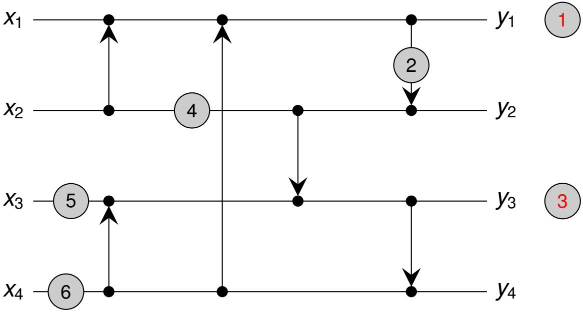

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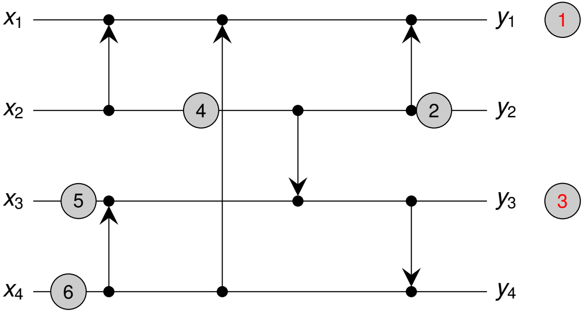

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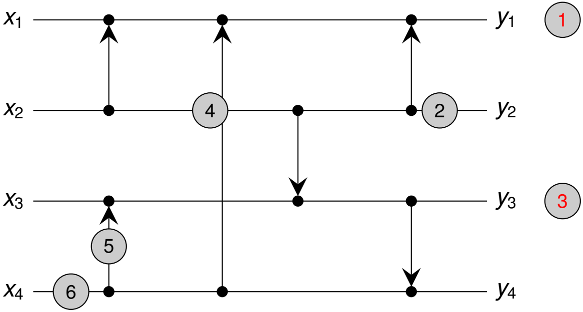

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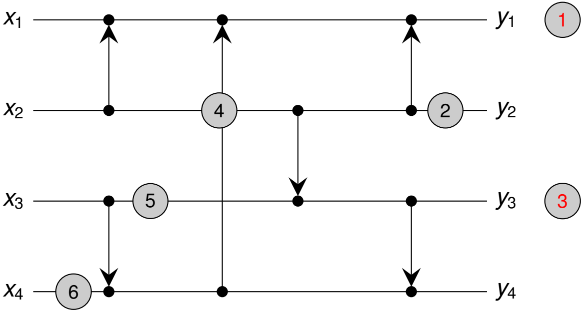

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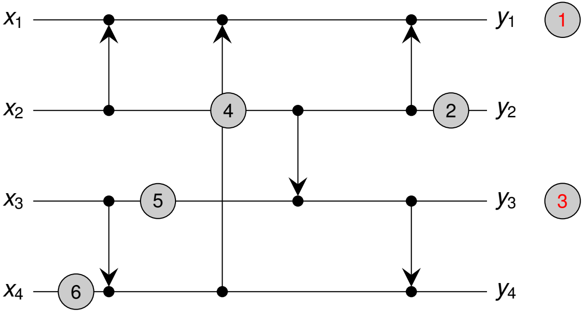

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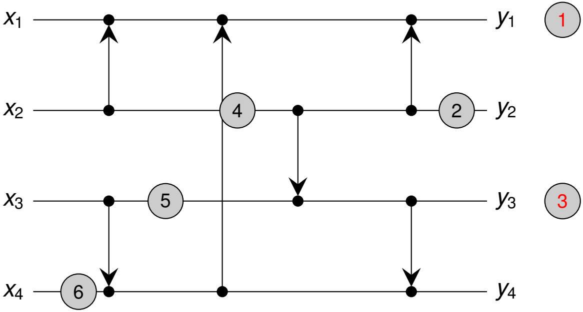

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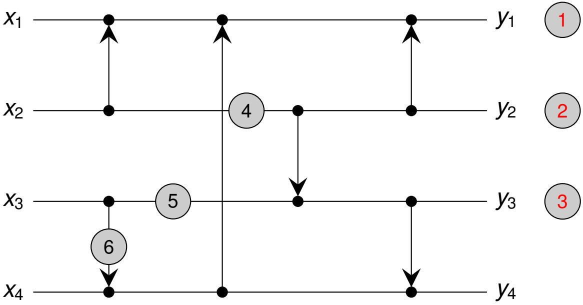

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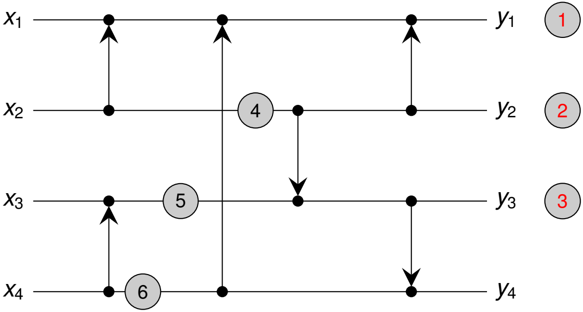

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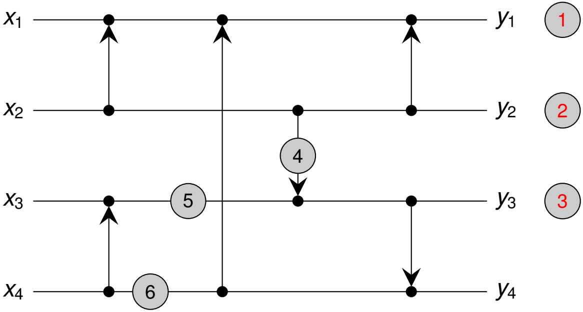

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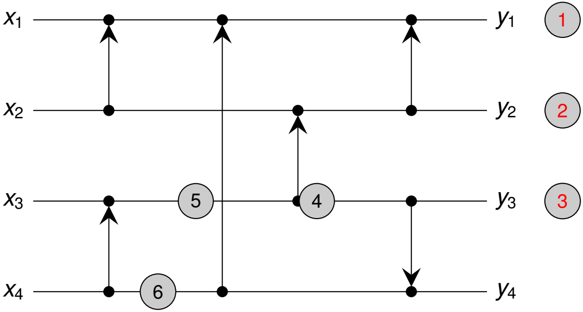

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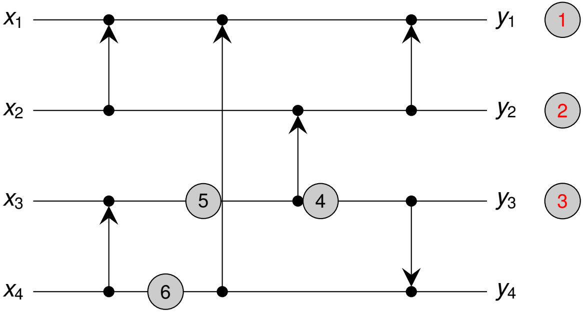

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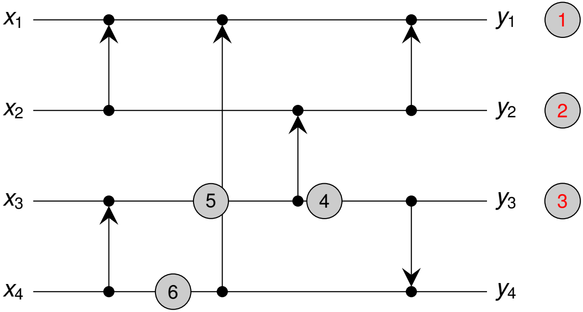

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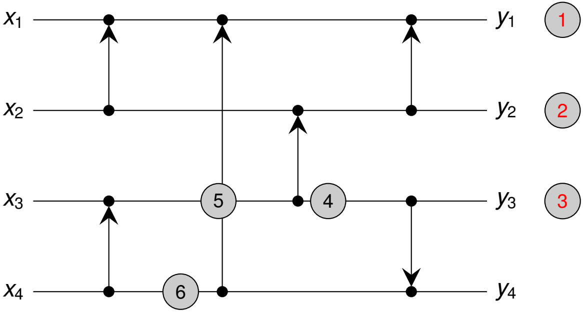

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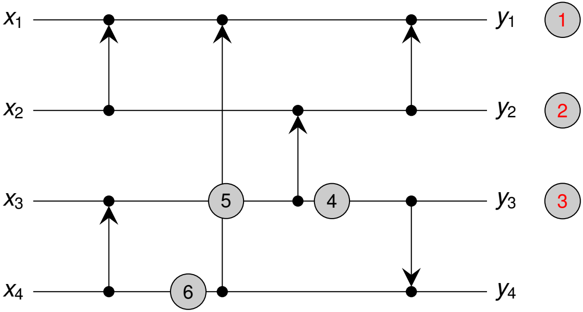

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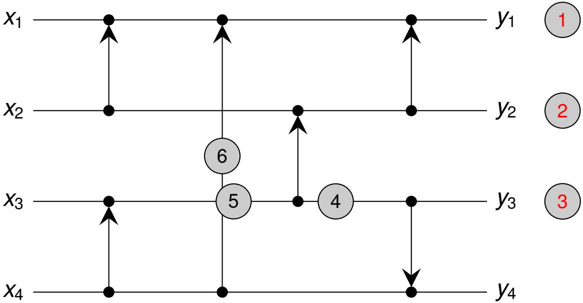

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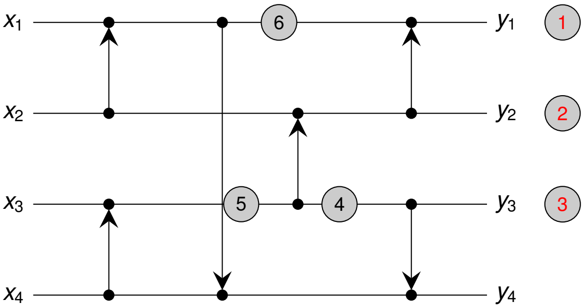

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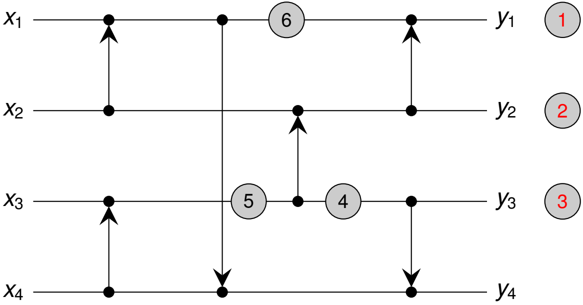

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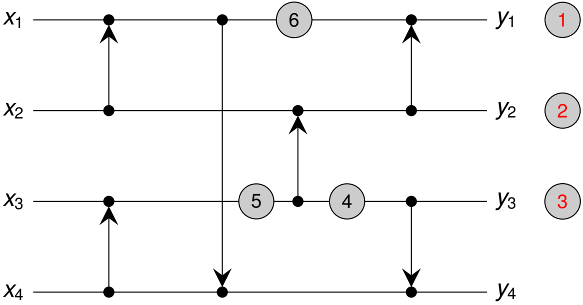

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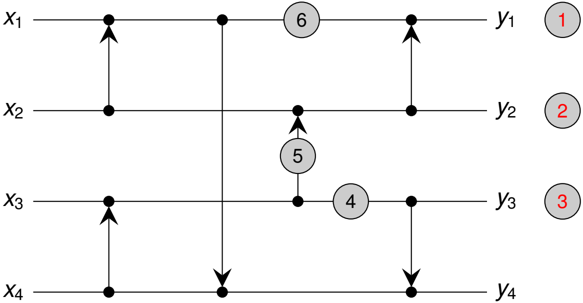

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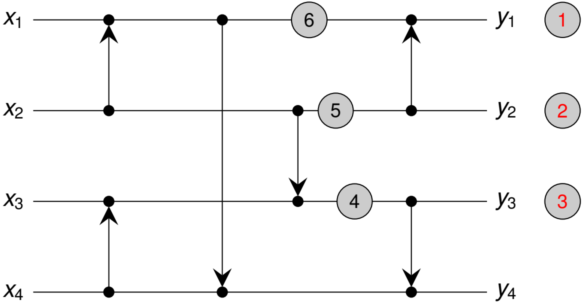

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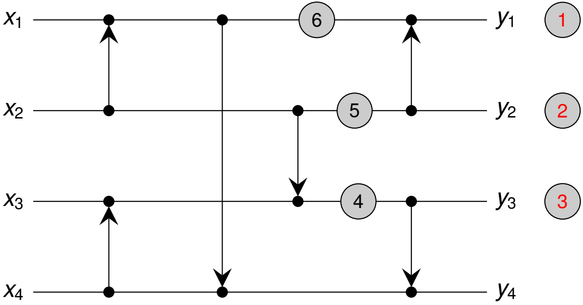

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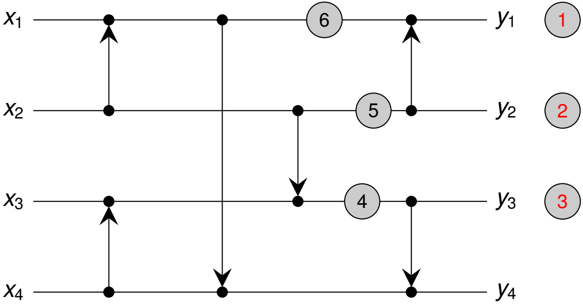

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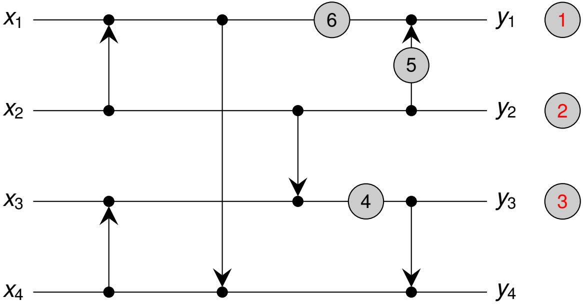

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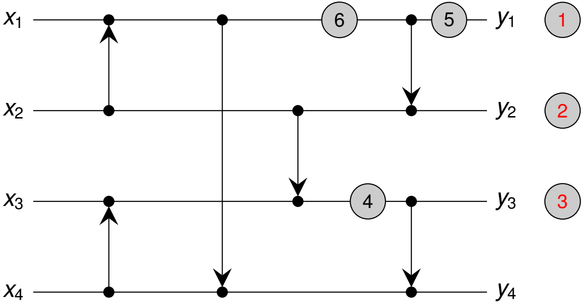

