$$\mathrm{Bij}(A,B) \subset (A \Rightarrow B) \subseteq (A \Rightarrow B) \subseteq \mathrm{Rel}(A,B) = \mathcal{P}(A \times B)$$

$\underbrace{(A \Rightarrow B)}_{\text{the set of functions from } A \text{ to } B}$

Functions (or maps)

Definition 120 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

A partial function $f : A \to B$ is total
whenever for all $a \in A$,

$f(a)\downarrow$

(i.e. $\exists\, b \in B$ s.t $a \mathrel{f} b$)

The unique $b \in B$
related to an $a \in A$
is denoted as

$f(a)$

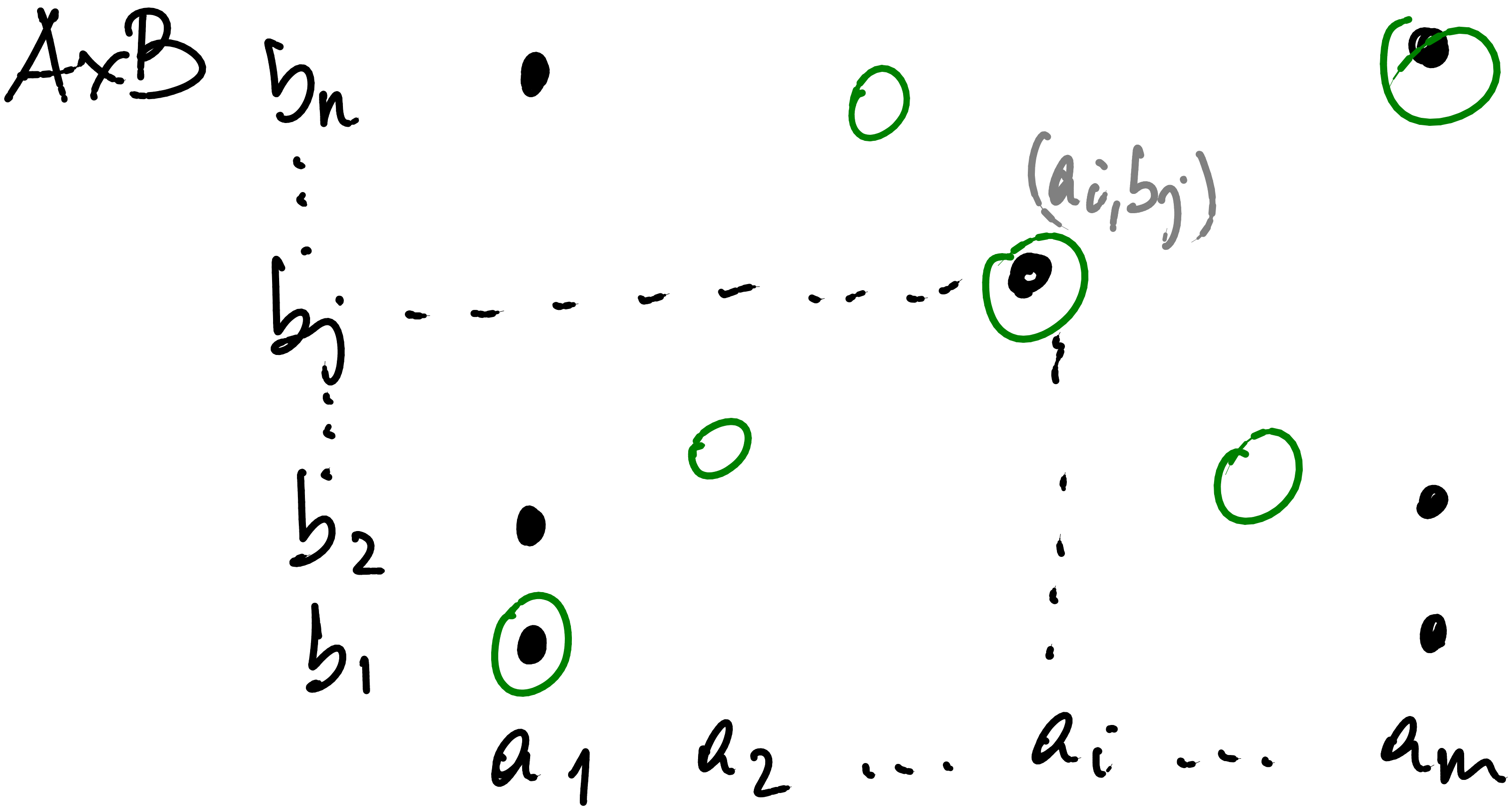
Theorem 121 *For all* $f \in \mathrm{Rel}(A, B)$*,*

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a \mathrel{f} b \quad .$$

Proposition 122 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) \;=\; \#B^{\#A} \quad .$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$ $B = \{b_1, \ldots, b_m\}$



In ML, $\circ : (\beta \to \gamma) * (\alpha \to \beta) \to \alpha \to \gamma$

Theorem 123 *The identity partial function is a function, and the composition of functions yields a function.*

NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

$$\mathrm{id}_A(a) = a$$

and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

$$(g \circ f)(a) = g\big(f(a)\big) \quad .$$

invertible processes or transformations.

Bijections

Definition 124 *A function* $f : A \to B$ *is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* f*) such that*

1. g *is a* retraction *(or* left inverse*) for* f*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. g *is a* section *(or* right inverse*) for* f*:*

$$f \circ g = \mathrm{id}_B \quad .$$

$f : A \to B$ is a bijection whenever $\exists\, g : B \to A$ s.t.

$g(f(a)) = a$

$\wedge$

$f(g(b)) = b$

$\forall a \in A, b \in B.$

$$
\begin{tikzcd}
\underline{\mathrm{Rel}}([m],[n]) \arrow[r, bend left, "\mathrm{mat}"] & (m\times n)\text{-matrices} \arrow[l, bend left, "\mathrm{rel}"] \\
R \arrow[r, mapsto] & \underline{\mathrm{mat}}(R) \\
\underline{\mathrm{rel}}(R) & M \arrow[l, mapsto]
\end{tikzcd}
$$

Bijections

Bijections between finite sets are permutations

Proposition 126 *For all finite sets* A *and* B,

$$\#\,\mathrm{Bij}(\mathrm{A},\mathrm{B}) = \begin{cases} 0 & \text{, if } \#\mathrm{A} \neq \#\mathrm{B} \\ n! & \text{, if } \#\mathrm{A} = \#\mathrm{B} = n \end{cases}$$

