Mathematical Methods for Computer Science

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB

Michaelmas Term 2014/15

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Exercise problems –

Fourier and related methods

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1. Using the Euclidean norm on an inner product space $V = \mathbb{R}^3$, for the following vectors $u, v \in V$ whose span is a linear subspace of V,

$$u = \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}} \right)$$

$$v = \left(\sqrt{3}, \ -\frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} \right)$$

demonstrate both whether u, v form an *orthogonal system,* and also whether they form an *orthonormal system.*

2. Let V be an inner product space spanned by an orthonormal system of vectors $\{e_1, e_2, \ldots, e_n\}$ so that $\forall i \neq j$ the inner product $\langle e_i, e_j \rangle = 0$, but every e_i is a unit vector so that $\langle e_i, e_i \rangle = 1$. We wish to represent a data set consisting of vectors $u \in \text{span}\{e_1, e_2, \ldots, e_n\}$ in this space as a linear combination of the orthonormal vectors: $u = \sum_{i=1}^{n} a_i e_i$. Derive how the coefficients a_i can be determined for any vector u, and comment on the computational advantage of representing the data in an orthonormal system.

3. Using complex exponentials, prove the following trigonometric identity, which describes the multiplicative modulation of one cosine wave by another as being simply the sum of a different pair of cosine waves:

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a+b)x) + \frac{1}{2}\cos((a-b)x)$$

4. Calculate the Fourier series of the function $f(x)$ $(x \in [-\pi, \pi])$ defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & -\pi \leq x < 0\,. \end{cases}$$

Find also the complex Fourier series for $f(x)$.

5. Suppose that $f(x)$ is a 2π-periodic function with complex Fourier series

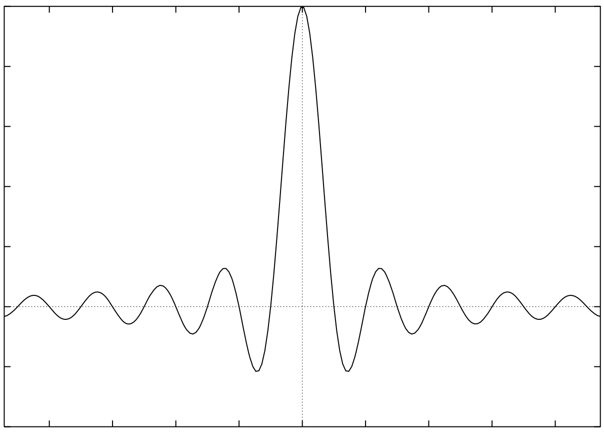
$$\sum_{n=-\infty}^{\infty} c_n e^{inx} .$$

Now consider the shifted version of $f(x)$ given by

$$g(x) = f(x - x_0)$$

where x_0 is a constant. Find the relationship between the complex Fourier coefficients of $g(x)$ in terms of those of $f(x)$. How do the magnitudes of the corresponding coefficients compare?

6. The function $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$ for $x \neq 0$ as plotted here plays an important role in the Sampling Theorem. By considering its Fourier Transform, show that this function is unchanged in form after convolution with itself, and show that it even remains unchanged in form after convolution with any higher frequency sinc function $\text{sinc}(ax)$ for $a > 1$, but that if $0 < a < 1$, then the result is instead that lower frequency sinc function $\text{sinc}(ax)$.

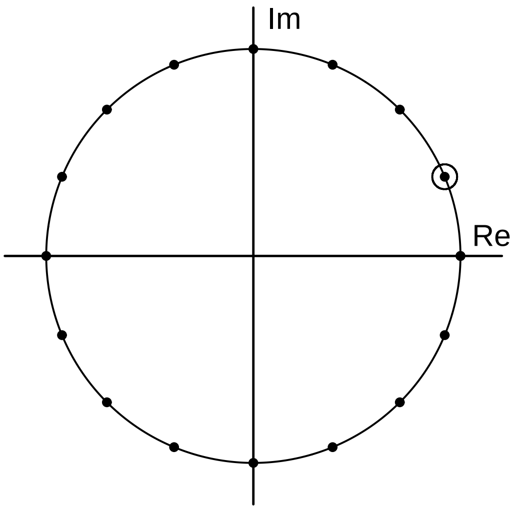


7. Show how Fourier methods facilitate solution of differential equations such as the following, in which the non-zero function $g(x)$ is known, its Fourier Transform $G(\omega)$ can be computed, and a, b, c are constant coefficients. Derive an expression for $f(x)$ that is a solution to this differential equation, assuming its Fourier Transform exists.

$$a\frac{d^2 f(x)}{dx^2} + b\frac{df(x)}{dx} + cf(x) = g(x)$$

8. For a function $f(x)$ whose Fourier Transform is $F(\omega)$, what is the Fourier Transform of $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ with respect to x? Explain how Fourier methods make it possible to define non-integer orders of derivatives, and name one scientific field in which it is useful to take half-order derivatives.

9. Using a diagram in the complex plane showing the N^{th} roots of unity, explain why all the values of complex exponentials that are needed for computing the Discrete Fourier Transform of N data points are powers of a primitive N^{th} root of unity (circled here for $N = 16$), and explain why such factorisation greatly reduces the number of multiplications required in a Fast Fourier Transform.



10. Show how a generating (or "mother") wavelet $\Psi(x)$ can spawn a family of "daughter" wavelets $\Psi_{jk}(x)$ by simple shifting and scaling ("dyadic") operations, and explain the advantages of representing continuous functions in terms of such a family of self-similar dilates and translates of a mother wavelet.