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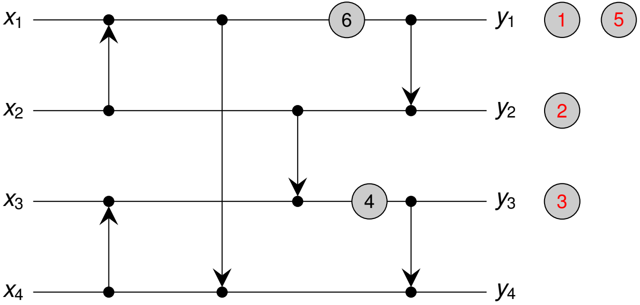

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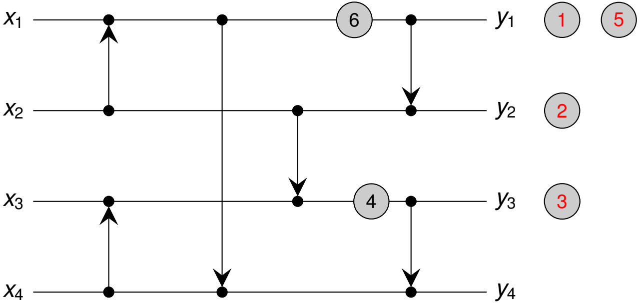

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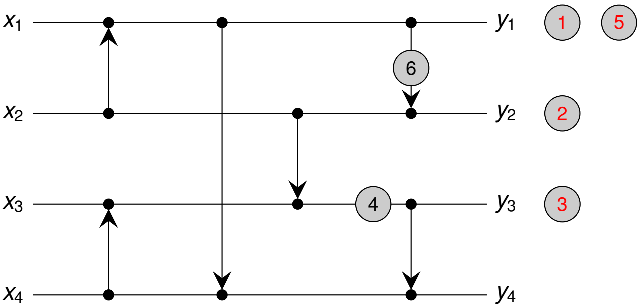

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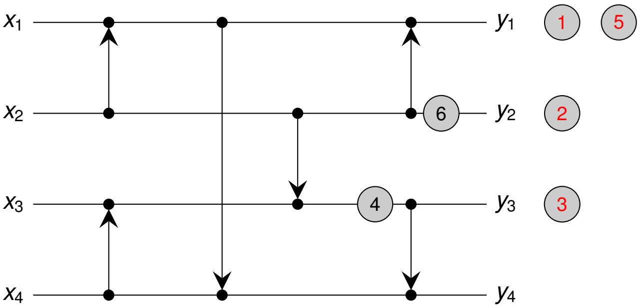

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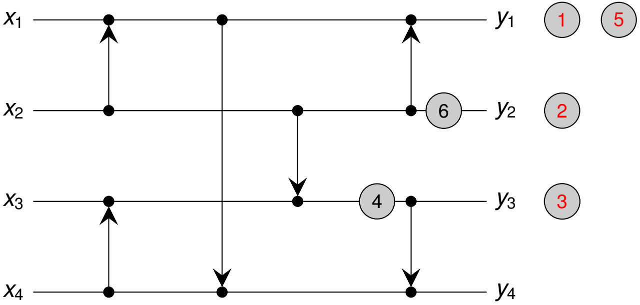

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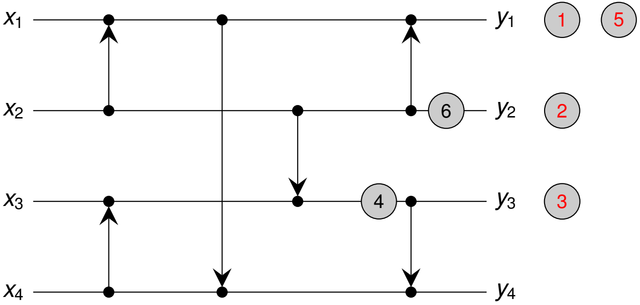

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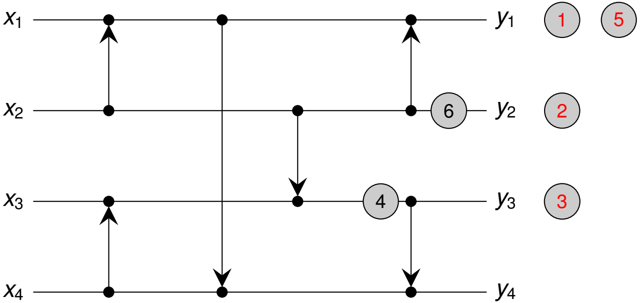

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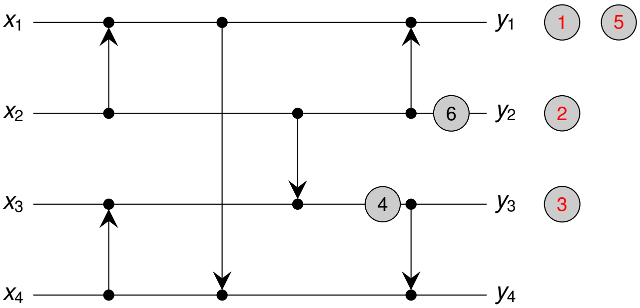

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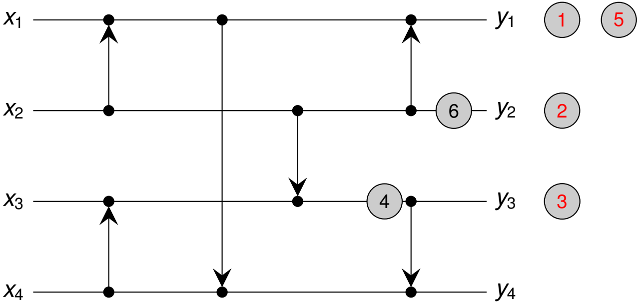

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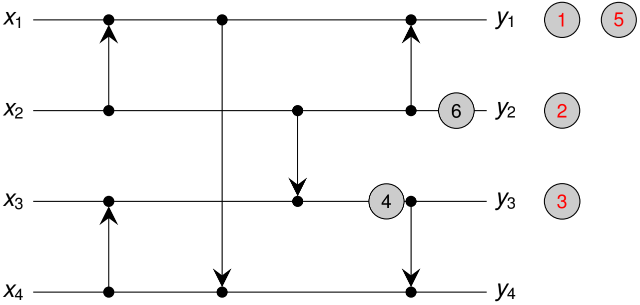

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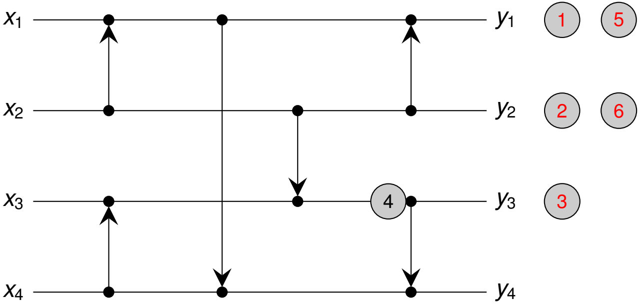

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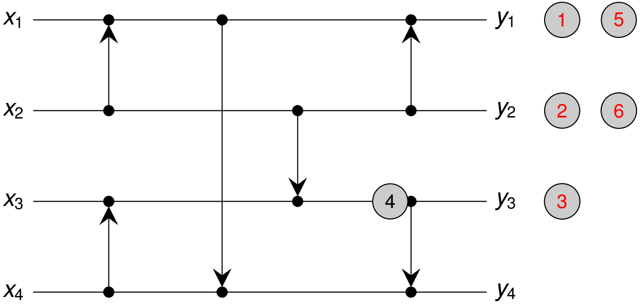

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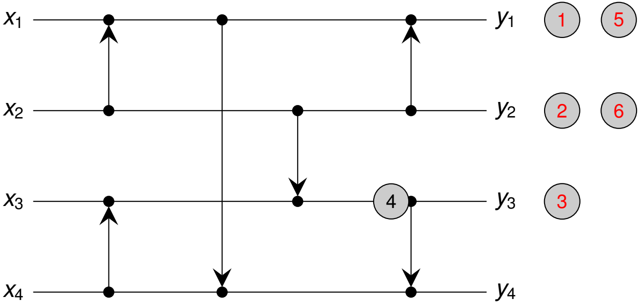

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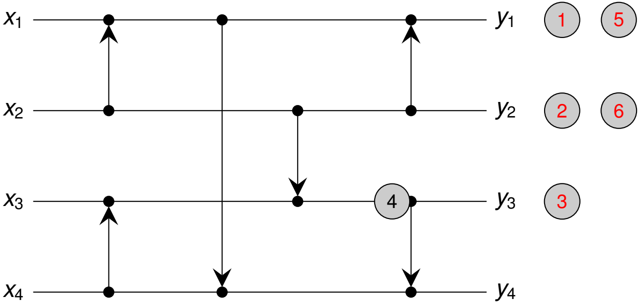

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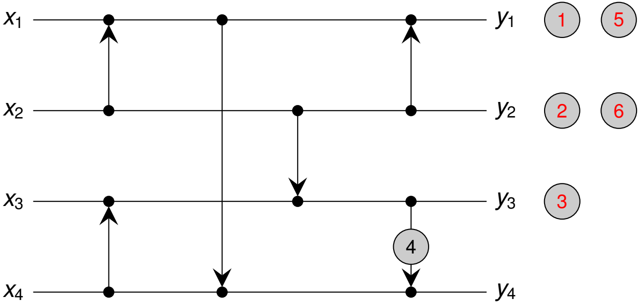

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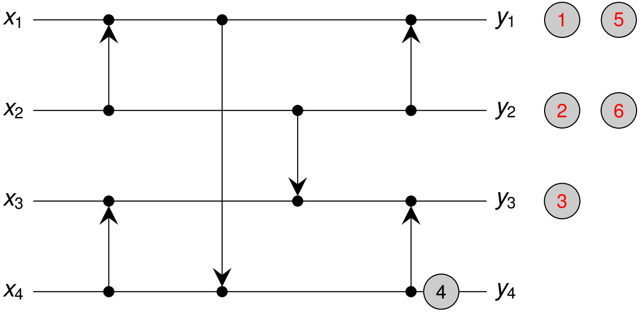

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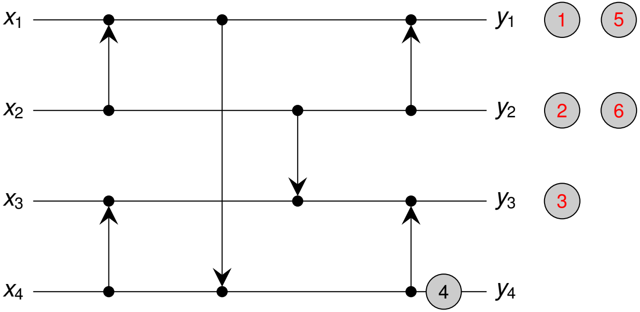

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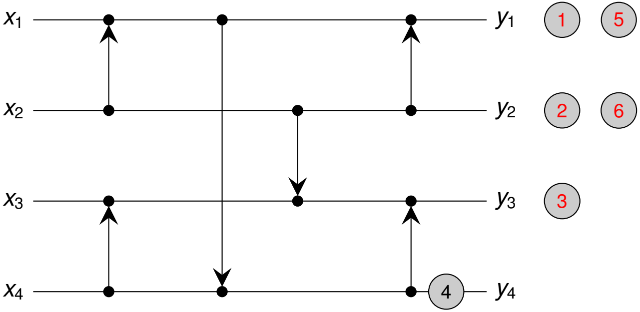

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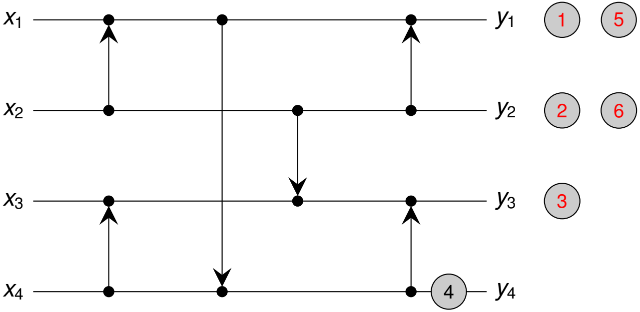

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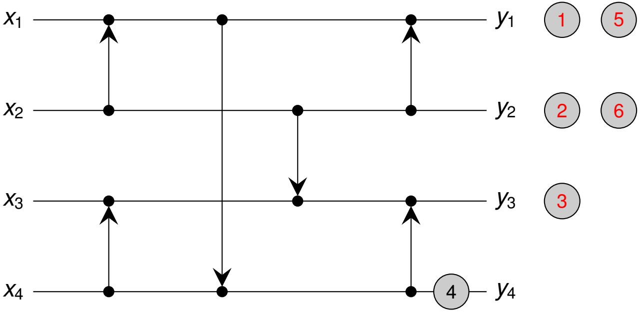

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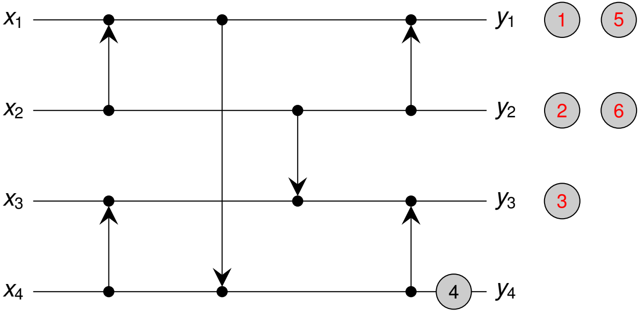

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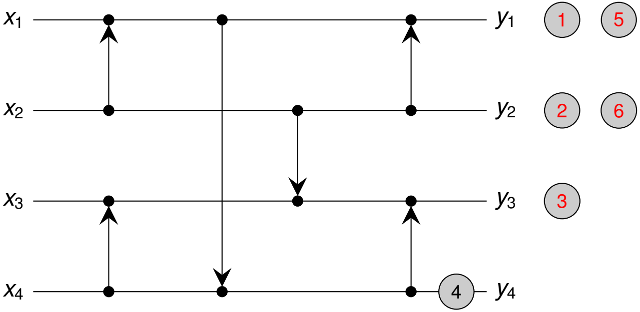

