$\in$ State $\qquad\qquad$ $/\!/$ $\ldots\ldots [\![B]\!] \ldots\ldots [\![C]\!] \ldots\ldots$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!](s)$

$$
= \begin{cases}
s & \text{if } [\![B]\!](s) = \text{false} \\
[\![C]\!](s) & \text{if } [\![B]\!](s) = \text{true and } [\![B]\!]([\![C]\!]s) = \text{false} \\
\vdots & \vdots \\
[\![C]\!]^{n+1}(s) & \text{if } [\![B]\!]([\![C]\!]^n s) = \text{true} \ \& \ [\![B]\!]([\![C]\!]^{n+1} s) = \text{false} \\
\vdots & \vdots \\
\uparrow & \text{if } \forall n \in \mathbb{N}\ [\![B]\!]([\![C]\!]^n s) = \text{true}
\end{cases}
$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!]^S = [\![\text{if } B \text{ then } C; \text{while } B \text{ do } C \text{ else skip}]\!]^S$

$= \text{if}\,([\![B]\!]s, [\![\text{while } B \text{ do } C]\!]([\![C]\!]s), s)$

A fixed point of a function h is an element x

$h(x) = x$.

For good fixed points we write $fix(h)$ for such an element

Idea: $[\![\text{while } B \text{ do } C]\!]$ is a fixed point. I.e.

$\overset{def}{/\!/}$

$fix(F_{[\![B]\!],[\![C]\!]})$

$h(fix\, h) = fix\, h$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$F_{[\![B]\!],[\![C]\!]} = \lambda f . \lambda s .$$

$$\underline{if}\,([\![B]\!]s,\ f([\![C]\!]s),\ s)\ .$$

$$[\![\text{while } B \text{ do } C]\!] = \underline{fix}\,(F_{[\![B]\!],[\![C]\!]})\ .$$

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$ and $c : \mathit{State} \rightharpoonup \mathit{State}$, we define

$$f_{b,c} : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$$

as

$$f_{b,c} = \lambda w \in (\mathit{State} \rightharpoonup \mathit{State}).\, \lambda s \in \mathit{State}.\, \mathit{if}\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$?

Approximating $[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!]_0 = \bot \sim$ empty partial function.

$[\![\text{while } B \text{ do } C]\!]_1 = f_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_0)$

$= \lambda s.\, \mathrm{if}([\![B]\!]s, \bot([\![C]\!]s), s)$

$= \lambda s.\, \mathrm{if}([\![B]\!]s, \uparrow, s)$

$[\![\text{while } B \text{ do } C]\!]_{n+1} = f_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_n)$

Approximating $[\![\textbf{while } B \textbf{ do } C]\!]$

$$[\![\text{while } B \text{ do } C]\!]_0$$

$$\sqsubseteq \quad [\![\text{while } B \text{ do } C]\!]_1 = f_{[\![B]\!],[\![C]\!]}\left([\![\text{while } B \text{ do } C]\!]_0\right)$$

$$\sqsubseteq \quad [\![\text{while } B \text{ do } C]\!]_2 = f^2_{[\![B]\!],[\![C]\!]}\left([\![\text{while } B \text{ do } C]\!]_0\right)$$

$$\sqsubseteq$$

$$\vdots$$

$$[\![\text{while } B \text{ do } C]\!] = \underline{\text{fix}}\left(f_{[\![B]\!],[\![C]\!]}\right)$$

the limit of the above

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$
\begin{aligned}
& {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot) \\
& \quad = \quad \lambda s \in \mathit{State}. \\
& \qquad \begin{cases}
[\![C]\!]^{k}(s) & \text{if } \exists\, 0 \leq k < n.\, [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\
& \text{and } \forall\, 0 \leq i < k.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \\
& \\
\uparrow & \text{if } \forall\, 0 \leq i < n.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true}
\end{cases}
\end{aligned}
$$

$$\dots\dots \llbracket B\rrbracket \dots \llbracket C\rrbracket \dots$$

State

$\llbracket$**while** B **do** $C\rrbracket$

$\in$

$\llbracket \text{while } B \text{ do } C\rrbracket(s)$

Claim: This is a fixed point for $F_{\llbracket B\rrbracket, \llbracket C\rrbracket}$.

$$= \begin{cases} s & \text{if } \llbracket B\rrbracket(s) = \text{false} \\ \llbracket C\rrbracket(s) & \text{if } \llbracket B\rrbracket(s) = \text{true and } \llbracket B\rrbracket(\llbracket C\rrbracket s) = \text{false} \\ \vdots & \vdots \\ \llbracket C\rrbracket^{n+1}(s) & \text{if } \llbracket B\rrbracket(\llbracket C\rrbracket^{n} s) = \text{true} \ \& \ \llbracket B\rrbracket(\llbracket C\rrbracket^{n+1} s) = \text{false} \\ \vdots & \vdots \\ \uparrow & \forall n \in \mathbb{N}\ \llbracket B\rrbracket(\llbracket C\rrbracket^{n}) = \text{true} \end{cases}$$

PARTIAL ORDERS

$(S, \preceq)$ $\quad \preceq \subseteq S \times S$

REFLEXIVE $\quad s \preceq s$

TRANSITIVE $\quad s_1 \preceq s_2 \;\&\; s_2 \preceq s_3$

$\Longrightarrow s_1 \preceq s_3$

ANTISYMMETRIC $\quad s \preceq s' \;\&\; s' \preceq s$

$\Longrightarrow s = s'$

For $f \in D$, $\operatorname{graph}(f) = \{ (x, fx) \mid fx \text{ is defined} \}$

$$D \stackrel{\text{def}}{=} (State \rightharpoonup State)$$

Has even more structure.

- **Partial order $\sqsubseteq$ on D:**

  $w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

  iff the graph of w is included in the graph of w'.

Hence $\sqsubseteq$ is a partial order.

- **Least element $\perp \in D$ w.r.t. $\sqsubseteq$:**

  $\perp$ = totally undefined partial function

  = partial function with empty graph

  (satisfies $\perp \sqsubseteq w$, for all $w \in D$).

$D = (\text{State} \rightharpoonup \text{State})$

$f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$

$\rightsquigarrow$ limit $f_\infty \in D$

$\bigcup_{n \in \omega} \underline{\text{graph}}(f_n)$

is a partial function

and we define f_∞ to be the partial function with this graph: $\underline{\text{graph}}(f_\infty) = \bigcup_n \underline{\text{graph}}(f_n)$.

Topic 2

Least Fixed Points

$(P, \sqsubseteq_P) \xrightarrow[f]{} (Q, \sqsubseteq_Q)$

$f : P \to Q$ is monotone $\overset{def}{\iff} \forall x \sqsubseteq_P y.\ f(x) \sqsubseteq_Q f(y).$

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

$D = (States \to States)$

NB:

$f_{[\![B]\!], [\![C]\!]} : D \to D$ is monotone.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x \ .$$

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.