Lambda calculus
(Advanced Functional Programming)

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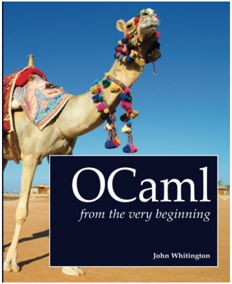
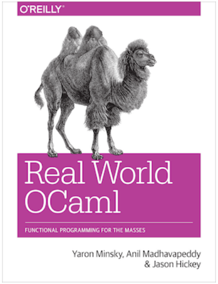
Course outline

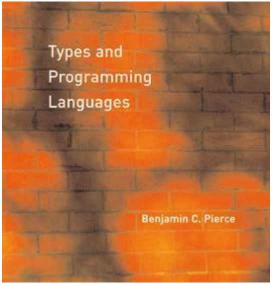
Books



OCaml from the very beginning
John Whitington
Coherent Press (2013)



Real World OCaml
Yaron Minsky,
Anil Madhavapeddy &
Jason Hickey
O'Reilly Media (2013)



Types and Programming Languages
Benjamin C. Pierce
MIT Press (2002)

Tooling

OPAM
OCaml package manager

Linux / OSX / VirtualBox

IOCaml

Fω

Fω interpreter

Philosophy and approach

- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, widely applicable techniques
- **current**: mostly the topics of ongoing research

Philosophy and approach

- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, widely applicable techniques
- **current**: mostly the topics of ongoing research
- **opinionated** (but you don't have to agree)

Mailing list

cl-acs-28@lists.cam.ac.uk

Announcements, questions and discussion. Feel free to post!

Have a question but feeling shy? Mail a lecturer instead and we'll anonymise and post your question:

jeremy.yallop@cl.cam.ac.uk leo.white@cl.cam.ac.uk

Exercises assessed and unassessed

Unassessed exercises:

Useful preparation for the assessed exercises, so we recommend that you work through them. Hand in for feedback, discuss freely on the mailing list.

Assessed exercises:

Mon 2 Feb	Mon 16 Feb	Mon 9 March
↓	↓	↓
Mon 9 Feb	Mon 2 March	Fri 24 April

Course structure

- **Technical background**
  Lambda calculus; type inference
- **Themes**
  Propositions as types; duality; parametricity and abstraction
- **(Fancy) types**
  Higher-rank and higher-kinded polymorphism; modules and functors; generalised algebraic types; rows
- **Applications**
  Monads and related concepts; domain-specific languages; datatype-generic programming; staged programming

Motivation & background

System Fω

Function composition in OCaml:

```
fun f g x -> f (g x)
```

Function composition in System Fω:

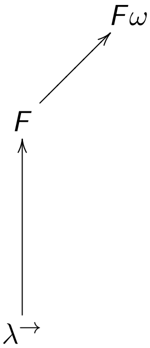
$$
\begin{array}{l}
\Lambda\alpha :: * . \\
\quad \Lambda\beta :: * . \\
\quad\quad \Lambda\gamma :: * . \\
\quad\quad\quad \lambda f : \alpha \to \beta . \\
\quad\quad\quad\quad \lambda g : \gamma \to \alpha . \\
\quad\quad\quad\quad\quad \lambda x :: \gamma . \; f \;\; (g \;\; x)
\end{array}
$$

What's the point of System Fω?

A framework for understanding language features and programming patterns:

- the elaboration language for type inference
- the proof system for reasoning with propositional logic
- the setting for dualities
- the background for parametricity properties
- the language underlying higher-order polymorphism in OCaml
- the elaboration language for modules
- the core calculus for GADTs

Roadmap



Inference rules

$$\frac{\text{premise 1} \qquad \text{premise 1} \qquad \ldots \qquad \text{premise N}}{\text{conclusion}}\ \text{rule name}$$

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$$\frac{\begin{array}{c}\text{all } M \text{ are } P\\ \text{all } S \text{ are } M\end{array}}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

Inference rules

$$\frac{\text{premise 1} \qquad \text{premise 1} \qquad \ldots \qquad \text{premise N}}{\text{conclusion}}\ \text{rule name}$$

$$\frac{\begin{array}{c}\text{all } M \text{ are } P \\ \text{all } S \text{ are } M\end{array}}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

$$\frac{\begin{array}{c}\text{all programs are buggy} \\ \text{all functional programs are programs}\end{array}}{\text{all functional programs are buggy}}\ \text{modus barbara}$$

Typing rules

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \to\text{-elim}$$

Terms, types, kinds

Kinds: K, K_1, K_2, ...