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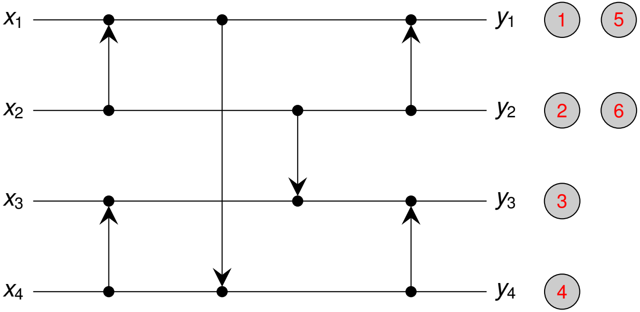

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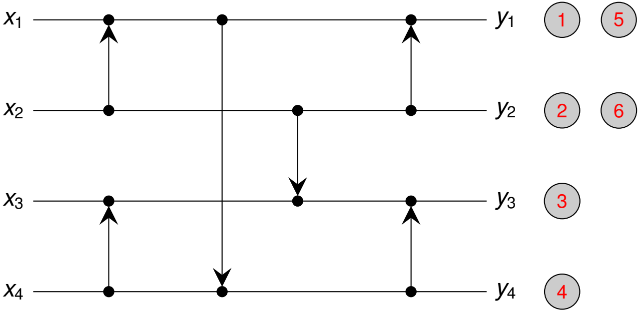
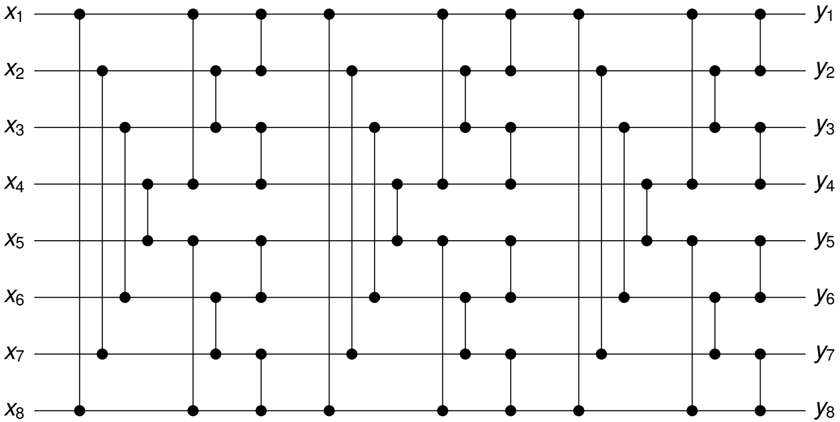


Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i+n, i+2 \cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



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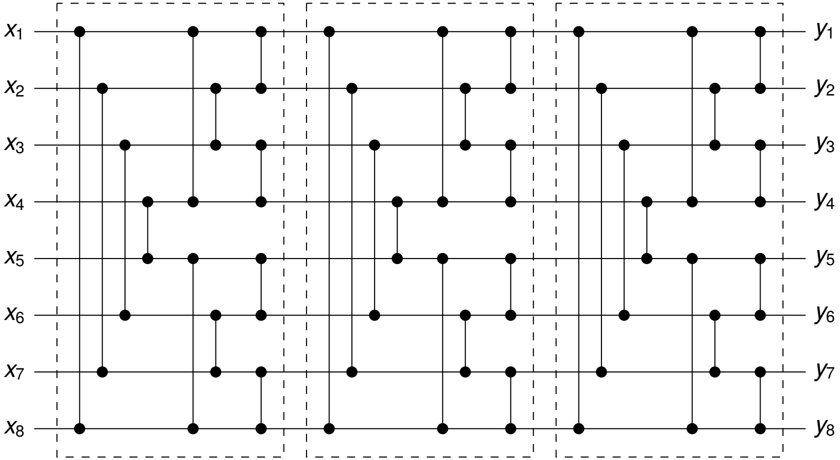

Consists of $\log n$ BLOCK[n] networks each of which has depth $\log n$

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

From Counting to Sorting

The converse is not true!

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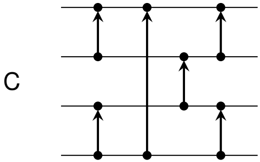

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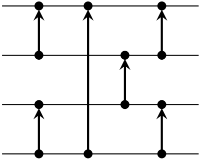
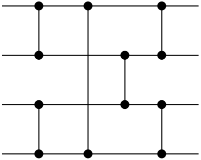
C   S

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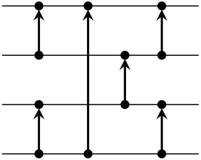

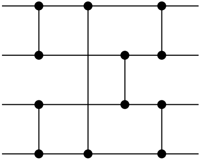

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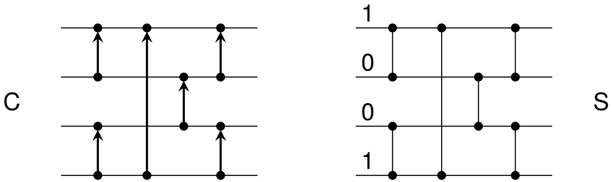

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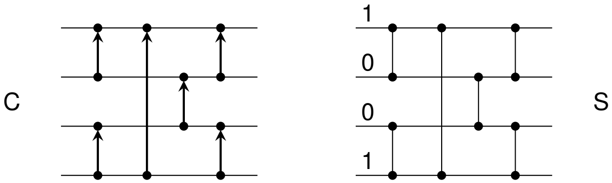

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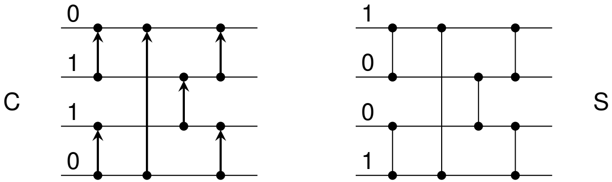

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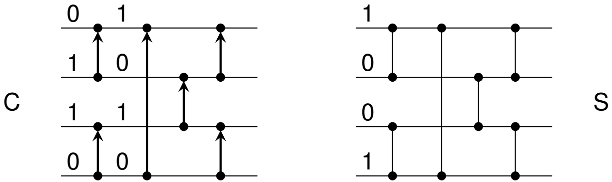

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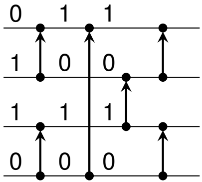

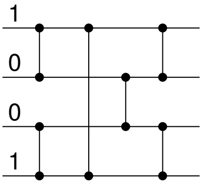

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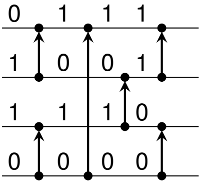

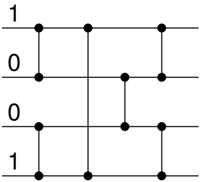

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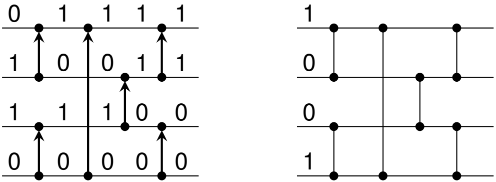

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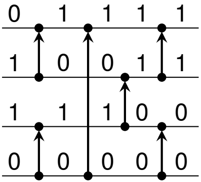

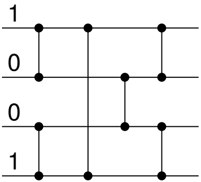

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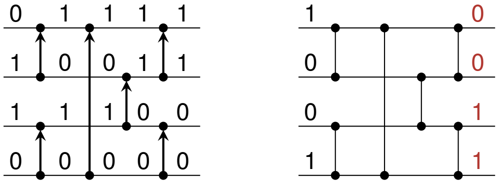

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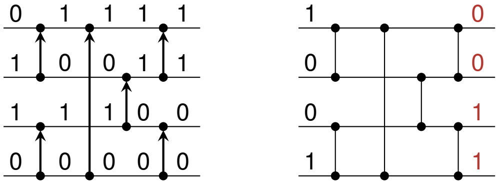
- By the Zero-One Principle, S is a sorting network. $\square$

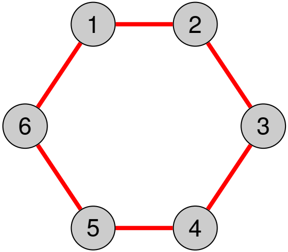
C

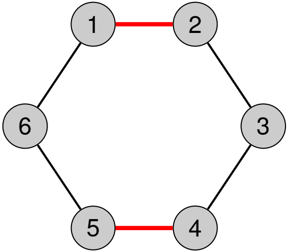


S

Outline

Communication Models: Diffusion vs. Matching





$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$M^{(t)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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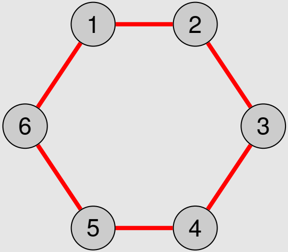

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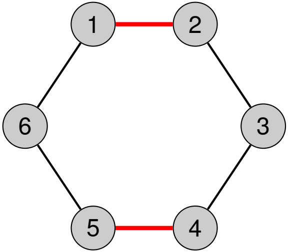

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Smoothness of the Load Distribution

- $x^t \in \mathbb{R}^n$ be a load vector at round t
- $\overline{x}$ denotes the average load

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- makespan: $\max_{i=1}^n x_i^t$
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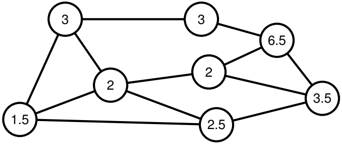

Smoothness of the Load Distribution

- $x^t \in \mathbb{R}^n$ be a load vector at round t
- $\overline{x}$ denotes the average load

Want that x^t converges for $t \to \infty$ to $(\overline{x}, \overline{x}, \ldots, \overline{x})$!

Metrics

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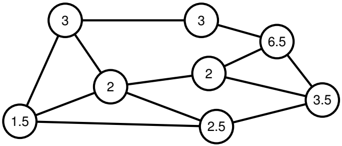

For this example:

- $\Phi^t = \sqrt{0^2 + 0^2 + 3.5^2 + 0.5^2 + 1^2 + 1^2 + 1.5^2 + 0.5^2} = \sqrt{17}$
- $\max_{i=1}^n x_i^t = 6.5$
- $\max_{i=1}^n x_i^t - \min_{i=1}^n x_i^t = 5$

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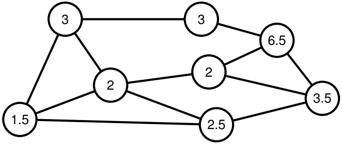

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Diffusion Matrix

Diffusion Matrix

Given an undirected, connected graph $G = (V, E)$ and a diffusion parameter $\alpha > 0$, the diffusion matrix M is defined as follows:

$$M_{ij} = \begin{cases} \alpha & \text{if } (i,j) \in E, \\ 1 - \alpha \deg(i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Diffusion Matrix

How to pick α for a d-regular graph?

- $\alpha = \frac{1}{d}$ may lead to oscillation (if graph is bipartite)
- $\alpha = \frac{1}{d+1}$ ensures convergence
- $\alpha = \frac{1}{2d}$ ensures convergence (and all eigenvalues of M are non-negative)

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This can be also seen as a random walk on G!

First-Order Diffusion: Load vector x^t satisfies

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1D grid

$$\gamma(M) \approx 1 - \frac{1}{n^2}$$

1D grid

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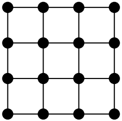
2D grid



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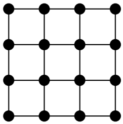

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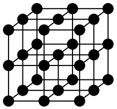

$\gamma(M) \approx 1 - \frac{1}{n^{2/3}}$

1D grid

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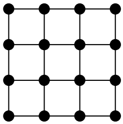

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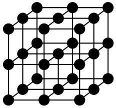

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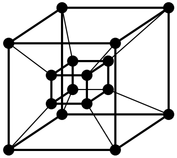
Hypercube

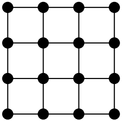


$$\gamma(M) \approx 1 - \frac{1}{\log n}$$

1D grid

$$\gamma(M) \approx 1 - \frac{1}{n^2}$$

2D grid



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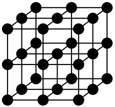

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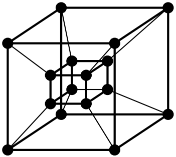

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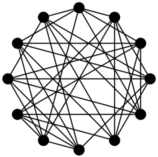
Random Graph

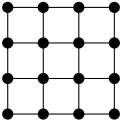


$$\gamma(M) < 1$$

1D grid

$\gamma(M) \approx 1 - \frac{1}{n^2}$

2D grid



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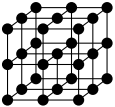

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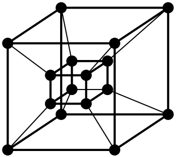

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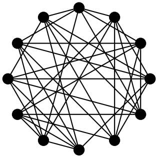
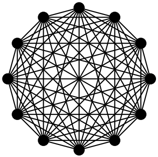


$\gamma(M) < 1$

Complete Graph

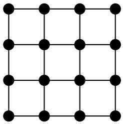


$\gamma(M) \approx 0$

1D grid

$\gamma(M) \approx 1 - \frac{1}{n^2}$

2D grid



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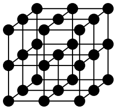

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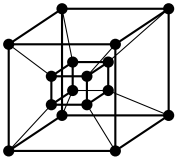

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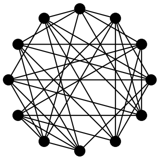

$\gamma(M) < 1$

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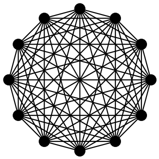
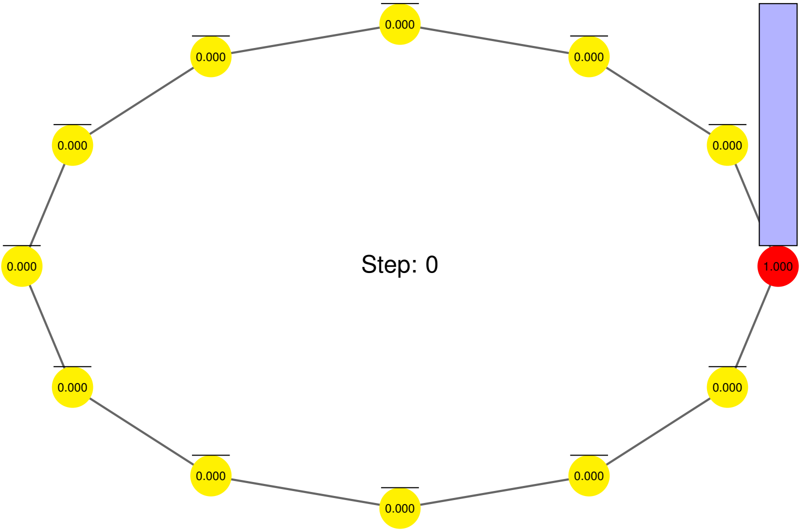


$\gamma(M) \approx 0$

$\gamma(M) \in (0,1]$ measures connectivity of G

Diffusion of Load on a Ring



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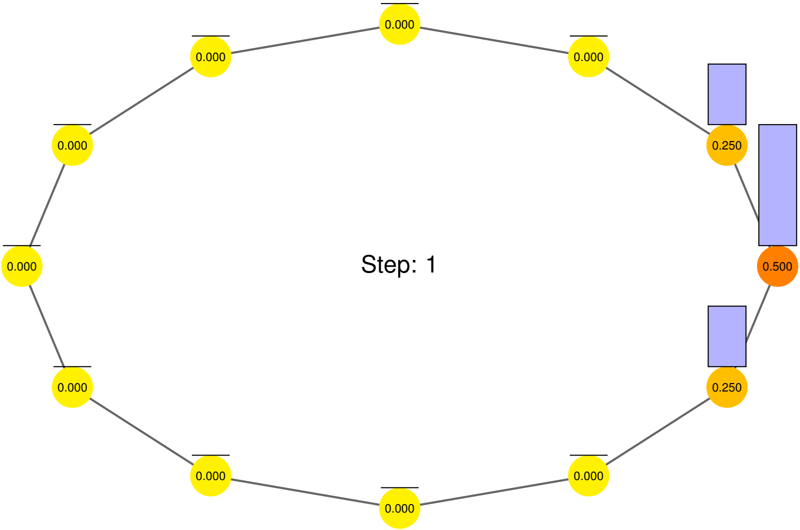

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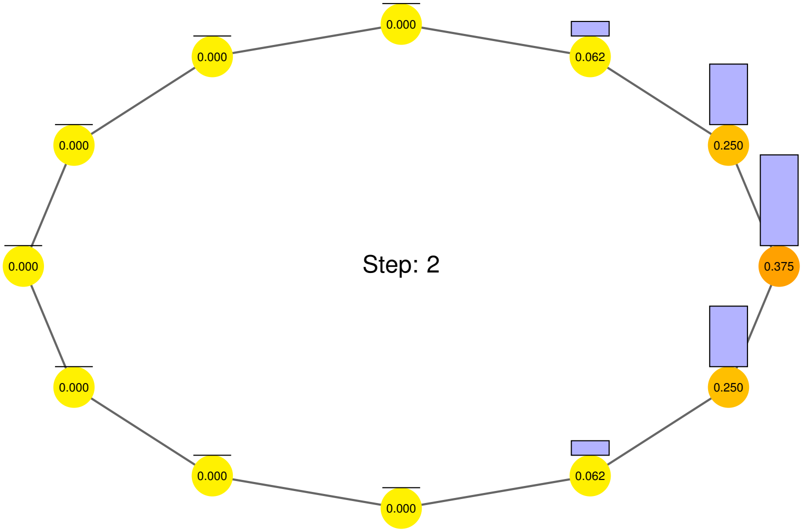

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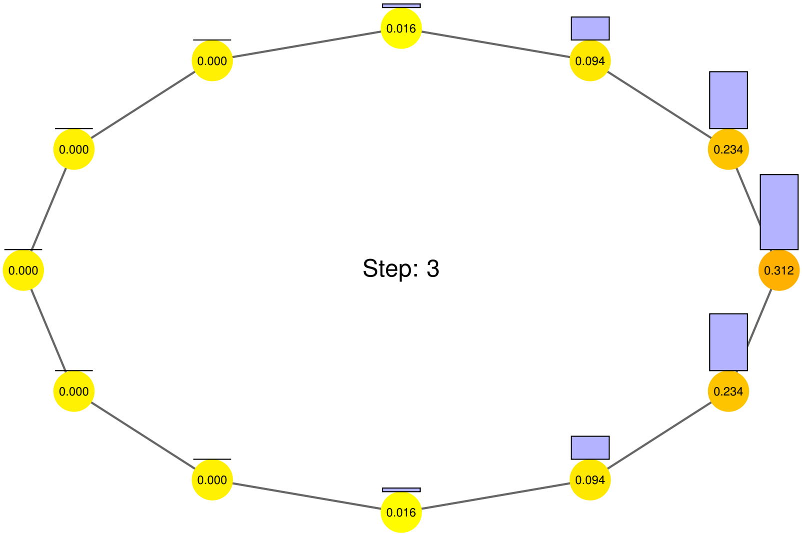

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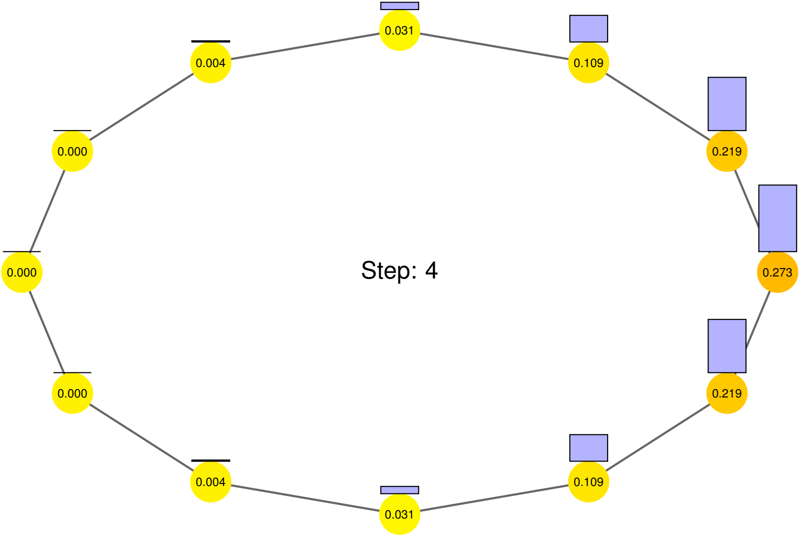

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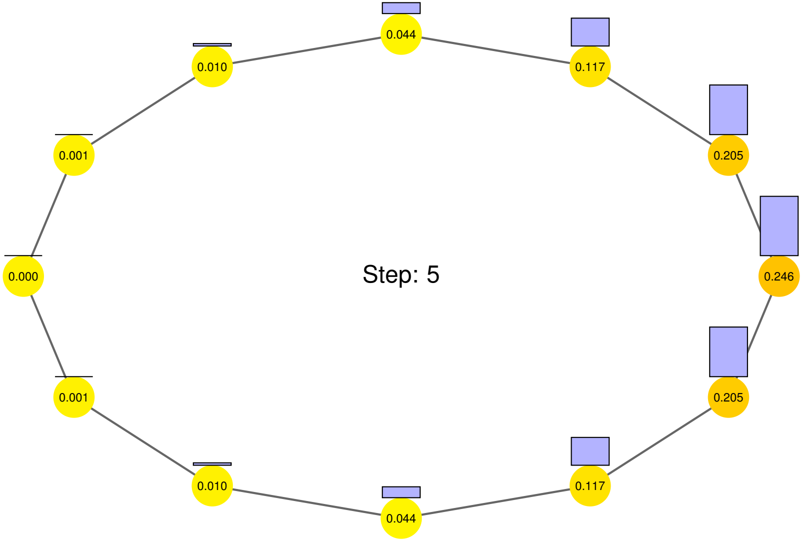

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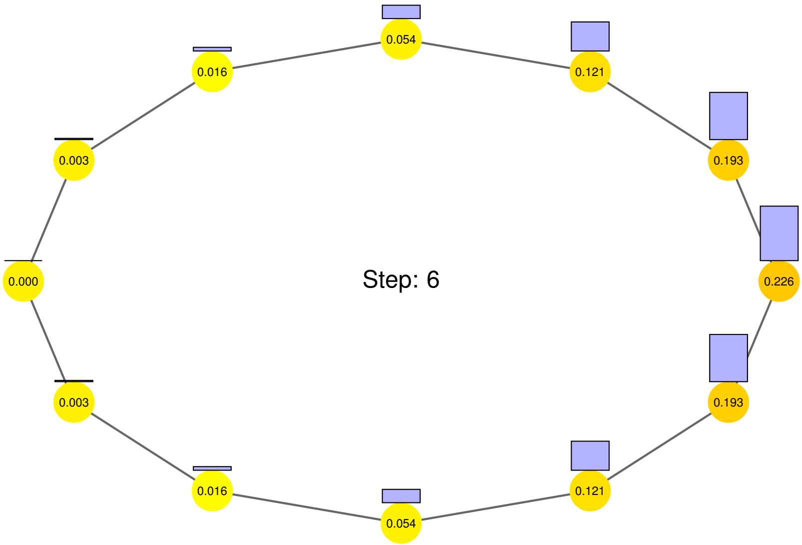

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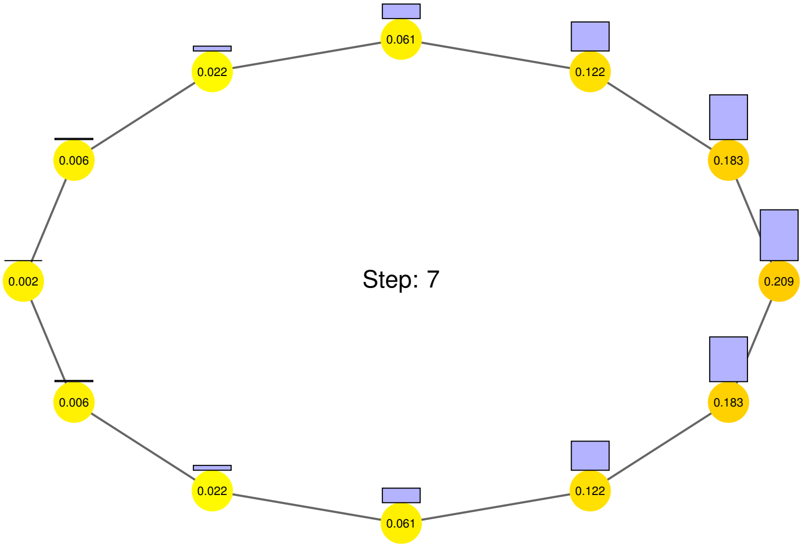

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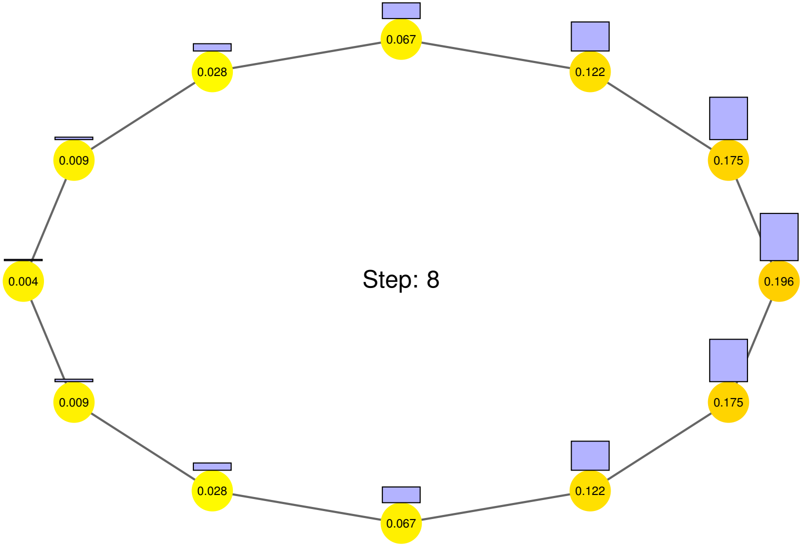

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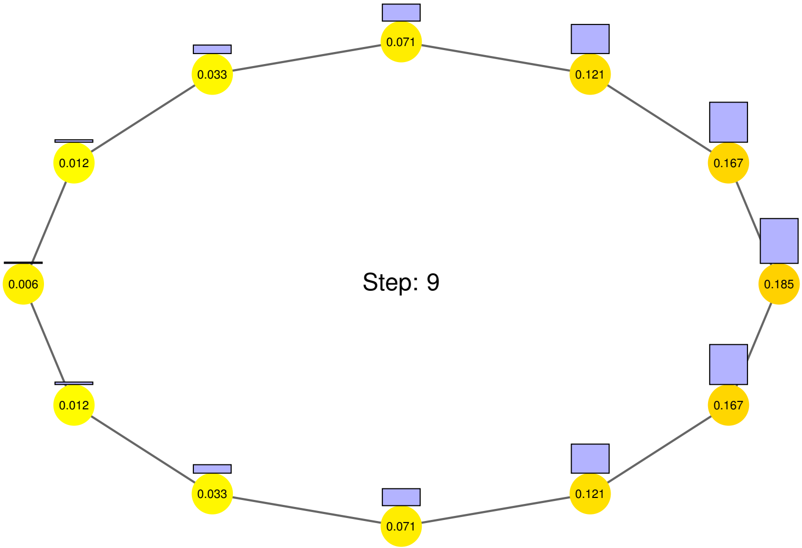

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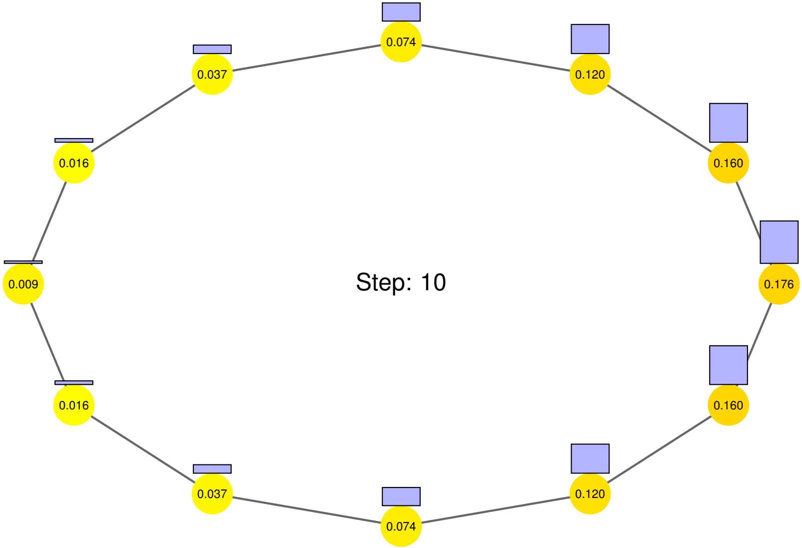

