# Existential quantification

Existential statements are of the form

> **there exists** an individual $x$ in the universe of discourse for which the property $P(x)$ holds

or, in other words,

> **for some** individual $x$ in the universe of discourse, the property $P(x)$ holds

or, in symbols,

$$\exists x.\, P(x)$$

**Example:** The Pigeonhole Principle.

Let $n$ be a positive integer. If $n + 1$ letters are put in $n$ pigeonholes then there will be a pigeonhole with more than one letter.

**Theorem 21 (Intermediate value theorem)** *Let $f$ be a real-valued continuous function on an interval $[a, b]$. For every $y$ in between $f(a)$ and $f(b)$, there exists $v$ in between $a$ and $b$ such that $f(v) = y$.*

**Intuition:**

**The main proof strategy for existential statements:**

To prove a goal of the form

$$\exists x.\, P(x)$$

find a ***witness*** for the existential statement; that is, a value of $x$, say $w$, for which you think $P(x)$ will be true, and show that indeed $P(w)$, i.e. the predicate $P(x)$ instantiated with the value $w$, holds.

**Proof pattern:**

In order to prove

$$\exists x.\, P(x)$$

1. Write: $\mathrm{Let}\ w = \ldots$ (the witness you decided on).
2. Provide a proof of $P(w)$.

**Scratch work:**

Before using the strategy

| Assumptions | Goal |
| --- | --- |
| | $\exists x.\, P(x)$ |
| $\vdots$ | |

After using the strategy

| Assumptions | Goals |
| --- | --- |
| | $P(w)$ |
| $\vdots$ | |

$w = \ldots$ (the witness you decided on)

**Proposition 22** *For every positive integer* $k$*, there exist natural numbers* $i$ *and* $j$ *such that* $4 \cdot k = i^2 - j^2$*.*

PROOF:

**The use of existential statements:**

To use an assumption of the form $\exists x.\, P(x)$, introduce a new variable $x_0$ into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.

**Theorem 24** *For all integers* $l$, $m$, $n$, *if* $l \mid m$ *and* $m \mid n$ *then* $l \mid n$.

PROOF:

# Unique existence

The notation

$$\exists!\, x.\, P(x)$$

stands for

the *unique existence* of an $x$ for which the property $P(x)$ holds .

That is,

$$\exists x.\, P(x) \ \wedge \ \Big(\forall y.\, \forall z.\, \big(P(y) \wedge P(z)\big) \implies y = z\Big)$$

# Disjunction

Disjunctive statements are of the form

$$\boxed{P \text{ or } Q}$$

or, in other words,

$$\boxed{\text{either } P,\ Q, \text{ or both hold}}$$

or, in symbols,

$$\boxed{P \vee Q}$$

**The main proof strategy for disjunction:**

To prove a goal of the form

$$P \vee Q$$

you may

1. try to prove $P$ (if you succeed, then you are done); or
2. try to prove $Q$ (if you succeed, then you are done); otherwise
3. break your proof into cases; proving, in each case, either $P$ or $Q$.

**Proposition 25** *For all integers* $n$, *either* $n^2 \equiv 0 \pmod 4$ *or* $n^2 \equiv 1 \pmod 4$.

PROOF: