Big unions

Definition 86 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, we let the* big union *(relative to $\mathcal{U}$) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$

Big intersections

Definition 88 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, we let the* big intersection *(relative to $\mathcal{U}$) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}.\, x \in A \} \quad .$$

$\mathcal{U} = \mathbb{R}$

closure property

Theorem 89 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$*. Hence,* $\bigcap \mathcal{F} = \mathbb{N}$*.*

characterises $\mathbb{N}$.

PROOF:

$\mathbb{R} \in \mathcal{F}$

$\mathbb{Q} \in \mathcal{F}$

$\mathbb{N} \in \mathcal{F}$

- $\bigcap \mathcal{F} \subseteq \mathbb{N}$ $\quad x \in \bigcap \mathcal{F}$ $\implies \forall S \in \mathcal{F}.\, x \in S$

- $\mathbb{N} \subseteq \bigcap \mathcal{F}$ $\quad$ But $\mathbb{N} \in \mathcal{F}$, hence $x \in \mathbb{N}$

$P(n)$

$\forall n \in \mathbb{N}.\, n \in \bigcap \mathcal{F} \iff \forall n \in \mathbb{N}.\, \boxed{\forall S \in \mathcal{F}.\, n \in S}$

Exercise

By induction on $n \in \mathbb{N}$.

Examples $\bigcup\{\} = \emptyset$

$\bigcup\{X\} = X$

$\bigcup\{X, Y\} = X \cup Y$

$x \in X \cup Y$ iff $x \in X$

$\vee\ x \in Y$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F} . x \in X \right) \quad .$$

Analogous to ~~datatype~~ $\alpha\ \beta$ disjoint union = one of α | two of β.

Disjoint unions

Definition 90 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

$A \uplus B = \{(1,a) \mid a \in A\} \cup \{(2,b) \mid b \in B\}$.

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1,a)\big) \vee \big(\exists b \in B.\, x = (2,b)\big)\,.$$

Remark: $\{*\} \times X = \{(*,x) \mid x \in X\}$

Proposition 92 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA: A and B are <u>disjoint</u>.

$$A = \{a_1, \ldots, a_m\}$$

$$\forall i, j.\ a_i \neq b_j$$

$$B = \{b_1 \ldots b_n\}$$

$$A \cup B = \{\overbrace{\overbrace{a_1, \ldots, a_m}^{m}, \overbrace{b_1, \ldots, b_n}^{n}}^{m+n}\}$$

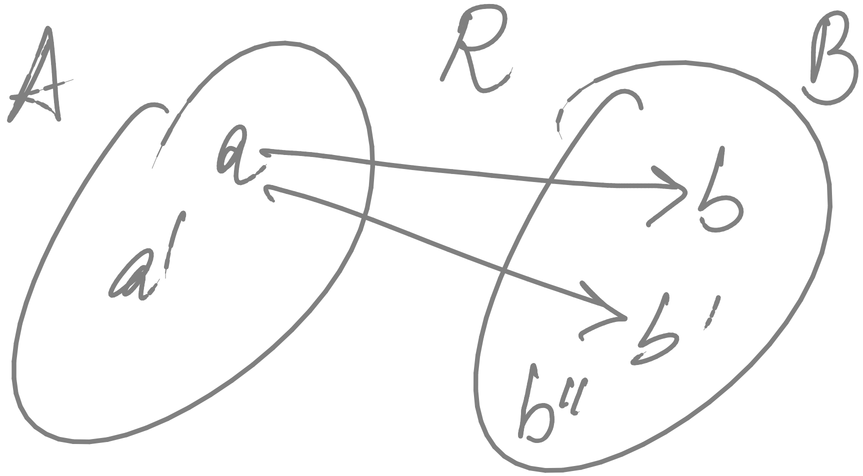
$\boxtimes$

Corollary 93 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$

$(a,b) \in R$ notation $a \, R \, b$

$(a,b') \in R$ $a \, R \, b'$



Relations

Definition 95 *A* <u>(binary) relation</u> R *from a set* A *to a set* B

$$R : A \mathrel{\rightarrow\!\!\!+\!\!\!\!\rightarrow} B \quad \textit{or} \quad R \in \mathrm{Rel}(A, B) \quad ,$$

is

|| def

$$R \subseteq A \times B \quad \textit{or} \quad R \in \mathcal{P}(A \times B) \quad .$$