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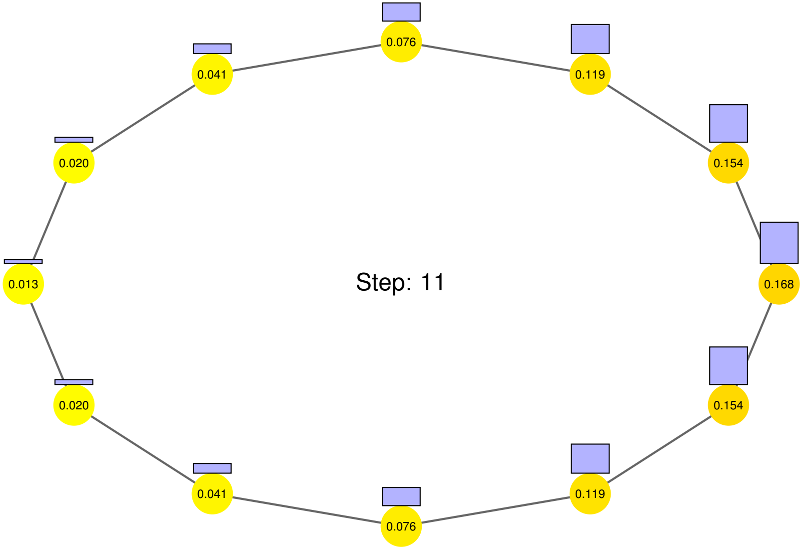

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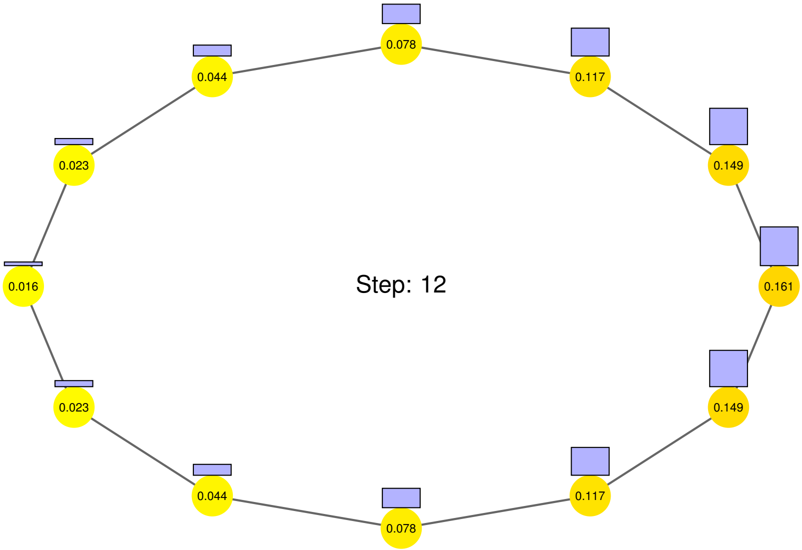

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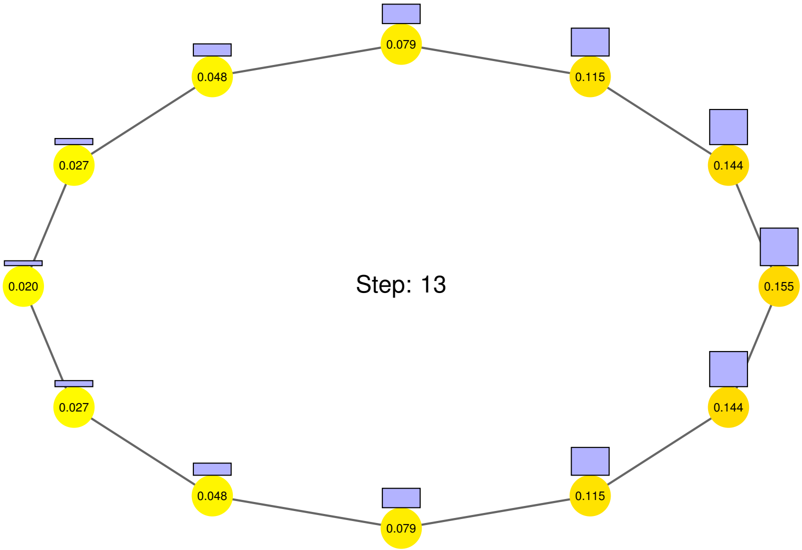

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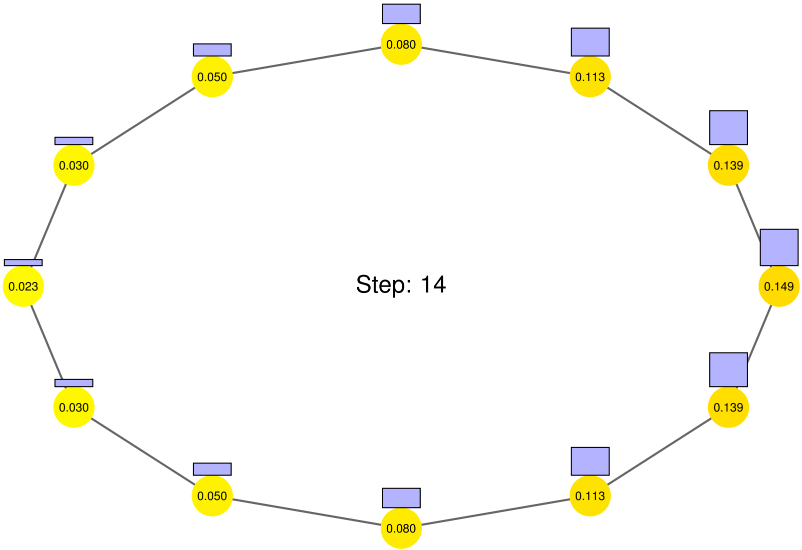

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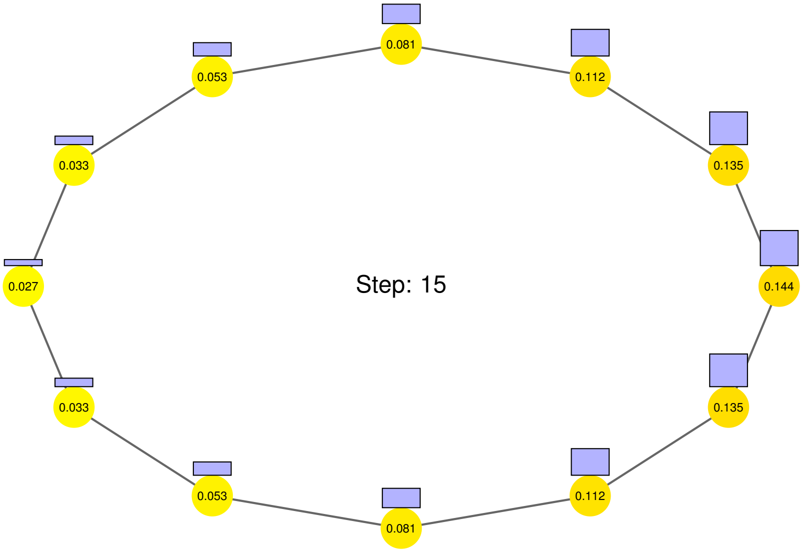

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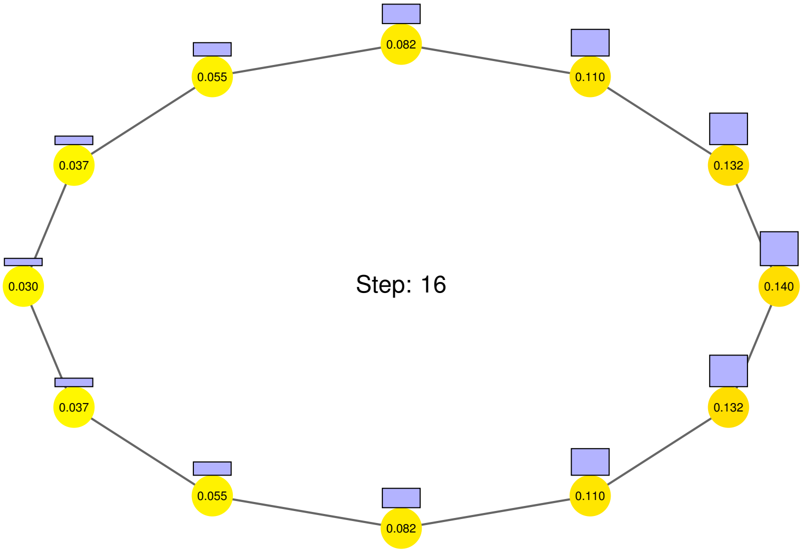

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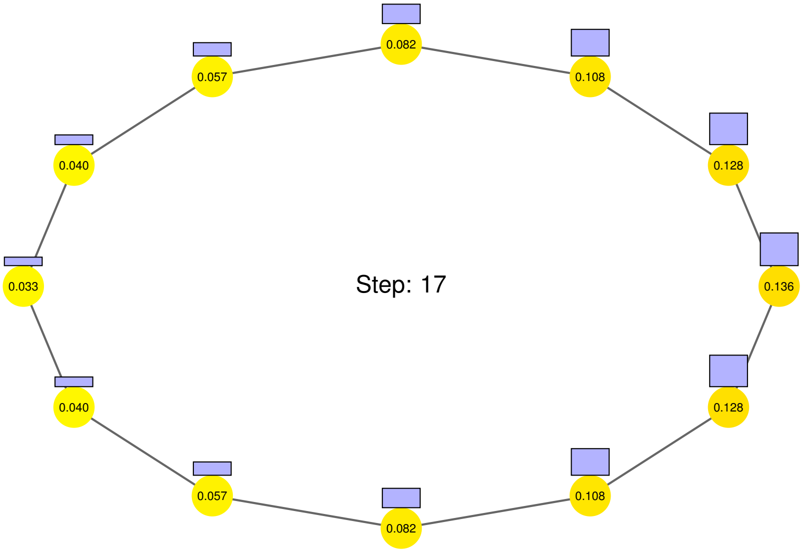

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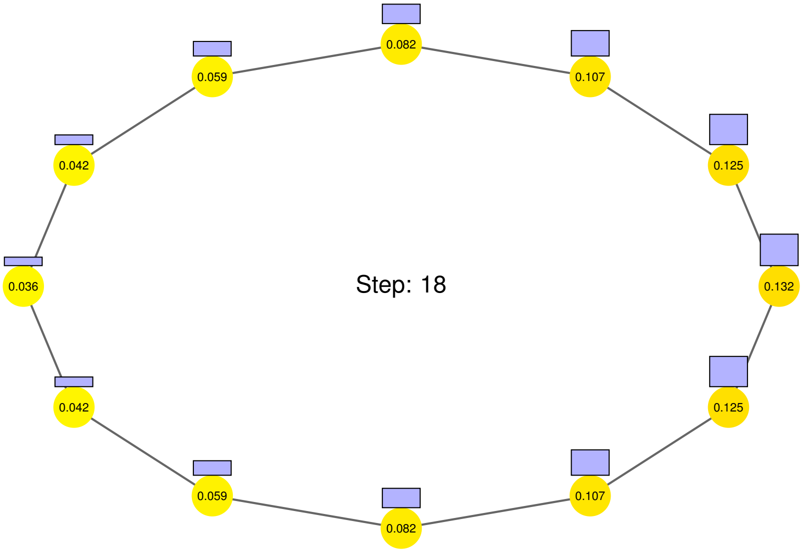

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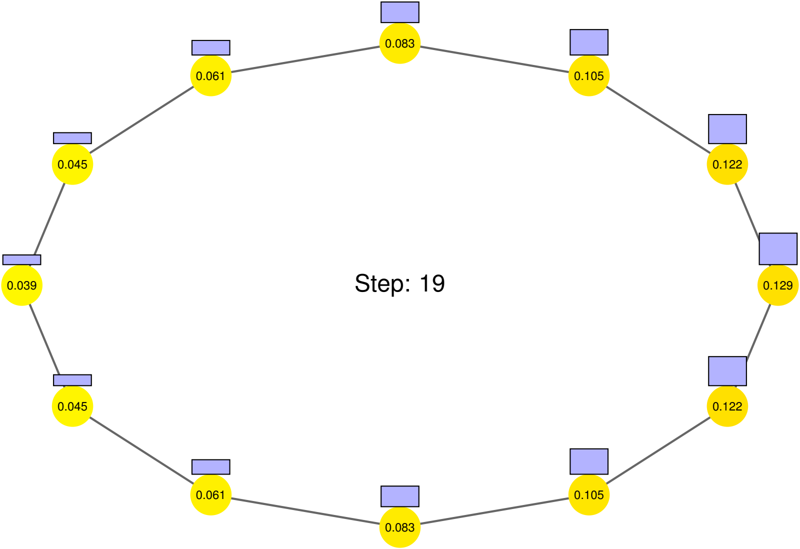

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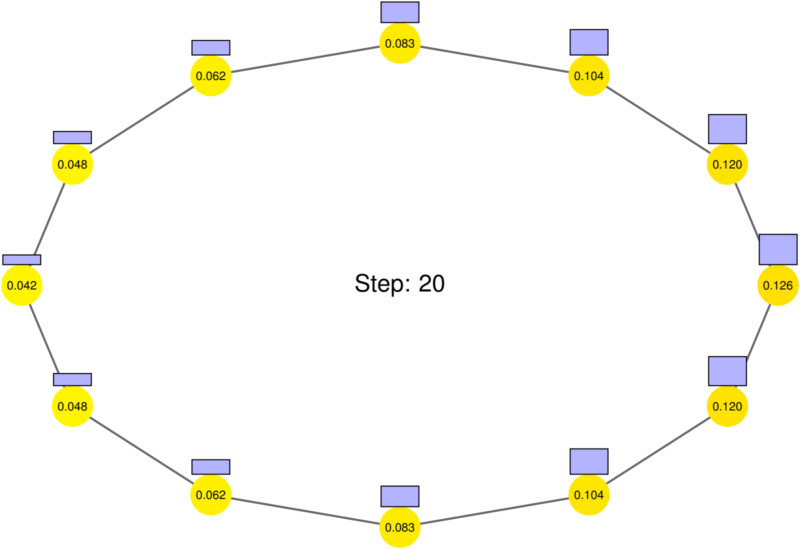