For bijection

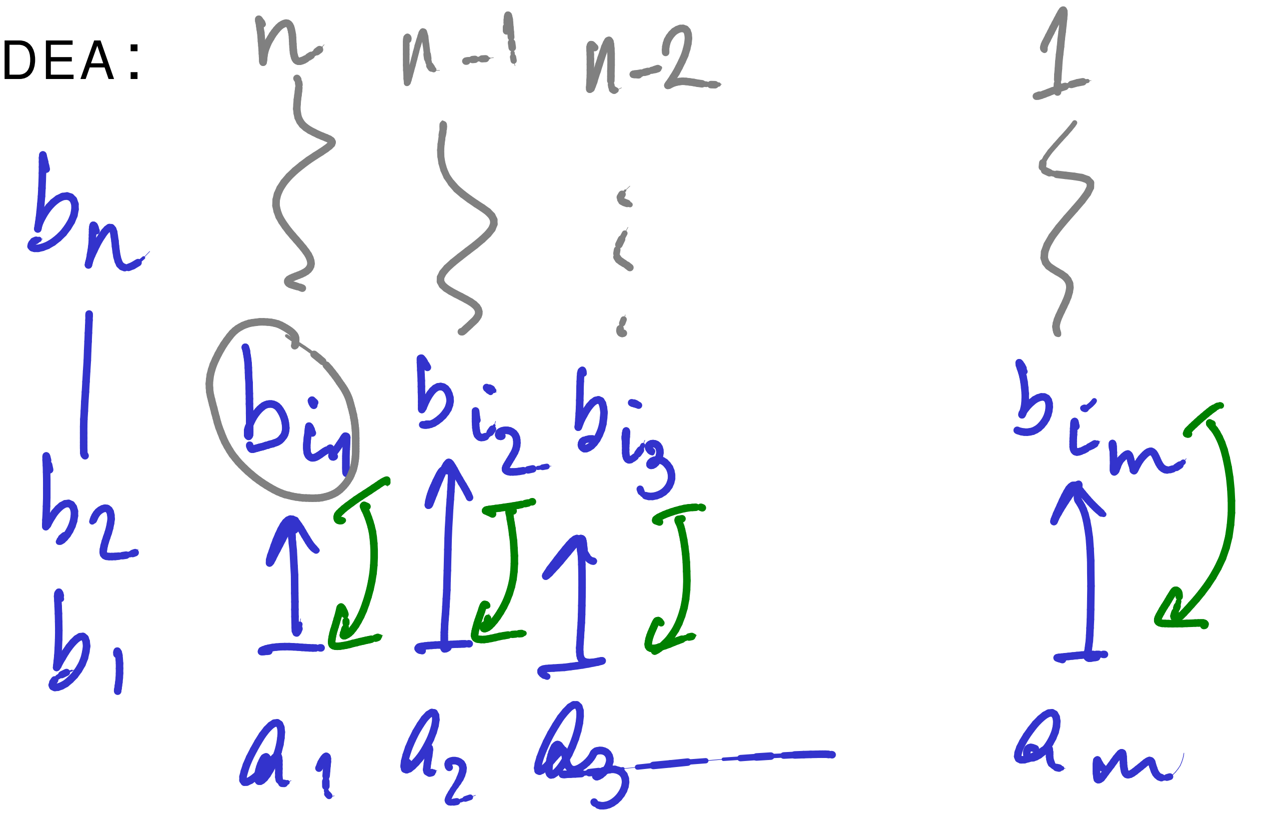
$a \mapsto b$

$a' \mapsto b'$

if $b = b'$

then $a = a'$

PROOF IDEA:

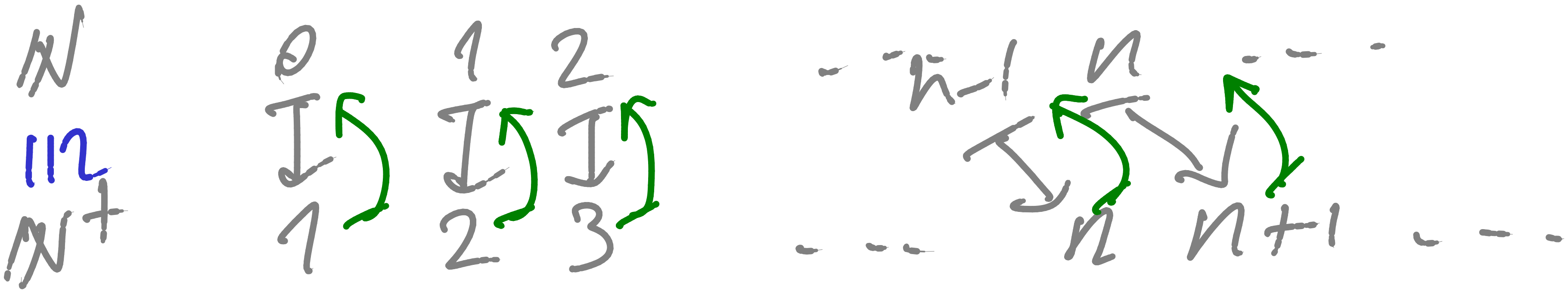


Theorem 127 *The identity function is a bijection, and the composition of bijections yields a bijection.*

If f is a bijection from A to B, by def, there is a g from B to A s.t. $g \circ f = id$ and $f \circ g = id$. In fact, such a g is unique! And we typically call it the inverse of f and denoted f^{-1}.

$f: A \to B$, $g: B \to C$ $\quad$ $g \circ f: A \to C$

given $f^{-1}: B \to A$, $g^{-1}: C \to B$ $\quad$ define $(g \circ f)^{-1}: C \to A$
$$= f^{-1} \circ g^{-1}$$



