

Topic 3

Constructions on Domains

Example

$\mathbb{N}_\perp, \mathbb{B}_\perp$

$\mathbb{B} = \{\text{true}, \text{false}\}$.

Discrete cpo's and flat domains

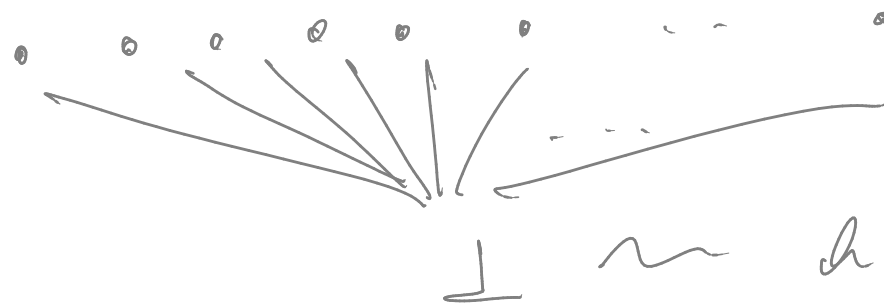
For any set X , the relation of equality

$$x \sqsubseteq x' \stackrel{\text{def}}{\iff} x = x' \quad (x, x' \in X)$$

makes (X, \sqsubseteq) into a cpo, called the **discrete** cpo with underlying set X .

Every set can be made into a domain.

X



\perp a new least element

Discrete cpo's and flat domains

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makes (X, \sqsubseteq) into a cpo, called the **discrete** cpo with underlying set X .

Let $X_{\perp} \stackrel{\text{def}}{=} X \cup \{\perp\}$, where \perp is some element not in X . Then

$$d \sqsubseteq d' \stackrel{\text{def}}{\iff} (d = d') \vee (d = \perp) \quad (d, d' \in X_{\perp})$$

makes (X_{\perp}, \sqsubseteq) into a domain (with least element \perp), called the **flat** domain determined by X .

Binary product of cpo's and domains

The **product** of two cpo's (D_1, \sqsubseteq_1) and (D_2, \sqsubseteq_2) has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order \sqsubseteq defined by

$$(d_1, d_2) \sqsubseteq (d'_1, d'_2) \stackrel{\text{def}}{\iff} d_1 \sqsubseteq_1 d'_1 \ \& \ d_2 \sqsubseteq_2 d'_2.$$

check this is a partial order

Claim If D_1 and D_2 are domains then the construction $D_1 \times D_2$ is also a domain.

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \quad x_2 \sqsubseteq_2 y_2}$$

Model the ML type constructor *

Given a chain in $D_1 \times D_2$ we need show it has a lub.

$$(x_0, y_0) \subseteq (x_1, y_1) \subseteq \dots \subseteq (x_n, y_n) \subseteq \dots$$

Need to define $\bigcup_n (x_n, y_n) = (x_\infty, y_\infty)$



$$x_0 \subseteq x_1 \subseteq \dots \subseteq x_n \subseteq \dots \quad \bigcup_n x_n \in D_1$$

$$y_0 \subseteq y_1 \subseteq \dots \subseteq y_n \subseteq \dots \quad \bigcup_n y_n \in D_2$$

Define
 $x_\infty = \bigcup_n x_n$
 $y_\infty = \bigcup_n y_n$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = \left(\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j} \right) .$$

If (D_1, \sqsubseteq_1) and (D_2, \sqsubseteq_2) are domains so is $(D_1 \times D_2, \sqsubseteq)$
and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \rightarrow F$ is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E. d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$

$$\forall d \in D, e, e' \in E. e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f\left(\bigsqcup_{m \geq 0} d_m, e\right) = \bigsqcup_{m \geq 0} f(d_m, e)$$

$$f\left(d, \bigsqcup_{n \geq 0} e_n\right) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$\forall f : (D \times E) \rightarrow F$

is monotone iff def

$\forall (d, e), (d', e') \in D \times E.$

$$\underbrace{(d, e) \sqsubseteq (d', e')} \Rightarrow f(d, e) \sqsubseteq f(d', e')$$

$$\begin{array}{l} \swarrow \text{def} \\ d \sqsubseteq d' \quad \text{in } D \end{array}$$

$$e \sqsubseteq e' \quad \text{in } E$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \quad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ cont})$$

$$\bigsqcup_m f(x_m, \bigsqcup_n y_n) = \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