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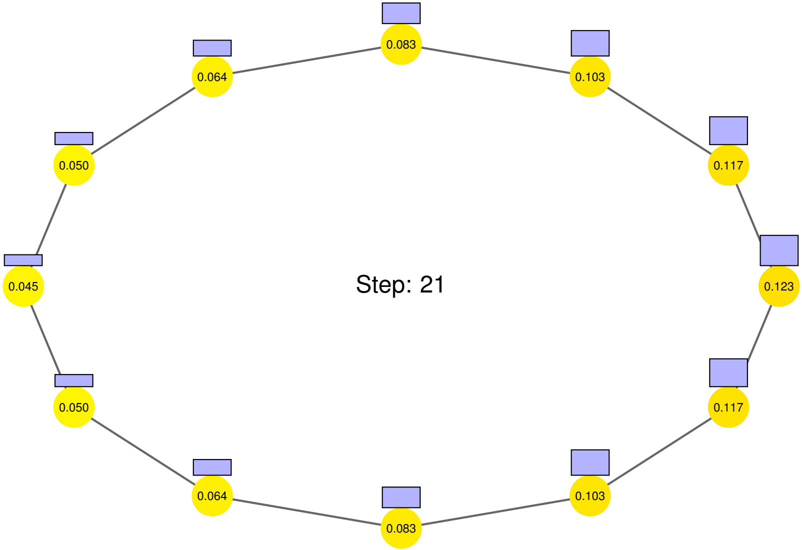

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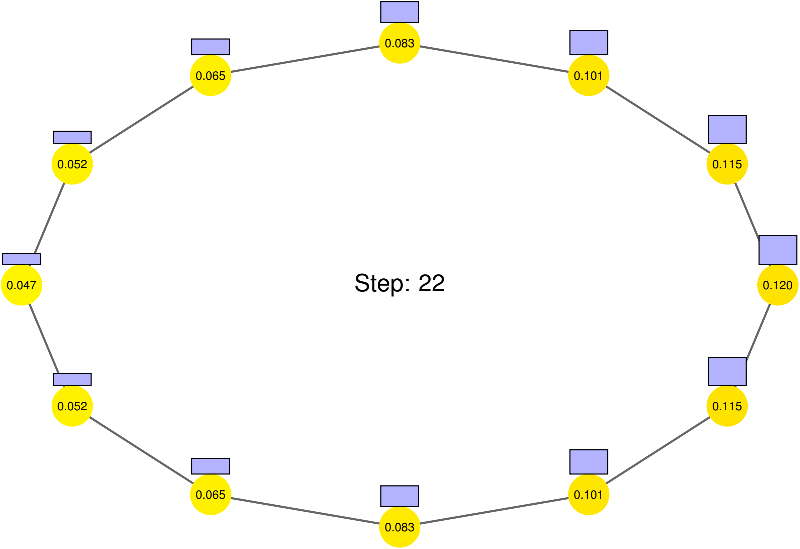

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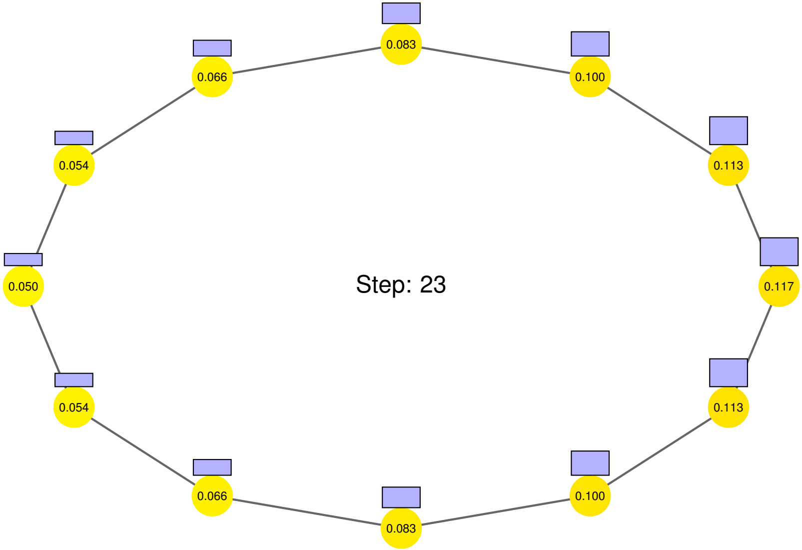

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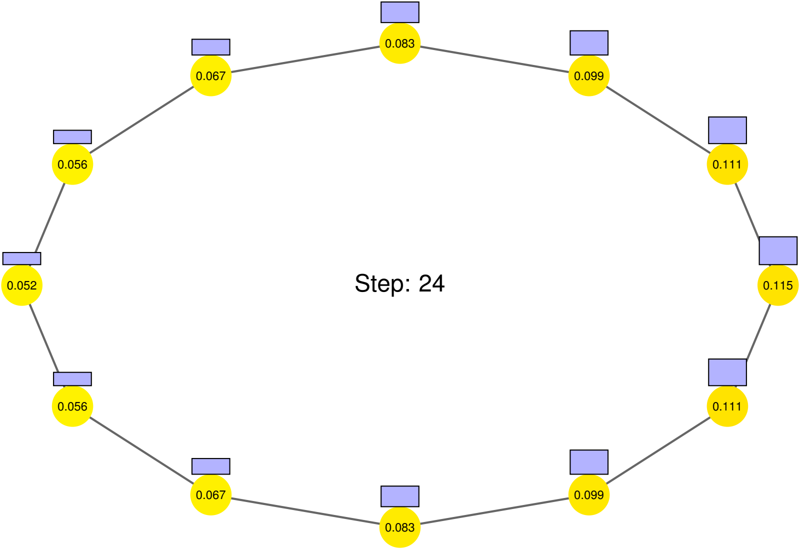

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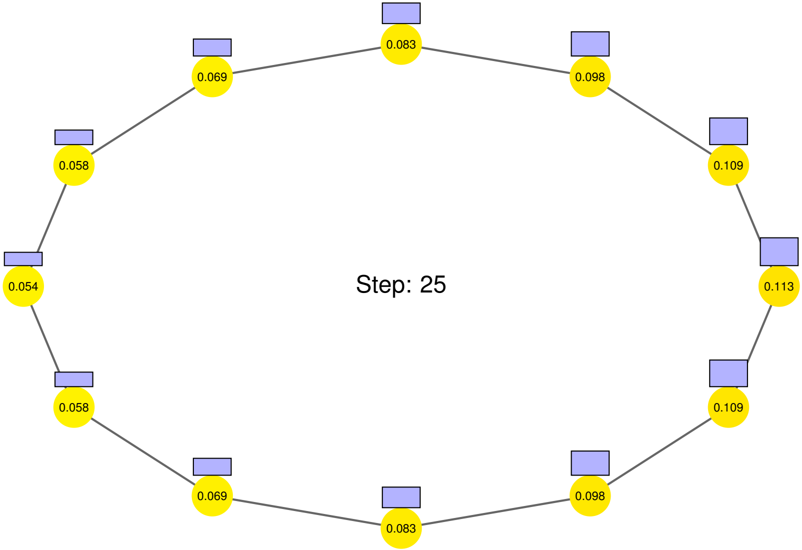

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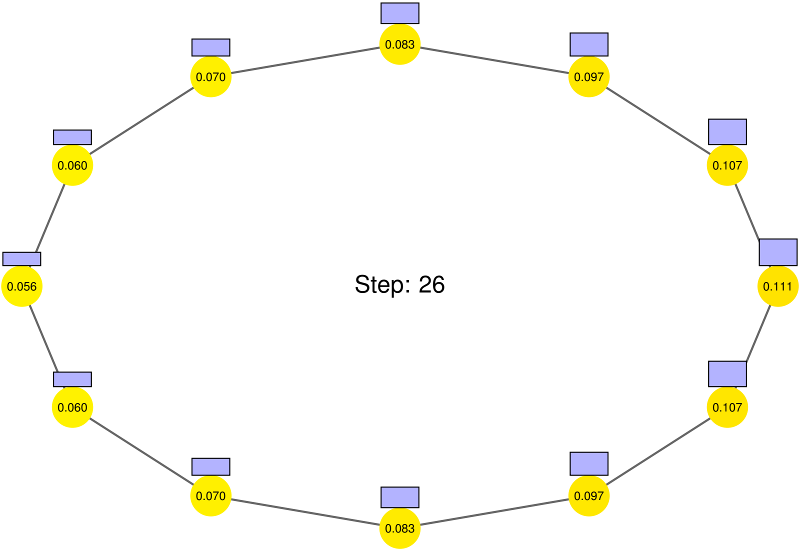

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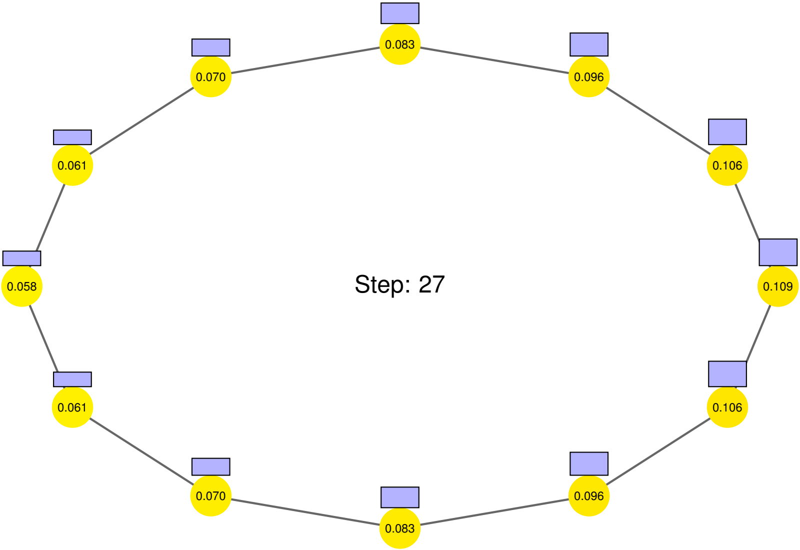

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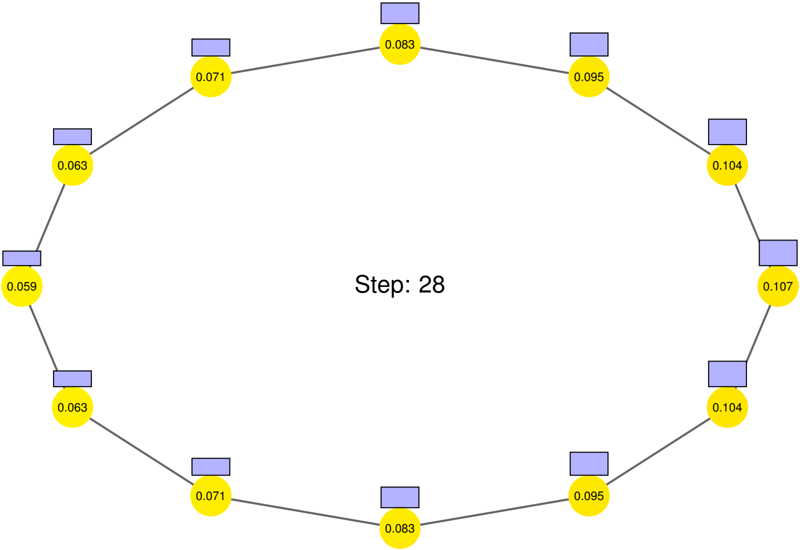

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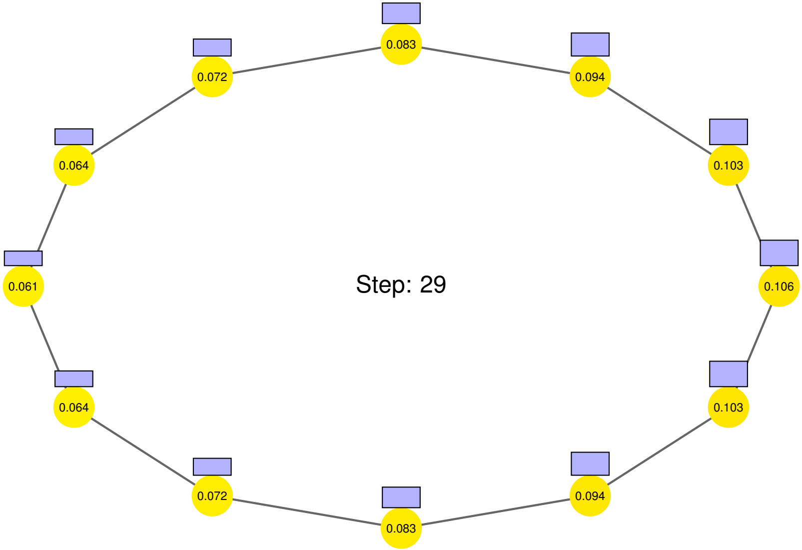

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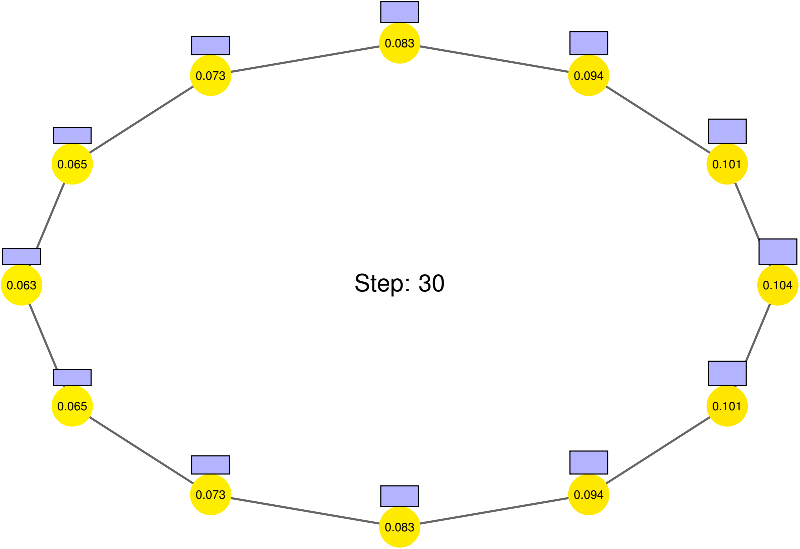
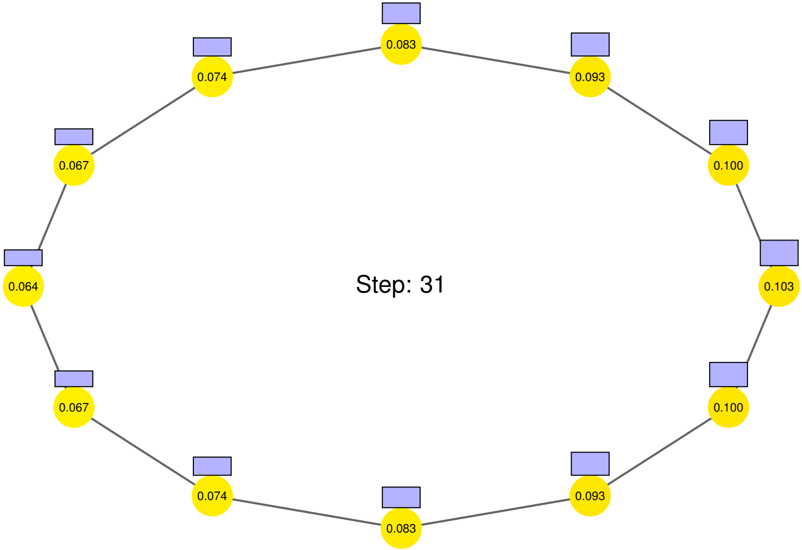


