Powerset axiom

$\boxed{\text{For any set, there is a set consisting of all its subsets.}}$

$$\mathcal{P}(U)$$

$$\forall X.\; X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

The powerset Boolean algebra

$$(\quad \mathcal{P}(U) \;,\quad \emptyset \;,\quad U \;,\quad \cup \;,\quad \cap \;,\quad (\cdot)^c \quad)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B \;=\; \{x \in U \mid x \in A \;\vee\; x \in B\} \;\in \mathcal{P}(U)$$

$$A \cap B \;=\; \{x \in U \mid x \in A \;\wedge\; x \in B\} \;\in \mathcal{P}(U)$$

$$A^c \;=\; \{x \in U \mid \neg(x \in A)\} \qquad \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A \ = \ A \ = \ \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathfrak{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathfrak{U} \cup A = \mathfrak{U}$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \;, \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) \;=\; A \;=\; A \cap (A \cup B)$$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathcal{U} \ , \quad A \cap A^{\mathrm{c}} = \emptyset$$

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$

Pairing axiom

> For every a and b, there is a set with a and b as its only elements.

$$\{ a , b \}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Ordered pairing

For every pair a and b, the set

$$\{\,\{\,a\,\}\,,\,\{\,a,b\,\}\,\}$$

is abbreviated as

$$\langle a, b\rangle$$

and referred to as an *ordered pair*.

Proposition 83 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff \big(a = x \wedge b = y\big) \quad .$$

PROOF:

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{ x \mid \exists\, a \in A, b \in B.\, x = (a, b) \}$$

where

$$\begin{aligned} &\forall a_1, a_2 \in A, b_1, b_2 \in B. \\ &\quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \end{aligned} \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

Proposition 85 *For all finite sets A and B,*

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA:

Big unions

Definition 86 *Let* U *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$*, we let the* big union *(relative to* U*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in U \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(U) \quad .$$