Definition 128 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

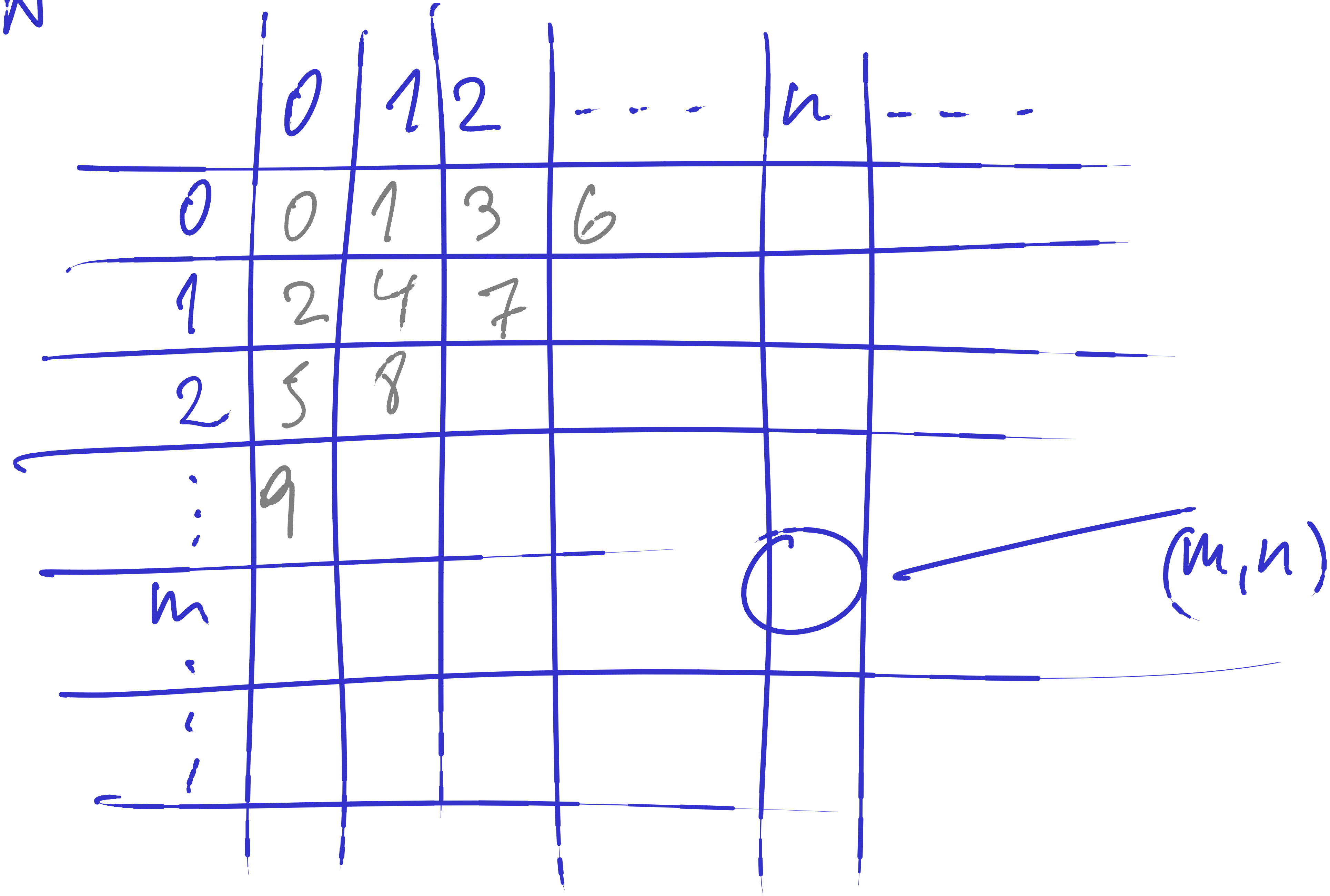
2. $\mathbb{N} \cong \mathbb{N}^+$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

$\mathbb{N} \not\cong \mathbb{R}$

$$
\begin{array}{lcccccccc}
\mathbb{Z} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots n \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{N} & \cdots & 4 & 2 & 0 & 1 & 3 & \cdots
\end{array}
$$

$\big\}$ a bijection.

$$\mathbb{N} \times \mathbb{N} \downarrow \mathbb{N}$$



Equivalence relations and set partitions

- Equivalence relations.

