Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$fix(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\ f(d) \sqsubseteq d \implies fix(f) \sqsubseteq d$$

$x \sqsubseteq d \Rightarrow f x \sqsubseteq f d \sqsubseteq d \Rightarrow f x \sqsubseteq d.$

$$\frac{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}{x \in \downarrow(d) \Rightarrow f(x) \in \downarrow(d)}$$

$$\frac{fix(f) \in \downarrow(d)}{fix(f) \sqsubseteq d}$$

admissible

$S = \downarrow(d) = \{ x \mid x \sqsubseteq d \}$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.
- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.
- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

$\mathbb{L} \to \mathbb{Z}$ where $\mathbb{L} = \{X, Y\}$.

Example (III): Partial correctness

Let $\mathcal{F} : \mathit{State} \rightharpoonup \mathit{State}$ be the denotation of

$$\mathbf{while}\ X > 0\ \mathbf{do}\ (Y := X * Y; X := X - 1)\ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$

$$\Longrightarrow\ \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$$

partial correctness.

recall $\mathcal{F} = fix(f)$,

$w \in S \Rightarrow f(w) \in S$

$\forall x y.\ w(x,y)\downarrow \Rightarrow w(x,y) = (0, !x \cdot y)$ ①

$\Rightarrow \forall x y.\ fw(x,y)\downarrow \Rightarrow fw(x,y) = (0, !x \cdot y)$ ②

Recall that

$$\mathcal{F} = fix(f)$$

where $f : (State \rightharpoonup State) \to (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x,y) \in State. \begin{cases} (x,y) & \text{if } x \leq 0 \\ w(x-1, x \cdot y) & \text{if } x > 0 \end{cases}$$

value of x ; value of y

Assume ① & ②. Case $x = 0$ ...

Case $x > 0$: $fw(x,y) = w(x-1, x \cdot y)$ // by ①

$(0, !x \cdot y) = (0, !(x-1) \cdot (x \cdot y))$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \ .$$

$$\overline{\mathit{fix}(f) \in S}$$

Topic 5

PCF

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

$[\![nat]\!] = \mathbb{N}_\perp$, $[\![bool]\!] = \mathbb{B}_\perp$, $[\![\tau \to \sigma]\!] = ([\![\tau]\!] \to [\![\sigma]\!])$.

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$M \quad ::= \quad \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M)$$

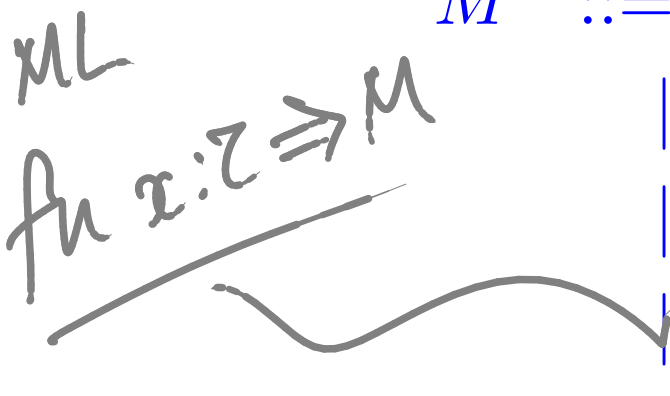
PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau\, .\, M \mid M\, M \mid \mathbf{fix}(M) \end{array}$$



where $x \in \mathbb{V}$, an infinite set of variables.

