Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \ \& \ n \leq n' \ \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \ \sqsubseteq \ \bigsqcup_{n\geq 0} d_{1,n} \ \sqsubseteq \ \bigsqcup_{n\geq 0} d_{2,n} \ \sqsubseteq \ \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \ \sqsubseteq \ \bigsqcup_{m\geq 0} d_{m,1} \ \sqsubseteq \ \bigsqcup_{m\geq 0} d_{m,3} \ \sqsubseteq \ \ldots$$

$$\begin{tikzcd}
\bigsqcup_m d_{m0} \arrow[r, phantom, "\sqsubseteq"] & \bigsqcup_m d_{m1} \arrow[r, phantom, "\sqsubseteq"] & \cdots \arrow[rr, phantom, "\text{claim}"] & & \bigsqcup_n \bigsqcup_m d_{mn} & = \bigsqcup_m \bigsqcup_n d_{mn} \\
\vdots \arrow[u, phantom, sloped, "\sqsupseteq"] & \vdots \arrow[u, phantom, sloped, "\sqsupseteq"] & & \bigsqcup_k d_{kk} \arrow[ur, phantom, sloped, "="] \arrow[urr, phantom, sloped, "= \ \text{claim}"] & & \vdots \arrow[u, phantom, sloped, "\sqsupseteq"] \\
d_{m0} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & d_{m1} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & \cdots \arrow[ur, phantom, sloped, "\cdots"] & \sqsubseteq d_{mn} \arrow[r, phantom, "\sqsubseteq"] & \cdots & \sqsubseteq \bigsqcup_n d_{mn} \arrow[u, phantom, sloped, "\sqsupseteq"] \\
\vdots & \vdots \arrow[ur, phantom, sloped, "\sqsubseteq"] & & \vdots & & \vdots \\
d_{10} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & d_{11} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] \arrow[ur, phantom, sloped, "\sqsubseteq"] & \cdots & \sqsubseteq d_{1n} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & \cdots & \sqsubseteq \bigsqcup_n d_{1n} \arrow[u, phantom, sloped, "\sqsupseteq"] \\
d_{00} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] \arrow[ur, phantom, sloped, "\sqsubseteq"] & d_{01} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & \cdots & \sqsubseteq d_{0n} \arrow[u, phantom, sloped, "\sqsupseteq"] \arrow[r, phantom, "\sqsubseteq"] & \cdots & \sqsubseteq \bigsqcup_n d_{0n} \arrow[u, phantom, sloped, "\sqsupseteq"]
\end{tikzcd}$$

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Moreover

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) \quad .$$

fn $l = \max(m,n)$

$$\begin{array}{c}
\forall m \forall n \; d_{mn} \sqsubseteq d_{l,l} \sqsubseteq \bigsqcup_k d_{kk} \\ \hline
\forall m \forall n \; d_{mn} \sqsubseteq \bigsqcup_k d_{kk} \\ \hline
\forall m . \; \bigsqcup_n d_{mn} \sqsubseteq \bigsqcup_k d_{kk} \\ \hline
\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}
\end{array}$$

$$\begin{array}{c}
\bigsqcup_m \bigsqcup_n d_{mn} \\
\sqsupseteq \\
\forall m \; \bigsqcup_n d_{m,n} \\
\sqsupseteq \\
\forall n \; d_{m,n}
\end{array}$$

$$\begin{array}{c}
\forall i \; x_i \sqsubseteq y \\ \hline
\bigsqcup_i x_i \sqsubseteq y \\ \hline
x_j \sqsubseteq \bigsqcup_i x_i
\end{array}$$

use

$$\begin{array}{c}
\hline
\forall k \; d_{kk} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{mn} \\ \hline
\bigsqcup_k d_{kk} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{mn}
\end{array}$$

$$\begin{array}{c}
\hline
\bigsqcup_m \bigsqcup_n d_{m,n} = \bigsqcup_k d_{kk}
\end{array}$$

Eg. $\mathbb{N}_\perp =$

$(\mathbb{N}_\perp \to \mathbb{N}_\perp)$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$\bigsqcup_n d_n \quad f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n)$$

in E.