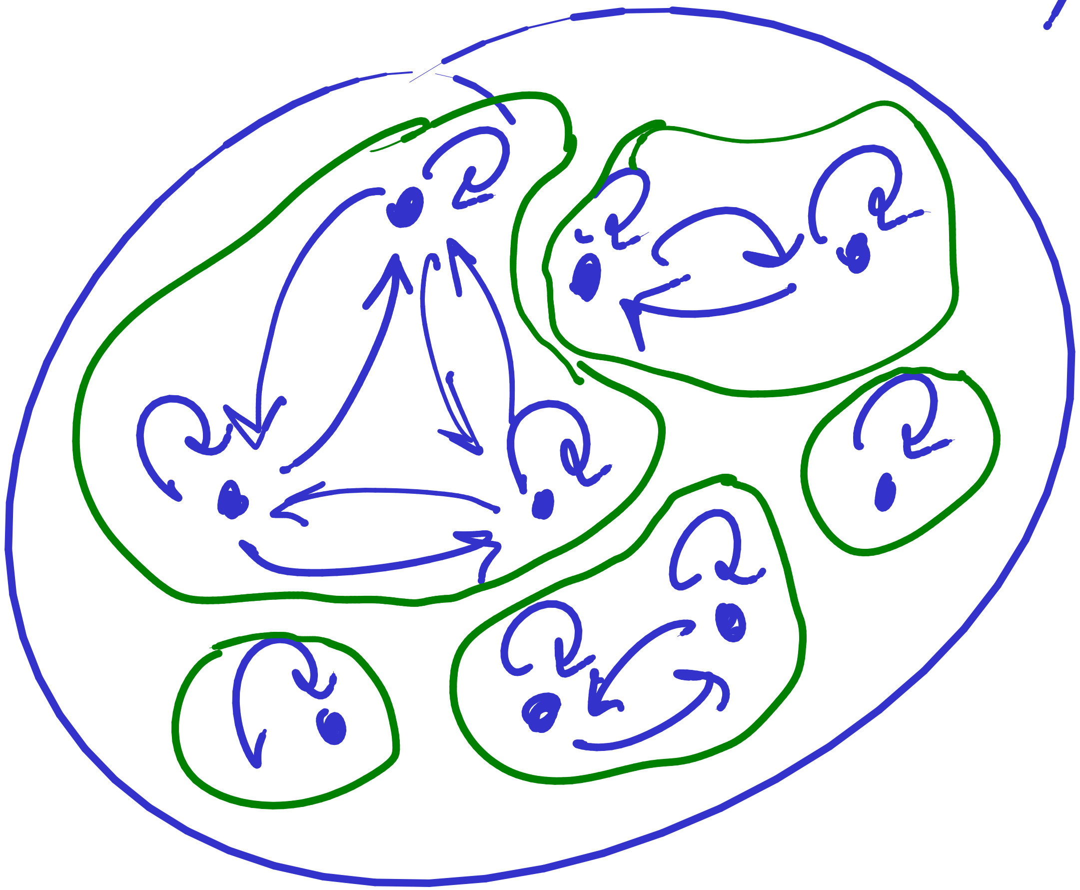
$E \subseteq A \times A.$

s.t. $\forall a \in A.\ a E a$ reflexivity

$\forall a, a', a'' \in A.\ a E a' \wedge a' E a'' \Rightarrow a E a''$ transitivity

$\forall a, a' \in A.\ a E a' \Rightarrow a' E a$ symmetric.

E A



- Set partitions. π

A partition on a set A is a set of subsets of A such that

(1) every $a \in A$ is in one of the subsets

$$\bigcup \pi = A$$

(2) Any two different subsets in π don't overlap. $\forall S, T \in \pi$

$$S \neq T \Rightarrow S \cap T = \emptyset.$$

Theorem 131 *For every set* A, $[a]_E = \{ x \in A \mid a E x \}$

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) \quad .$$

PROOF:

① $\underline{\mathrm{Part}}(A) \longrightarrow \underline{\mathrm{EqRel}}(A)$

$\pi \longmapsto E_\pi \subseteq A \times A$

$a \, E_\pi \, a'$ iff $\exists S \in \pi.\ a \in S \wedge a' \in S$

② $\mathrm{EqRel}(A) \longrightarrow \underline{\mathrm{Part}}(A)$

$E \longmapsto A/_E = \{ [a]_E \mid a \in A \}$