Notation 96 *One typically writes* $a \, R \, b$ *for* $(a, b) \in R$*.*

Examples:

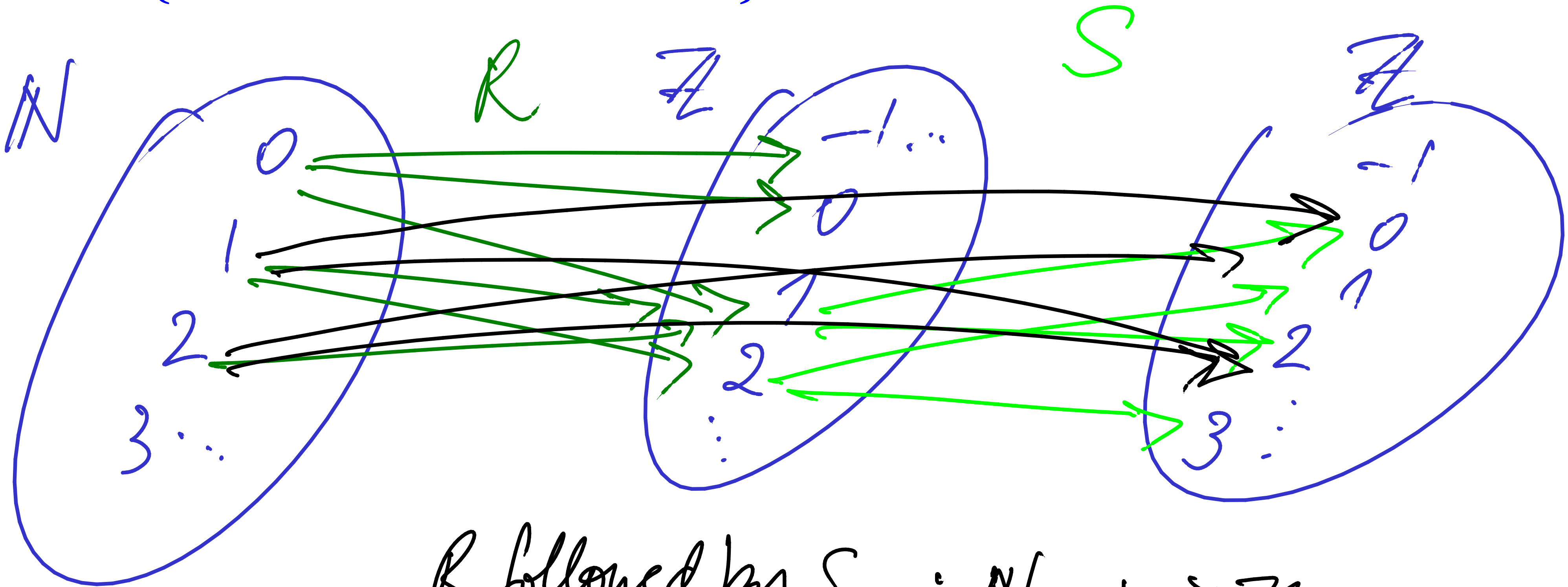
- Empty relation.
  $\emptyset : A \rightarrow\!\!\!\!\!\!+\!\!\!\rightarrow B$ $\qquad$ $(a \; \emptyset \; b \iff \textbf{false})$
- Full relation.
  $(A \times B) : A \rightarrow\!\!\!\!\!\!+\!\!\!\rightarrow B$ $\qquad$ $(a \; (A \times B) \; b \iff \textbf{true})$
- Identity (or equality) relation.
  $\mathrm{id}_A = \{ \, (a, a) \mid a \in A \, \} : A \rightarrow\!\!\!\!\!\!+\!\!\!\rightarrow A$ $\qquad$ $(a \; \mathrm{id}_A \; a' \iff a = a')$
- Integer square root.
  $R_2 = \{ \, (m, n) \mid m = n^2 \, \} : \mathbb{N} \rightarrow\!\!\!\!\!\!+\!\!\!\rightarrow \mathbb{Z}$ $\qquad$ $(m \; R_2 \; n \iff m = n^2)$