K is a kind

Environments: Γ

Γ is an environment

Types: A, B, C, ...

$\Gamma \vdash A :: K$

Terms: L, M, N, ...

$\Gamma \vdash M : A$

$$\lambda^{\rightarrow}$$

(simply typed lambda calculus)

$\lambda^{\rightarrow}$ by example

In $\lambda^{\rightarrow}$:

$\lambda x : A . x$

$\lambda f : B \rightarrow C .$
$\quad \lambda g : A \rightarrow B .$
$\quad\quad \lambda x : A . f \; (g \; x)$

In OCaml:

```
fun x -> x

fun f g x -> f (g x)
```

Kinds in $\lambda^{\rightarrow}$

$$\frac{}{* \text{ is a kind}}\ *\text{-kind}$$

Kinding rules (type formation) in $\lambda^{\rightarrow}$

$$\frac{}{\Gamma \vdash \mathcal{B} :: *} \text{kind-}\mathcal{B}$$

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \rightarrow B :: *} \text{kind-}\rightarrow$$

A kinding derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathcal{B} :: *}\,\text{kind-}\mathcal{B} \qquad \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\,\text{kind-}\mathcal{B}}{\Gamma \vdash \mathcal{B} \to \mathcal{B} :: *}\,\text{kind-}\to \qquad \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\,\text{kind-}\mathcal{B}}{\Gamma \vdash (\mathcal{B} \to \mathcal{B}) \to \mathcal{B} :: *}\,\text{kind-}\to$$

Environment formation rules

$$\frac{}{\cdot \text{ is an environment}}\ \Gamma\text{-}\cdot$$

$$\frac{\Gamma \text{ is an environment} \qquad \Gamma \vdash A :: *}{\Gamma, x : A \text{ is an environment}}\ \Gamma\text{-}:$$

Typing rules (term formation) in $\lambda^{\rightarrow}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{tvar}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \rightarrow B}\ \rightarrow\text{-intro} \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}\ \rightarrow\text{-elim}$$

A typing derivation for the identity function

$$\frac{\cdot, x : A \vdash x : A}{\cdot \vdash \lambda x : A . x : A \to A} \to\text{-intro}$$

Products by example

In $\lambda^{\rightarrow}$ with products:

$\lambda p : (A \rightarrow B) \times A .$
$\quad$ **fst** p (**snd** p)

$\lambda x : A . \langle x , x \rangle$

$\lambda f : A \rightarrow C .$
$\quad \lambda g . B \rightarrow C .$
$\quad\quad \lambda p . A \times B .$
$\quad\quad\quad \langle$ f **fst** p , g **snd** p $\rangle$

$\lambda p . A \times B . \langle$ **snd** p , **fst** p $\rangle$

In OCaml:

```
fun (f, p) -> f p

fun x -> (x, x)

fun f g (x, y) -> (f x, g y)

fun (x, y) -> (y, x)
```

Kinding and typing rules for products

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \times B :: *} \text{ kind-}\times$$

$$\frac{\begin{array}{c}\Gamma \vdash M : A \\ \Gamma \vdash N : B\end{array}}{\Gamma \vdash \langle M, N \rangle : A \times B} \times\text{-intro}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{fst}\ M : A} \times\text{-elim-1}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{snd}\ M : B} \times\text{-elim-2}$$

Sums by example

In $\lambda^{\to}$ with sums:

```
λf:A→C.
  λg:B→C.
    λs:A+B.
      case s of
        x.f x
      | y.g y
```

```
λs:A+B.
  case s of
    x.inr [B] x
  | y.inr [A] y
```

In OCaml:

```
fun f g s ->
  match s with
    Inl x -> f x
  | Inr y -> g y
```

```
function
  Inl x -> Inr x
| Inr y -> Inl y
```

Kinding and typing rules for sums

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A + B :: *}\ \text{kind-}+$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \text{inl}\ [B]\ M : A + B}\ +\text{-intro-1}$$

$$\frac{\Gamma \vdash N : B}{\Gamma \vdash \text{inr}\ [A]\ N : A + B}\ +\text{-intro-2}$$

$$\frac{\begin{array}{c}\Gamma \vdash L : A + B \\ \Gamma, x : A \vdash M : C \\ \Gamma, y : B \vdash N : C\end{array}}{\Gamma \vdash \text{case}\ L\ \text{of}\ x.M \mid y.N : C}\ +\text{-elim}$$

System F

(polymorphic lambda calculus)

System F by example

$$\Lambda\alpha :: * . \lambda \mathsf{x} : \alpha . \mathsf{x}$$

$$
\begin{array}{l}
\Lambda\alpha :: * . \\
\quad \Lambda\beta :: * . \\
\quad\quad \Lambda\gamma :: * . \\
\quad\quad\quad \lambda \mathsf{f} : \beta \to \gamma . \\
\quad\quad\quad\quad \lambda \mathsf{g} : \alpha \to \beta . \\
\quad\quad\quad\quad\quad \lambda \mathsf{x} : \alpha . \mathsf{f}\ (\mathsf{g}\ \mathsf{x})
\end{array}
$$

$$\Lambda\alpha :: * . \Lambda\beta :: * . \lambda \mathsf{p} : (\alpha \to \beta) \times \alpha . \mathbf{fst}\ \mathsf{p}\ (\mathbf{snd}\ \mathsf{p})$$

New kinding rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash A :: *}{\Gamma \vdash \forall\alpha{::}K.A :: *}\ \text{kind-}\forall$$

$$\frac{\alpha{::}K \in \Gamma}{\Gamma \vdash \alpha :: K}\ \text{tyvar}$$

New environment rule for System F

$$\frac{\Gamma \text{ is an environment} \qquad K \text{ is a kind}}{\Gamma, \alpha{::}K \text{ is an environment}} \ \Gamma\text{-}{::}$$

New typing rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash M : A}{\Gamma \vdash \Lambda\alpha{::}K.M : \forall\alpha{::}K.A} \ \forall\text{-intro}$$

$$\frac{\Gamma \vdash M : \forall\alpha{::}K.A \qquad \Gamma \vdash B :: K}{\Gamma \vdash M\ [B] : A[\alpha := B]} \ \forall\text{-elim}$$

Existential types

What's the point of existentials?

- $\forall$ and $\exists$ in logic are closely connected to polymorphism and existentials in type theory
- As in logic, $\forall$ and $\exists$ for types are closely related
- Module types can be viewed as a kind of existential type
- OCaml's variant types now support existential variables

Existential intuition

Existentials
correspond to
abstract types

Kinding rules for existentials

$$\frac{\Gamma, \alpha{::}K \vdash A :: *}{\Gamma \vdash \exists \alpha{::}K.A :: *}\ \text{kind-}\exists$$

Typing rules for existentials

$$\frac{\Gamma \vdash M : A[\alpha := B] \qquad \Gamma \vdash \exists\alpha{::}K.A :: *}{\Gamma \vdash \text{pack } B, M \text{ as } \exists\alpha{::}K.A : \exists\alpha{::}K.A}\ \exists\text{-intro}$$

$$\frac{\begin{array}{c}\Gamma \vdash M : \exists\alpha{::}K.A \\ \Gamma, \alpha{::}K, x : A \vdash M' : B\end{array}}{\Gamma \vdash \text{open } M \text{ as } \alpha, x \text{ in } M' : B}\ \exists\text{-elim}$$