Euclid's infinitude of primes

Theorem 78 *The set of primes is infinite.*

PROOF:

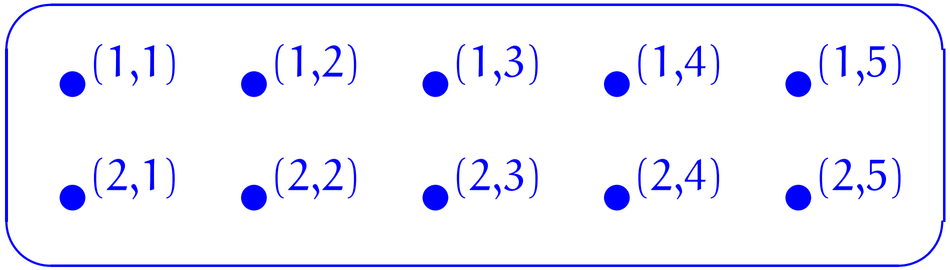
Sets

Objectives

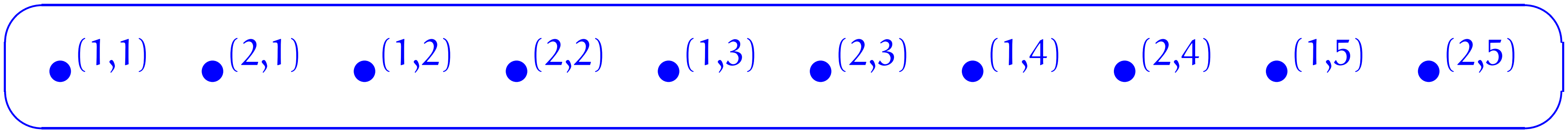
To introduce the basics of the theory of sets and some of its uses.

Abstract sets

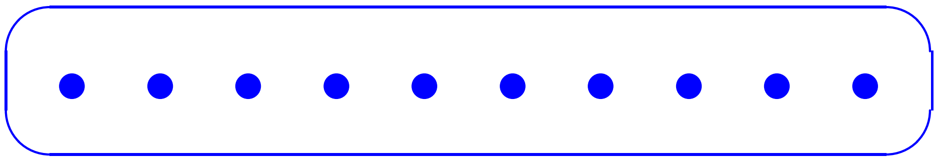
It has been said that a set is like a mental "bag of dots", except of course that the bag has no shape; thus,



may be a convenient way of picturing a certain set for some considerations, but what is apparently the same set may be pictured as



or even simply as



for other considerations.

Naive Set Theory

We are not going to be formally studying Set Theory here; rather, we will be *naively* looking at ubiquitous structures that are available within it.

Extensionality axiom

Two sets are equal if they have the same elements.

Thus,

$$\forall \text{ sets } A, B.\ A = B \iff (\ \forall x.\, x \in A \iff x \in B\ ) \quad .$$

Example:

$$\{0\} \neq \{0, 1\} = \{1, 0\} \neq \{2\} = \{2, 2\}$$

Subsets and supersets

Separation principle

For any set A and any definable property P, there is a set containing precisely those elements of A for which the property P holds.

$$\{ x \in A \mid P(x) \}$$

Russell's paradox

Empty set

$\emptyset$ or $\{\}$

defined by

$$\forall x.\, x \notin \emptyset$$

or, equivalently, by

$$\neg(\exists x.\, x \in \emptyset)$$

Cardinality

The *cardinality* of a set specifies its size. If this is a natural number, then the set is said to be *finite*.

Typical notations for the cardinality of a set S are $\#S$ or $|S|$.

Example:

$$\#\emptyset = 0$$

Powerset axiom

For any set, there is a set consisting of all its subsets.

$$\mathcal{P}(U)$$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

Hasse diagrams

Proposition 81 *For all finite sets* U*,*

$$\#\,\mathcal{P}(\mathsf{U}) = 2^{\#\mathsf{U}} \quad .$$

PROOF IDEA: