L11: Algebraic Path Problems with applications to Internet Routing
Lectures 10 — 11

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Recall our basic iterative algorithm

$$
\begin{array}{rcl}
\mathbf{A}^{\langle 0 \rangle} & = & \mathbf{I} \\
\mathbf{A}^{\langle k+1 \rangle} & = & \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I}
\end{array}
$$

A closer look ...

$$
\begin{array}{rcl}
\mathbf{A}^{\langle k+1 \rangle}(i,j) & = & \mathbf{I}(i,j) \oplus \bigoplus_{u} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \\
 & = & \mathbf{I}(i,j) \oplus \bigoplus_{(i,u) \in E} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j)
\end{array}
$$

This is the basis of distributed Bellman-Ford algorithms — a node i computes routes to a destination j by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state **M**?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$\begin{aligned} \mathbf{A}_{\mathbf{M}}^{\langle 0 \rangle} &= \mathbf{M} \\ \mathbf{A}_{\mathbf{M}}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

Lemma 6.4

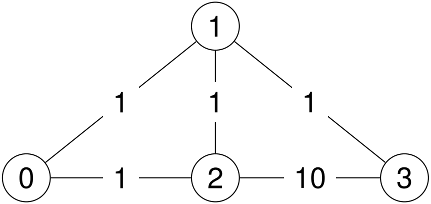
For $1 \leq k$,

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

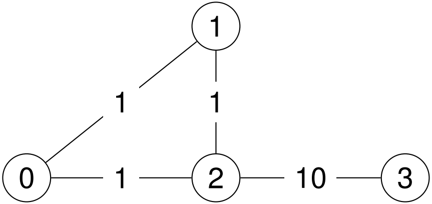
$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{*}$$

RIP-like example — counting to convergence (1)



Adjacency matrix $\mathbf{A}_1$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & 1 \\
1 & 1 & \infty & 10 \\
\infty & 1 & 10 & \infty
\end{array}\right]
\end{array}
$$

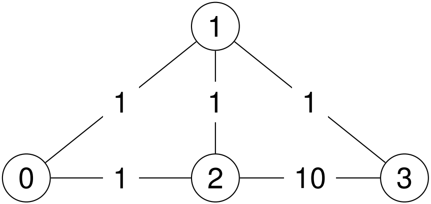


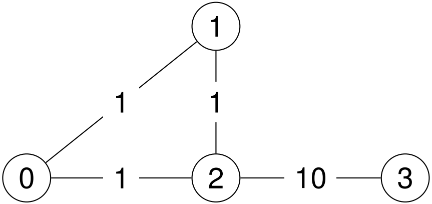
Adjacency matrix $\mathbf{A}_2$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & \infty \\
1 & 1 & \infty & 10 \\
\infty & \infty & 10 & \infty
\end{array}\right]
\end{array}
$$

See RFC 1058.

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 2 \\
2 & 1 & 2 & 0
\end{array}\right]
\end{array}
$$

The solution $\mathbf{A}_2^*$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
0 & 1 & 1 & 11 \\
1 & 0 & 1 & 11 \\
1 & 1 & 0 & 10 \\
11 & 11 & 10 & 0
\end{array}\right]
\end{array}
$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent $\mathbf{A_2}_{\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

RIP-like example — counting to convergence (5)

$$\mathbf{B}_6 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 7 \\ 1 & 1 & 0 & 1 & 7 \\ 2 & 1 & 1 & 0 & 7 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$

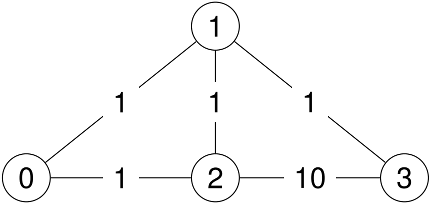
$$\mathbf{B}_9 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 10 \\ 1 & 1 & 0 & 1 & 10 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

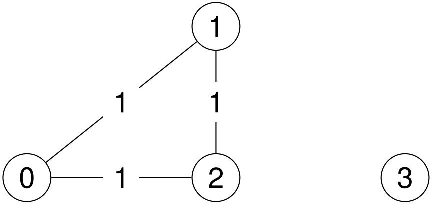
$$\mathbf{B}_7 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 8 \\ 1 & 1 & 0 & 1 & 8 \\ 2 & 1 & 1 & 0 & 8 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_{10} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 1 & 11 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_8 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 9 \\ 1 & 1 & 0 & 1 & 9 \\ 2 & 1 & 1 & 0 & 9 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

RIP-like example — counting to infinity (1)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

The solution $\mathbf{A}_3^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & \infty \\ 1 & 0 & 1 & \infty \\ 1 & 1 & 0 & \infty \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A}_{3\,\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$\mathbf{B}_0 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{376} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 377 \\ 1 & 0 & 1 & 377 \\ 1 & 1 & 0 & 377 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 999 \\ 1 & 0 & 1 & 999 \\ 1 & 1 & 0 & 999 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

RIP-like example — What's going on?

Recall

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}(i,\ j) = \mathbf{A}^k\mathbf{M}(i,\ j) \oplus \mathbf{A}^*(i,\ j)$$

- $\mathbf{A}^*$(i, j) may be arrived at very quickly
- but $\mathbf{A}^k\mathbf{M}(i,\ j)$ may be better until a very large value of k is reached (counting to convergence)
- or it may always be better (counting to infinity).

Solutions?

- RIP: $\infty = 16$
- We will explore various ways of adding paths to metrics and eliminating those paths with loops ....

Starting in an arbitrary state? No!

Let's use our friend

$$\text{add\_zero}(\infty,\ \text{min\_plus}\ \vec{\times}\ \text{sep}(G))$$

Problem:

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \left[\begin{array}{cccc} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (999, \{\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (999, \{\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (999, \{\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{array}\right] \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

Starting in an arbitrary state?

Solution: use another reduction!

$$
\begin{array}{rcl}
r(\infty) & = & \infty \\
r(s, W) & = & \begin{cases} \infty & \text{if } W = \{\} \\ (s, W) & \text{otherwise} \end{cases}
\end{array}
$$

Now use this instead

$$\mathrm{red}_r(\mathrm{add\_zero}(\infty,\ \mathrm{min\_plus} \vec{\times} \mathrm{sep}(G)))$$

Starting in an arbitrary state?

$\mathbf{B}_0$ and $\mathbf{B}_1$

$$
\begin{array}{c}
\begin{array}{cccc} \quad 0 & \quad 1 & \quad 2 & \quad 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3 \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (1, \{[(1,3)]\} \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3 \\
(2, \{[(3,1),(1,0)]\}) & (1, \{[(3,1)]\}) & (2, \{[(3,1),(1,2)]\}) & (0, \{\epsilon\})
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccc} \quad 0 & \quad 1 & \quad 2 & \quad 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$

Starting in an arbitrary state?

$\mathbf{B}_2$ and $\mathbf{B}_3$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (3, \{[(0,2),(2,1),(1,3)]\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (3, \{[(2,0),(0,1),(1,3)]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & \infty \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & \infty \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{bmatrix}
\end{array}
$$

Homework 2: Recall

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) = (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

Theorem

If $\oplus_S$ is commutative, idempotent, and selective, then

$$\mathrm{LD}(S \vec{\times} T) \iff \mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge (\mathrm{LC}(S) \vee \mathrm{LK}(T))$$

Where

Property	Definition
LD	$\forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$
LC	$\forall a, b, c : c \otimes a = c \otimes b \implies a = b$
LK	$\forall a, b, c : c \otimes a = c \otimes b$

Homework 2: Problem 1 (40 points)

Prove this

$$\textsc{ld}(S \vec{\times} T) \implies \textsc{ld}(S) \wedge \textsc{ld}(T) \wedge (\textsc{lc}(S) \vee \textsc{lk}(T))$$

Homework 2: Recall the operation for inserting a one

$$\text{add\_one}(\overline{1},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{1}\}, \oplus_{\overline{1}},\ \otimes_{\overline{1}})$$

where

$$a \oplus_{\overline{1}} b \quad = \quad \begin{cases} \text{inr}(\overline{1}) & (\text{if } b = \text{inr}(\overline{1})) \\ \text{inr}(\overline{1}) & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \oplus y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

$$a \otimes_{\overline{1}} b \quad = \quad \begin{cases} a & (\text{if } b = \text{inr}(\overline{1})) \\ b & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \otimes y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(x)) \end{cases}$$

Homework 2: Problem 2 (60 points)

Problem 2

Find predicates **P** and **Q** and a proof of a theorem of the following form:
If **P**($\oplus$, $\otimes$), then

$$\text{LD}(\mathrm{add\_one}(\overline{1},\ (S,\ \oplus,\ \otimes))) \iff \mathbf{Q}(S,\ \oplus,\ \otimes)$$

Note: this is a bit open-ended.
Hint: As with the lexicographic product, you may need some auxiliary property or properties (as lexicigraphic needed LC and LK).
Full marks for a complete proof with **the most general result** (weakest assumption **P**).