application

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

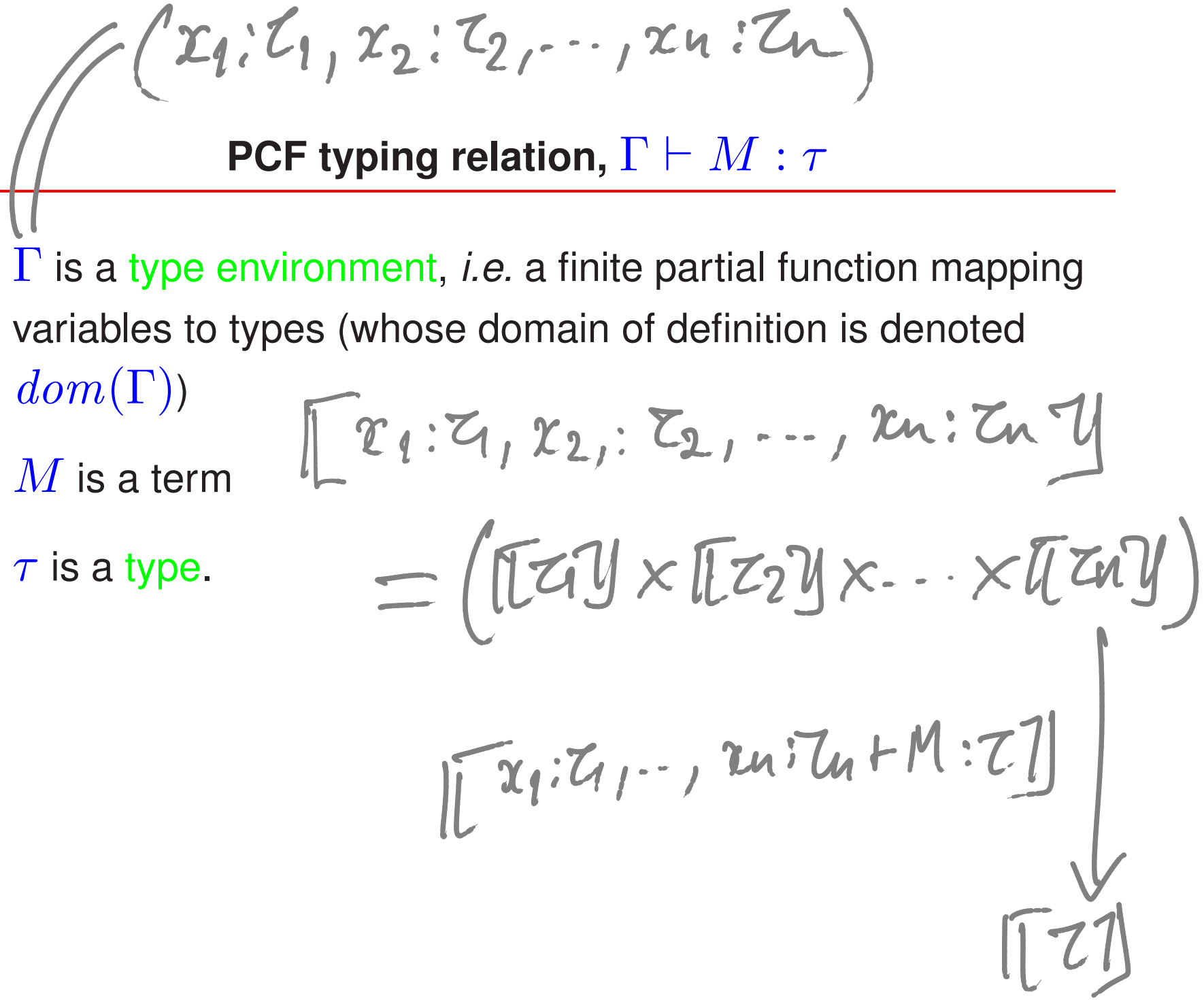
$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau\, .\, M \mid M\, M \mid \mathbf{fix}(M) \end{array}$$

where $x \in \mathbb{V}$, an infinite set of variables.

Technicality: We identify expressions up to α-conversion of bound variables (created by the $\mathbf{fn}$ expression-former): by definition a PCF term is an α-equivalence class of expressions.

PCF typing relation, $\Gamma \vdash M : \tau$

- Γ is a type environment, *i.e.* a finite partial function mapping variables to types (whose domain of definition is denoted $dom(\Gamma)$)
- M is a term
- τ is a type.



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Notation:

$M : \tau$ means M is closed and $\emptyset \vdash M : \tau$ holds.

$\mathrm{PCF}_\tau \stackrel{\mathrm{def}}{=} \{M \mid M : \tau\}$.

PCF typing relation (sample rules)

$$(:_{\text{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

$$(:_{\text{app}}) \quad \frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'}$$

$$(:_{\text{fix}}) \quad \frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathbf{fix}(M) : \tau}$$

Partial recursive functions in PCF

- Primitive recursion.

$$\begin{cases} h(x,0) = f(x) \\ h(x,y+1) = g(x,y,h(x,y)) \end{cases}$$

$$\underline{\text{fix}}\Big(\underline{\text{fn}}\ h \cdot \underline{\text{fn}}\ z_1 \cdot \underline{\text{fn}}\ z_2 \cdot \underline{\text{if}}\ \underline{\text{zero}}\ (z_2)\ \underline{\text{Then}}\ f\ (z_1)\ \underline{\text{else}}\ g\ z_1\ (\underline{\text{pred}}\ z_2)\ (h\ z_1\ (\underline{\text{pred}}\ z_2))\Big)$$

In PCF we can write programs of type $(nat \to bool) \to bool$ that can be thought of as taking input boolean streams.

Partial recursive functions in PCF

- Primitive recursion.

$$\begin{cases} h(x,0) = f(x) \\ h(x,y+1) = g(x,y,h(x,y)) \end{cases}$$

- Minimisation.

$$m(x) = \text{ the least } y \geq 0 \text{ such that } k(x,y) = 0$$

$m(x) = t(x,0)$

$t(x,y) =$ if $k(x,y) = 0$ then y else $t(x,y+1)$

PCF evaluation relation

takes the form

$$M \Downarrow_\tau V$$

where

- τ is a PCF type
- $M, V \in \mathrm{PCF}_\tau$ are closed PCF terms of type τ
- V is a value,

$$V ::= \mathbf{0} \mid \mathbf{succ}(V) \mid \mathbf{true} \mid \mathbf{false} \mid \mathbf{fn}\, x : \tau \,.\, M.$$

PCF evaluation (sample rules)

$$(\Downarrow_{\mathrm{val}}) \quad V \Downarrow_\tau V \qquad (V \text{ a value of type } \tau)$$

$$(\Downarrow_{\mathrm{cbn}}) \quad \frac{M_1 \Downarrow_{\tau\to\tau'} \mathbf{fn}\, x : \tau \,.\, M_1' \qquad M_1'[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$$

call by name

$$(\Downarrow_{\mathrm{fix}}) \quad \frac{M(\mathbf{fix}(M)) \Downarrow_\tau V}{\mathbf{fix}(M) \Downarrow_\tau V}$$