Internal diagrams

Example:

$$R = \{\ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1)\ \} : \mathbb{N} \leftrightarrow \mathbb{Z}$$

$$S = \{\ (1,0), (1,2), (2,1), (2,3)\ \} : \mathbb{Z} \leftrightarrow \mathbb{Z}$$



R followed by S $: \mathbb{N} \leftrightarrow \mathbb{Z}$

Relational composition

$R : A \mapsto B \qquad S : B \mapsto C$

$(S \circ R) : A \mapsto C$

Df $\quad a \in A,\ c \in C$

$a\,(S \circ R)\,c \overset{\text{df}}{\iff} \exists b \in B.\ aRb \wedge bSa.$

Theorem 98 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$, $S : B \relbar\joinrel\mapstochar\joinrel\rightarrow C$, *and* $T : C \relbar\joinrel\mapstochar\joinrel\rightarrow D$,

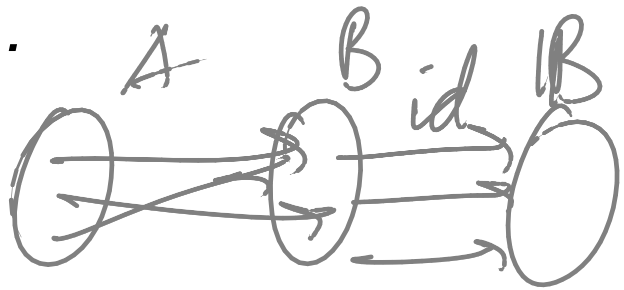
  $$(T \circ S) \circ R \;=\; T \circ (S \circ R)$$

  ⌊ justifies the notation $T \circ S \circ R$

- *Neutral element.*

  For all $R : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$,

  $$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$



Relational extensionality

$$R = S : A \longrightarrow\!\!\!\!\!\!\!+\ B$$

iff

$$\forall a \in A.\, \forall b \in B.\ a\,R\,b \iff a\,S\,b$$

Example $(\{\text{true}, \text{false}\}, 0 = \text{false}, 1 = \text{true}, \oplus = \vee, \odot = \wedge)$

Relations and matrices

NB rows and columns are enumerated from 0

Definition 99

1. *For positive integers* m *and* n, *an* $(m \times n)$-matrix M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.

$(m \times n)$-matrix M

$(n \times \ell)$-matrix N

$(m \times \ell)$-matrix $(N \bullet M)$

$M =$ i, j: $M_{i,j} \in S$

$$(N \bullet M)_{i,j} = \bigoplus_k (N_{k,j} \odot M_{i,k})$$

Theorem 100 *Matrix multiplication is associative and has the identity matrix as neutral element.*

Def $[n] = \{0, \ldots, n-1\}$ $\qquad \#[n] = n$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

$R : [m] \nrightarrow [n] \rightsquigarrow \underline{mat}(R)$

$(\underline{mat}(R))_{i,j} = true$

$\overset{df}{\iff} (i,j) \in R$

$\underline{rel}(M) \leftsquigarrow M$

$(i,j) \in rel(M)$

$\overset{df}{\iff} M_{i,j} = \underline{true}$

$$
\begin{tikzcd}
R \arrow[r, rightsquigarrow] \arrow[d, equal] & \mathrm{mat}(R) \arrow[dl, rightsquigarrow] \\
\underline{\mathrm{rel}}(\underline{\mathrm{mat}}(R)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & M \arrow[dl, rightsquigarrow] \arrow[d, equal] \\
\underline{\mathrm{rel}}(M) \arrow[r, rightsquigarrow] & \underline{\mathrm{mat}}(\underline{\mathrm{rel}}(M))
\end{tikzcd}
$$