Diffusion of Load on a Ring



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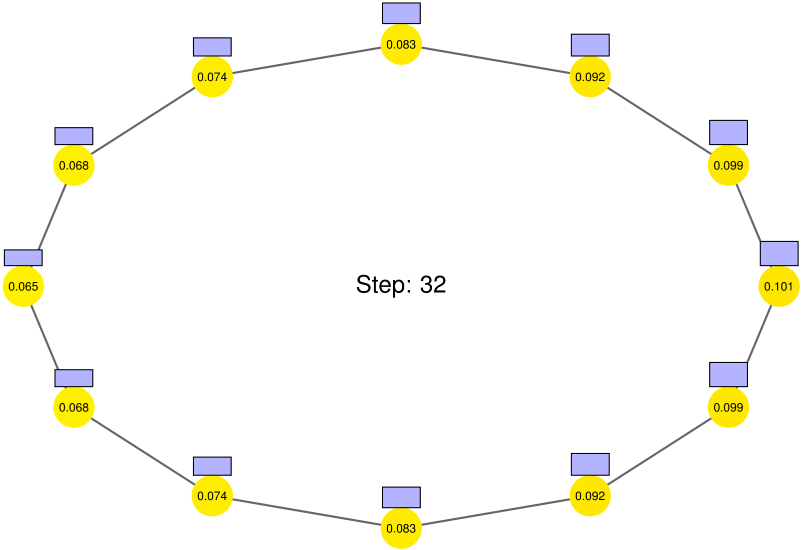

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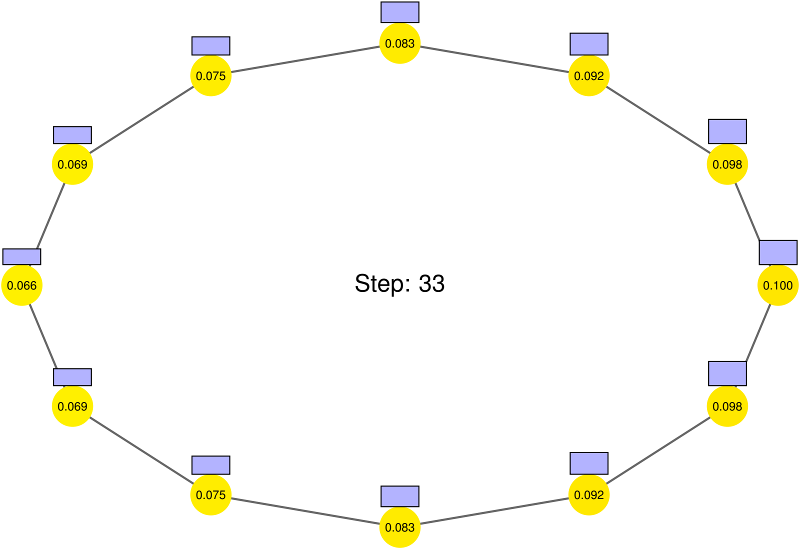

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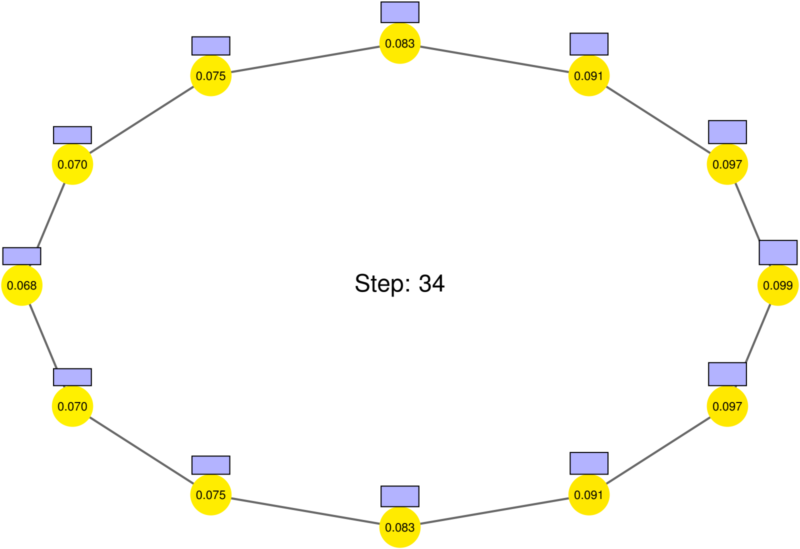

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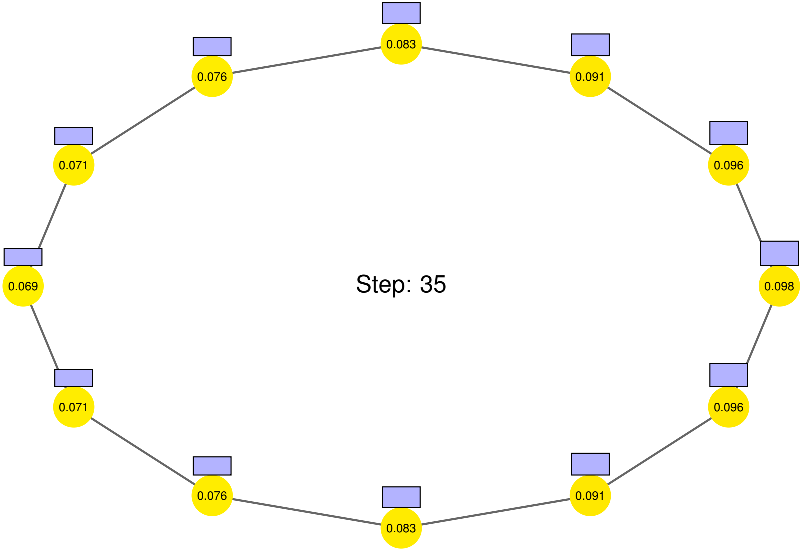

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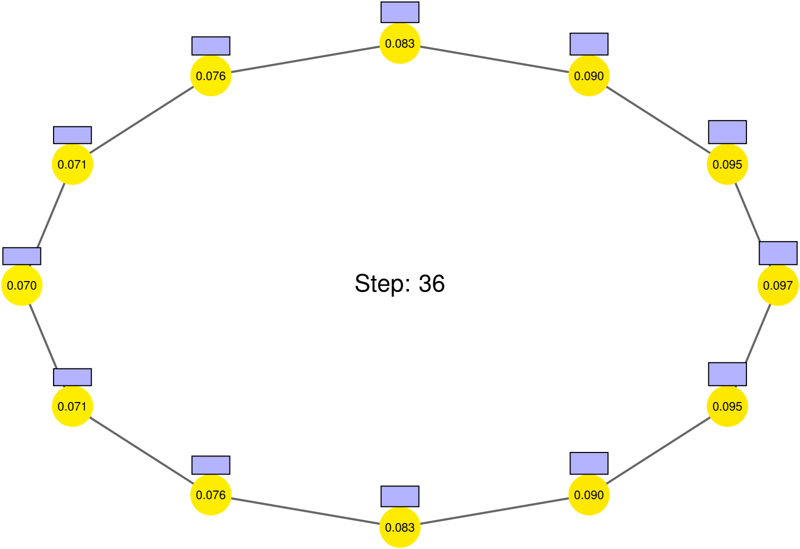

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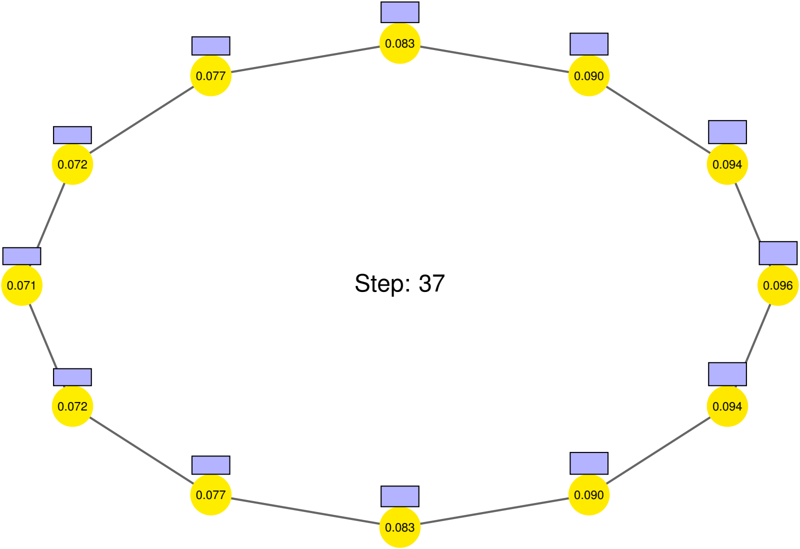

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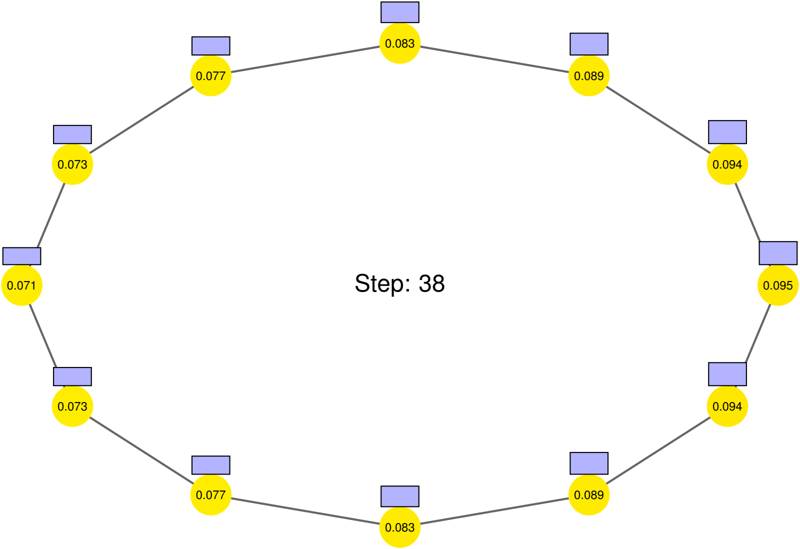

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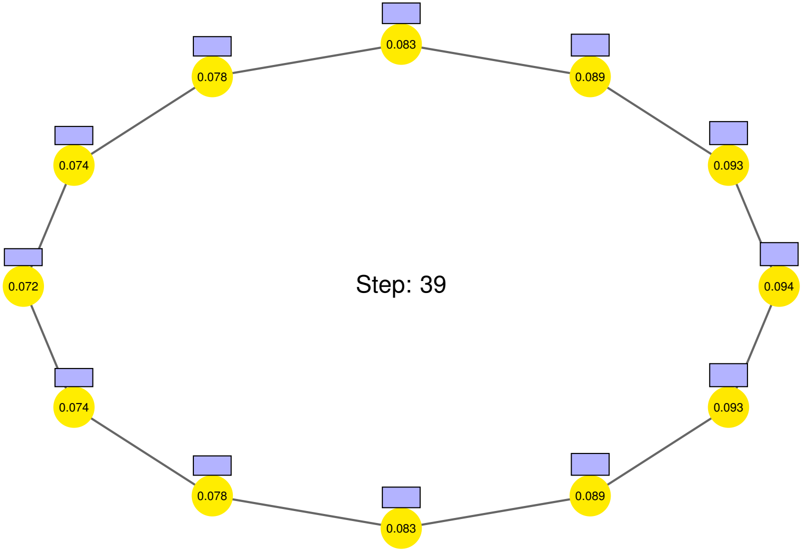

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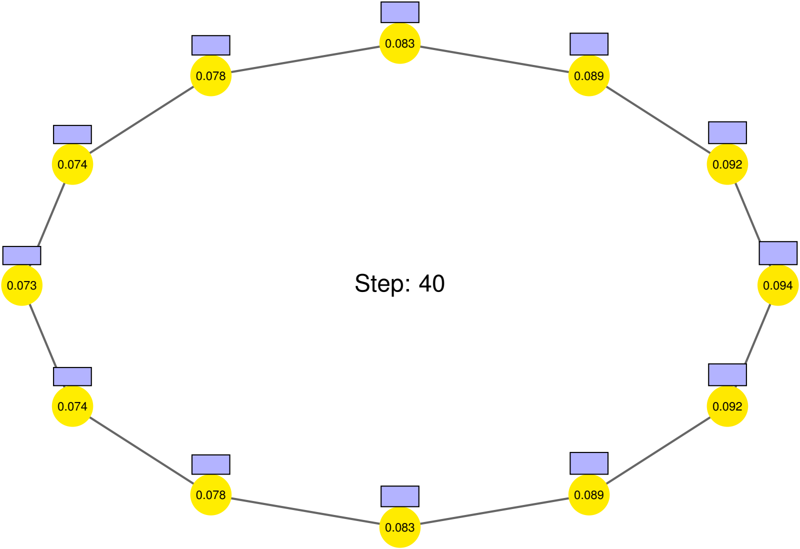