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

$$f(\bigsqcup_n d_n) \sqsupseteq \bigsqcup_n f(d_n)$$

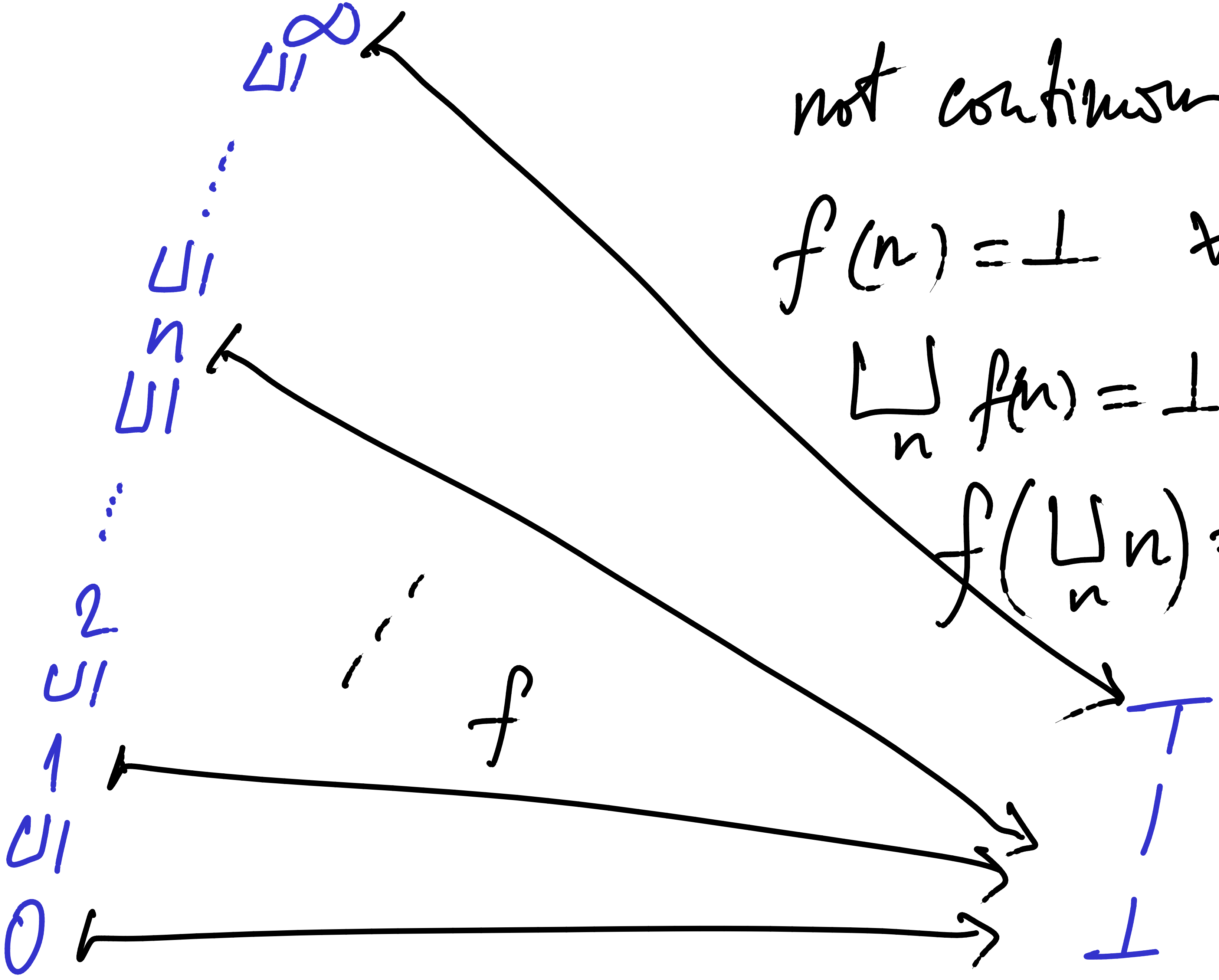
$$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots$$

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq \cdots$$

CONTINUITY:

$$f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$$

$$D \xrightarrow{f} E$$



not continuous.

$$f(n) = \bot \quad \forall n \neq \infty$$

$$\bigsqcup_n f(n) = \bot \neq$$

$$f\left(\bigsqcup_n n\right) = f(\infty) = \top$$

$\mathbb{N}_\perp \longrightarrow \mathbb{N}_\perp$ $\lambda n.0$ ~ constantly 0 function

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and

  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

  $$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

non-termination

$$D \xrightarrow{f} D \qquad \frac{\bot \sqsubseteq f(\bot)}{\bot \sqsubseteq f(\bot) \sqsubseteq ff(\bot)}$$

$\underline{fix}(f) \in D$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

$$\bigsqcup_n f^n \bot$$

$$\bot \sqsubseteq f\bot \sqsubseteq ff\bot \sqsubseteq \cdots f^n(\bot) \sqsubseteq \cdots$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f(\mathit{fix}(f)) = \mathit{fix}(f)$, and hence is the least fixed point of f.

CLAIM: is $\underline{fix}(f)$

(1) $f(\mathrm{fix}\, f) \overset{?}{\sqsubseteq} \mathrm{fix}\, f$

the chai

$\underbrace{\bot \sqsubseteq f\bot \sqsubseteq f^2\bot \sqsubseteq \cdots}$

I.e $f(\sqcup_n f^n\bot) \overset{?}{\sqsubseteq} \sqcup_n f^n\bot$

$$f(\sqcup_n f^n\bot) \underset{\text{by cont}}{=} \sqcup_n f(f^n\bot)$$

$$= \sqcup_n \underbrace{f^{n+1}(\bot)}_{\text{the Chain } f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots}$$

$= \sqcup_n f^n(\bot)$

(2) $\forall d.\ f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d$

Let $f(d) \sqsubseteq d$. <u>RTP</u> $\bigsqcup_n f^n(\bot) \sqsubseteq d$

(BC)$_{\underline{n=0}}$ $\dfrac{\checkmark}{\bot \sqsubseteq d}$

(IS) Assume $f^n(\bot) \sqsubseteq d$

Need show $f^{n+1}(\bot) \sqsubseteq d$

$f^{n+1}(\bot) = f(f^n \bot) \sqsubseteq f(d) \sqsubseteq d$

By induction

$$\forall n.\ f^n(\bot) \sqsubseteq d$$

$$\bigsqcup_n f^n(\bot) \sqsubseteq d$$

union of partial functions.

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$[\![\textbf{while } B \textbf{ do } C]\!]$$

continuous!

$$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$$

the empty partial function.

$$= \bigsqcup_{n\geq 0} f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot)$$

$$= \quad \lambda s \in \mathit{State}.\ \begin{cases} [\![C]\!]^{k}(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$

Intuition for fix

$fact\ n = \underline{if}\ n = 0\ \underline{then}\ 1$
$\qquad\qquad \underline{else}\ n \times fact(n-1)$

$F = \lambda f . \lambda n .\ \underline{if}\ n = 0\ \underline{then}\ 1$
$\qquad\qquad \underline{else}\ n \times f(n-1)$

$fix(F) = fact$

Datatypes

functions $(\alpha \to \beta)$

products $(\alpha * \beta)$

datatypes

bool, nat, ...

enumerated

inductive.

recursive datatype

$D = $ a fn of $D \to D$

Topic 3

Constructions on Domains