Function cpo's and domains

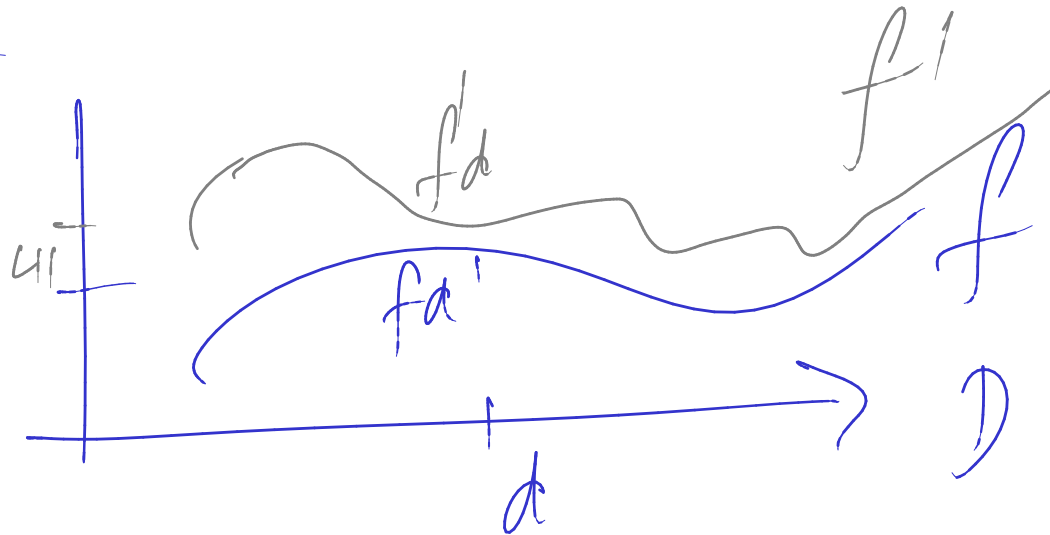
Given cpo's (D, \sqsubseteq_D) and (E, \sqsubseteq_E) , the **function cpo** $(D \rightarrow E, \sqsubseteq)$ has underlying set

$$(D \rightarrow E) \stackrel{\text{def}}{=} \{f \mid f : D \rightarrow E \text{ is a continuous function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\iff} \forall d \in D. f(d) \sqsubseteq_E f'(d)$.

Model the ML type constructor \rightarrow

Claim: If D and E are domains then so is $(D \rightarrow E)$.



Check $(D \rightarrow E)$ is a domain whenever D and E are.

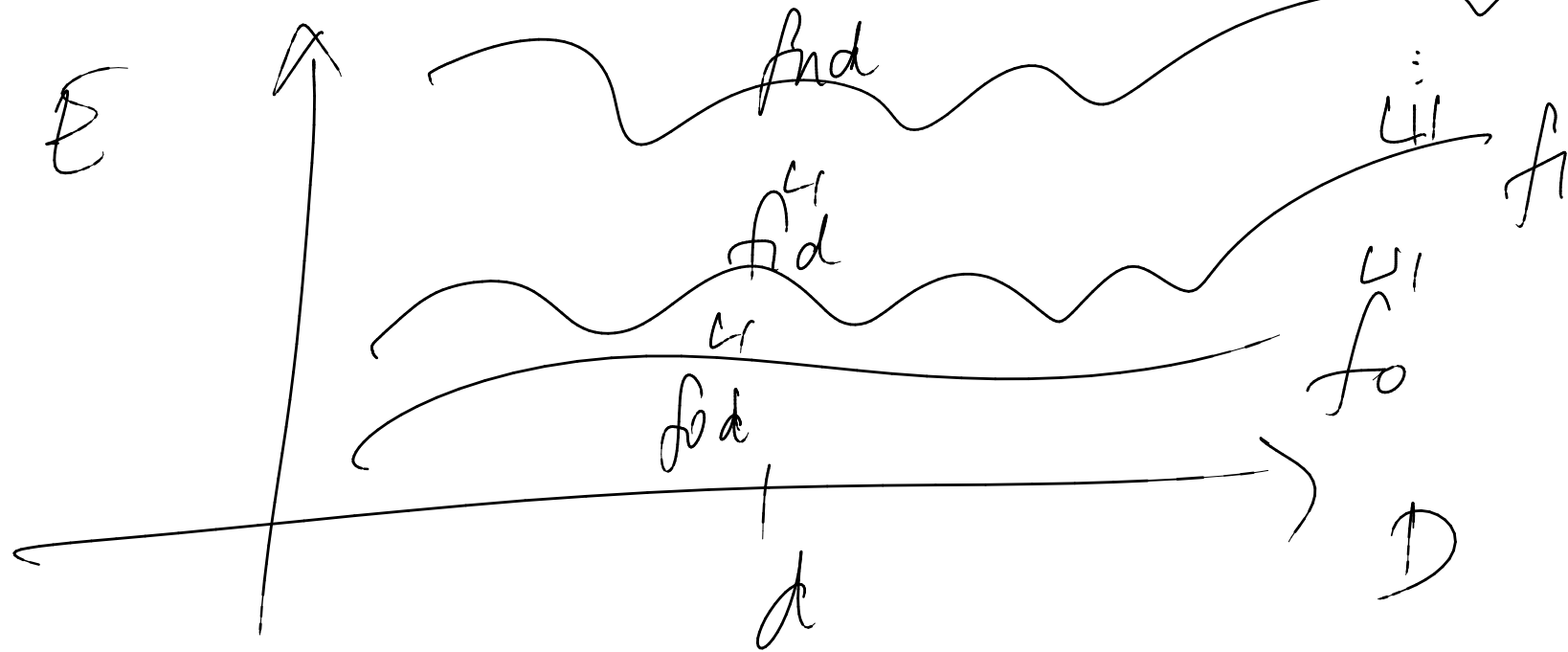
⑥ Want a function $\perp_{(D \rightarrow E)}$ s.t.

