Full abstraction

A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.

- The domain model of PCF is *not* fully abstract.

  In other words, there are contextually equivalent PCF terms with different denotations.

Failure of full abstraction, idea

We will construct two closed terms

$$T_1, T_2 \in \mathrm{PCF}_{(bool \to (bool \to bool)) \to bool}$$

such that

$$T_1 \cong_{\mathrm{ctx}} T_2$$

and

$$[\![T_1]\!] \neq [\![T_2]\!]$$

- We achieve $T_1 \cong_{\mathrm{ctx}} T_2$ by making sure that

$$\forall\, M \in \mathrm{PCF}_{bool\to(bool\to bool)}\,(\, T_1\, M \not\Downarrow_{bool}\ \&\ T_2\, M \not\Downarrow_{bool}\,)$$

Hence,

$$[\![T_1]\!]([\![M]\!]) = \bot = [\![T_2]\!]([\![M]\!])$$

for all $M \in \mathrm{PCF}_{bool\to(bool\to bool)}$.

- We achieve $[\![T_1]\!] \neq [\![T_2]\!]$ by making sure that

$$[\![T_1]\!](por) \neq [\![T_2]\!](por)$$

for some *non-definable* continuous function

$$por \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot))\ .$$

Parallel-or function

is the unique continuous function $por : \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)$ such that

$$\begin{aligned} por\ true\ \perp &= true \\ por\ \perp\ true &= true \\ por\ false\ false &= false \end{aligned}$$

In which case, it necessarily follows by monotonicity that

$$\begin{aligned} por\ true\ true &= true \\ por\ true\ false &= true \\ por\ false\ true &= true \end{aligned} \qquad \begin{aligned} por\ false\ \perp &= \perp \\ por\ \perp\ false &= \perp \\ por\ \perp\ \perp &= \perp \end{aligned}$$

Undefinability of parallel-or

Proposition. *There is no closed PCF term*

$$P : bool \to (bool \to bool)$$

satisfying

$$[\![P]\!] = por : \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp) \ .$$

NB: One can build a domain-theoretic model of stable domains & functions, and note that *por* is not stable; hence not definable.

Parallel-or test functions

For $i = 1, 2$ define

$$T_i \stackrel{\text{def}}{=} \mathbf{fn}\, f : bool \to (bool \to bool)\,.\ \mathbf{if}\ (f\ \mathbf{true}\ \Omega)\ \mathbf{then}\ \mathbf{if}\ (f\ \Omega\ \mathbf{true})\ \mathbf{then}\ \mathbf{if}\ (f\ \mathbf{false}\ \mathbf{false})\ \mathbf{then}\ \Omega\ \mathbf{else}\ B_i\ \mathbf{else}\ \Omega\ \mathbf{else}\ \Omega$$

where $B_1 \stackrel{\text{def}}{=} \mathbf{true}$, $B_2 \stackrel{\text{def}}{=} \mathbf{false}$,

and $\Omega \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\, x : bool\,.\, x)$.

NB $\lceil \Omega \rceil$
$= \mathbf{fix}(\lambda x.x)$
$= \bot$

$\mathbf{fix}(\mathbf{fn}\, x.x) \Downarrow \mathbf{fn}\, x.x$

$x[\mathbf{fix}(\mathbf{fn}\, x.x)/x] \Downarrow V$

$(\mathbf{fn}\, x.x)\,(\mathbf{fix}(\mathbf{fn}\, x.x)) \Downarrow V$

$\mathbf{fix}(\mathbf{fn}\, x.x) \Downarrow V$

contradicts the minimality of the derivation

Claim: $\Omega \not\Downarrow$ a minimal derivation

Failure of full abstraction

Proposition.

$$T_1 \cong_{\mathrm{ctx}} T_2 : (bool \to (bool \to bool)) \to bool$$

$$[\![T_1]\!] \neq [\![T_2]\!] \in (\mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)) \to \mathbb{B}_\perp$$

PCF+por

Expressions $$M ::= \cdots \mid \mathbf{por}(M, M)$$

Typing $$\frac{\Gamma \vdash M_1 : bool \quad \Gamma \vdash M_2 : bool}{\Gamma \vdash \mathbf{por}(M_1, M_2) : bool}$$

Evaluation

$$\frac{M_1 \Downarrow_{bool} \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{true}} \qquad \frac{M_2 \Downarrow_{bool} \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{true}}$$

$$\frac{M_1 \Downarrow_{bool} \mathbf{false} \quad M_2 \Downarrow_{bool} \mathbf{false}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{false}}$$

Plotkin's full abstraction result

The denotational semantics of PCF+por is given by extending that of PCF with the clause

$$[\![\Gamma \vdash \mathbf{por}(M_1, M_2)]\!](\rho) \stackrel{\text{def}}{=} por\big([\![\Gamma \vdash M_1]\!](\rho)\big)\big([\![\Gamma \vdash M_2]\!](\rho)\big)$$

This denotational semantics is fully abstract for contextual equivalence of PCF+por terms:

$$\Gamma \vdash M_1 \cong_{\text{ctx}} M_2 : \tau \;\Leftrightarrow\; [\![\Gamma \vdash M_1]\!] = [\![\Gamma \vdash M_2]\!].$$