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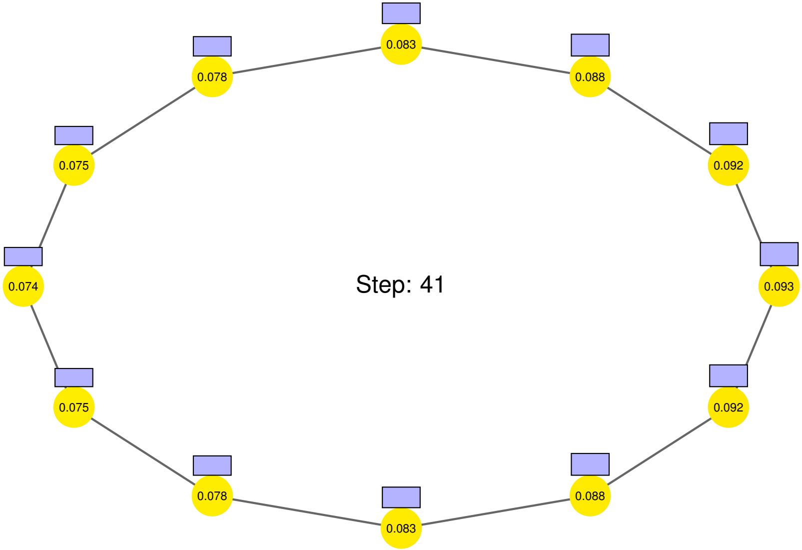

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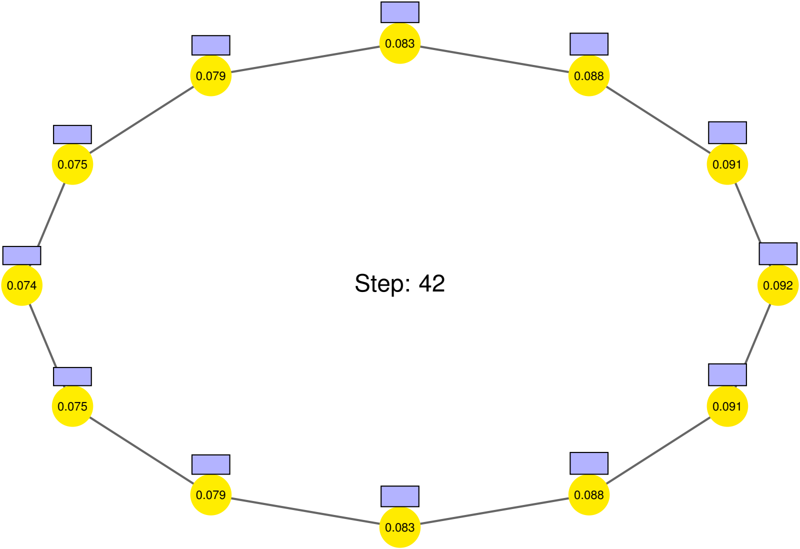

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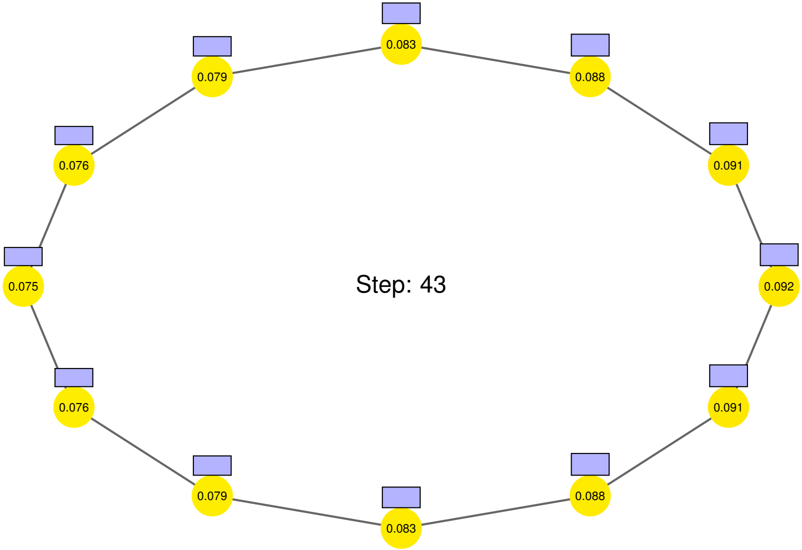

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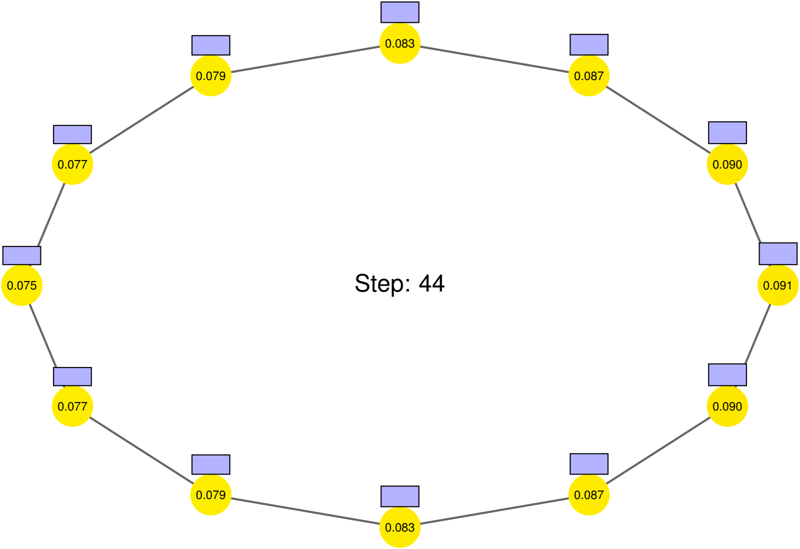

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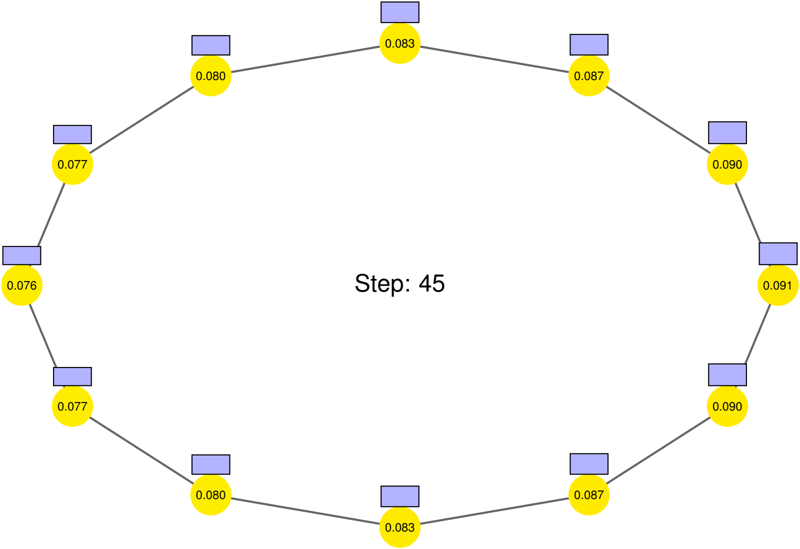

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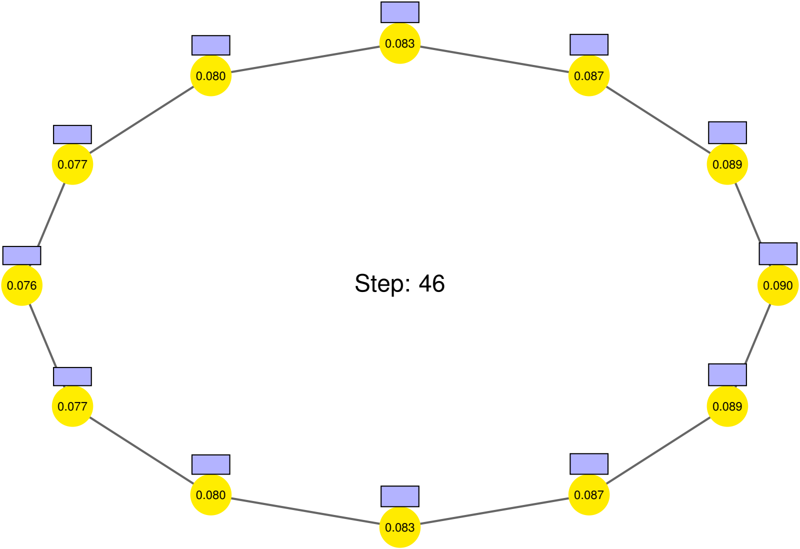

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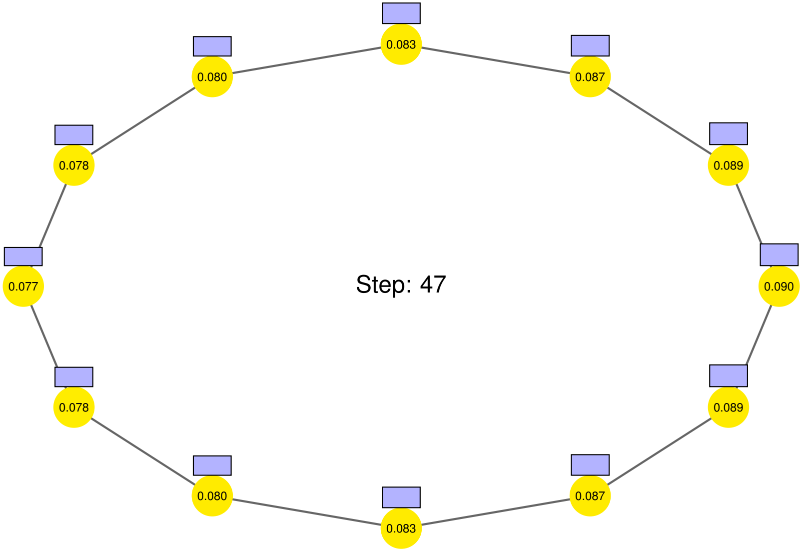

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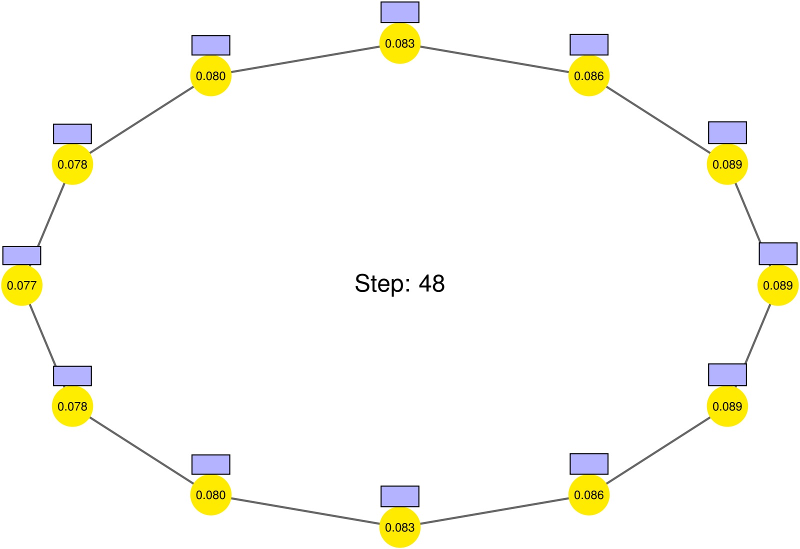

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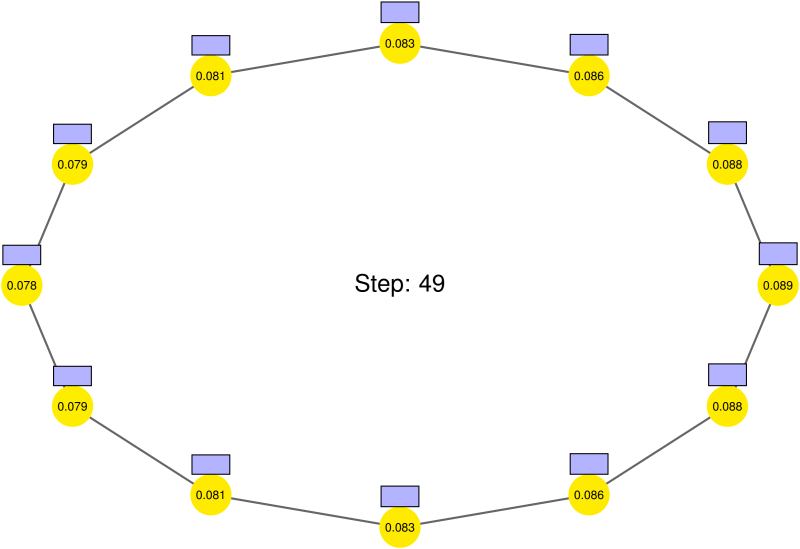

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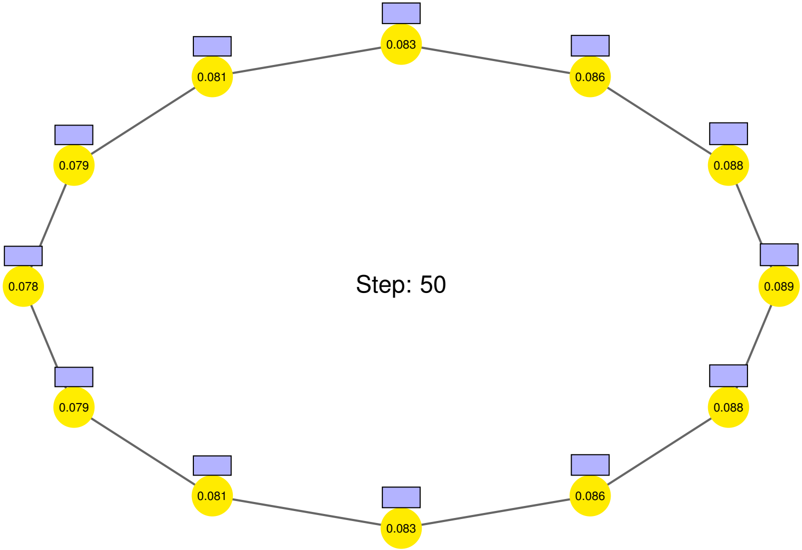

Convergence of the Quadratic Error (Upper Bound)

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Let $\gamma(M) := \max_{\mu_i \neq 1} |\mu_i|$, where $\mu_1 = 1 > \mu_2 \geq \cdots \geq \mu_n \geq -1$ are the eigenvalues of M. Then for any iteration t,

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Summary and Outlook: Idealised versus Discrete Case

Idealised Case

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Linear System

- corresponds to Markov chain
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