$(D \rightarrow E) \ni \perp \sqsubseteq f$ for all cont $f: D \rightarrow E$

$$\forall d. \perp(d) \sqsubseteq f(d)$$

Can be guaranteed by setting $\perp_{(D \rightarrow E)} = \lambda d. \perp E$

Check $(D \rightarrow \mathbb{R})$ has lub of $\{f_n\}$ chains $f_\infty(d) = \bigcup_n f_n(d)$ $f_\infty = \bigcup_n f_n$



Def: $f_\infty = \text{def } \lambda d \in D. \bigcup_n (f_n(d))$

f_∞ is continuous, and it is a lub of $(f_n)_n$

Want to show f_∞ is continuous.

$$f_\infty(\cup_i d_i) \stackrel{?}{=} \cup_i (f_\infty(d_i))$$

// by def

// by def

$$\cup_n (f_n(\cup_i d_i))$$

$$\cup_i (\cup_n (f_n(d_i)))$$

by
cont

$$\cup_n \cup_i (f_n(d_i))$$

$$\cup_k f_k(d_k)$$



Function cpo's and domains

Given cpo's (D, \sqsubseteq_D) and (E, \sqsubseteq_E) , the **function cpo** $(D \rightarrow E, \sqsubseteq)$ has underlying set

$$(D \rightarrow E) \stackrel{\text{def}}{=} \{f \mid f : D \rightarrow E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\iff} \forall d \in D . f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \rightarrow E)} g \quad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d) .$$

- A derived rule:

$$\left(\bigsqcup_n f_n \right) \left(\bigsqcup_m x_m \right) = \bigsqcup_k f_k(x_k)$$

If E is a domain, then so is $D \rightarrow E$ and $\perp_{D \rightarrow E}(d) = \perp_E$, all $d \in D$.

Continuity of composition

For cpo's D, E, F , the composition function

$$\circ : ((E \rightarrow F) \times (D \rightarrow E)) \longrightarrow (D \rightarrow F)$$

defined by setting, for all $f \in (D \rightarrow E)$ and $g \in (E \rightarrow F)$,

$$g \circ f = \lambda d \in D. g(f(d))$$

is continuous.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \rightarrow D)$ possesses a least fixed point, $fix(f) \in D$.

Proposition. *The function*

$$fix : (D \rightarrow D) \rightarrow D$$

$$f \longmapsto \underline{fix}(f)$$

least pre-fixed
point of f
equiv.

$$\bigsqcup_n f^n(\perp)$$

is continuous.

Show: $\underline{fix}(\bigsqcup_n f_n) \stackrel{?}{=} \bigsqcup_n \underline{fix}(f_n)$

Topic 4

Scott Induction

$\bigcup_n P_n \perp = \text{fix}(f)$ → Idea: Body of a recursive definition

Scott's Fixed Point Induction Principle

Let $f : D \rightarrow D$ be a continuous function on a domain D .

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S , i.e. that

$$\text{fix}(f) \in S,$$

it suffices to prove

$$\forall d \in D (d \in S \Rightarrow f(d) \in S).$$

The recursive definition

A property we are interested in

$$\frac{\forall d. d \in S \Rightarrow f(d) \in S}{\text{fix}(f) \in S}$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called **chain-closed** iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$ in D

$$(\forall n \geq 0 . d_n \in S) \Rightarrow \left(\bigsqcup_{n \geq 0} d_n \right) \in S$$

If D is a domain, $S \subseteq D$ is called **admissible** iff it is a chain-closed subset of D and $\perp \in S$.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

*chain closed by
definition of
sub.*

$$\downarrow(d) \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\}$$

of D is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of $D \times D$ are chain-closed.

