$[\![\textbf{while } B \textbf{ do } C]\!]$

$\in$ State

// .... $[\![B]\!]$ .... $[\![C]\!]$ ....

$[\![\text{while } B \text{ do } C]\!](s)$

$$
= \begin{cases}
s & \text{if } [\![B]\!](s) = \text{false} \\
[\![C]\!](s) & \text{if } [\![B]\!](s) = \text{true and} \\
 & \quad [\![B]\!]([\![C]\!]s) = \text{false} \\
\vdots & \vdots \\
[\![C]\!]^{n+1}(s) & \text{if } [\![B]\!]([\![C]\!]^{n} s) = \text{true} \\
 & \quad \& \ [\![B]\!]([\![C]\!]^{n+1} s) = \text{false} \\
\vdots & \vdots \\
\uparrow & \text{if } \forall n \in \mathbb{N} \ [\![B]\!]([\![C]\!]^{n} s) = \text{true}
\end{cases}
$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!]^S = [\![\text{if } B \text{ then } C; \text{while } B \text{ do } C \text{ else skip}]\!]^S$

$= \text{if}([\![B]\!]s, [\![\text{while } B \text{ do } C]\!]([\![C]\!]s), s)$

A fixed point of a function h is an element x
$h(x) = x$.

For good fixed points we write
$\text{fix}(h)$ for such an element

Idea: $[\![\text{while } B \text{ do } C]\!]$ is a fixed point. I.e.

$//\text{def}$

$\text{fix}(F_{[\![B]\!], [\![C]\!]})$

$h(\text{fix } h) = \text{fix } h$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$F_{[\![B]\!],[\![C]\!]} = \lambda f . \lambda s .$$

$$\underline{\text{if}}\,([\![B]\!]s,\ f([\![C]\!]s),\ s) .$$

$$[\![\text{while } B \text{ do } C]\!] = \underline{\text{fix}}\,(F_{[\![B]\!],[\![C]\!]}) .$$

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : State \to \{true, false\}$ and $c : State \rightharpoonup State$, we define

$$f_{b,c} : (State \rightharpoonup State) \to (State \rightharpoonup State)$$

as

$$f_{b,c} = \lambda w \in (State \rightharpoonup State).\, \lambda s \in State.\, if\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$?

Approximating $[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\text{while } B \text{ do } C]\!]_0 = \perp \rightsquigarrow$ empty partial function.

$[\![\text{while } B \text{ do } C]\!]_1 = f_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_0)$

$= \lambda s.\ \text{if}([\![B]\!]s, \perp([\![C]\!]s), s)$

$= \lambda s.\ \text{if}([\![B]\!]s, \uparrow, s)$

$[\![\text{while } B \text{ do } C]\!]_{n+1} = f_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_n)$

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$[\![\text{while } B \text{ do } C]\!]_0$

$\sqsubseteq [\![\text{while } B \text{ do } C]\!]_1 = f_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_0)$

$\sqsubseteq [\![\text{while } B \text{ do } C]\!]_2 = f^2_{[\![B]\!],[\![C]\!]}([\![\text{while } B \text{ do } C]\!]_0)$

$\sqsubseteq$

$\vdots$

$[\![\text{while } B \text{ do } C]\!] = \mathrm{fix}(f_{[\![B]\!],[\![C]\!]})$

the limit of the above

Approximating $[\![\textbf{while } B \textbf{ do } C]\!]$

$$
\begin{aligned}
& f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot) \\
& \quad = \quad \lambda s \in \mathit{State}. \\
& \qquad \begin{cases}
[\![C]\!]^{k}(s) & \text{if } \exists\, 0 \leq k < n.\, [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\
 & \text{and } \forall\, 0 \leq i < k.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \\
 & \\
\uparrow & \text{if } \forall\, 0 \leq i < n.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true}
\end{cases}
\end{aligned}
$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$\in$ State

$\ne \ldots [\![B]\!] \ldots [\![C]\!] \ldots$

<u>Claim</u>: This is a fixed point for $F_{[\![B]\!],[\![C]\!]}$.

$$[\![\text{while } B \text{ do } C]\!](s) = \begin{cases} s & \text{if } [\![B]\!](s) = \text{false} \\ [\![C]\!](s) & \text{if } [\![B]\!](s) = \text{true and } [\![B]\!]([\![C]\!]s) = \text{false} \\ \vdots & \vdots \\ [\![C]\!]^{n+1}(s) & \text{if } [\![B]\!]([\![C]\!]^n s) = \text{true} \ \& \ [\![B]\!]([\![C]\!]^{n+1} s) = \text{false} \\ \vdots & \vdots \\ \uparrow & \forall n \in \mathbb{N}\ [\![B]\!]([\![C]\!]^n) = \text{true} \end{cases}$$

PARTIAL ORDERS

$(S, \preceq)$ $\qquad \preceq \subseteq S \times S$

REFLEXIVE $\qquad s \preceq s$

TRANSITIVE $\qquad s_1 \preceq s_2 \;\&\; s_2 \preceq s_3$

$\qquad \Longrightarrow s_1 \preceq s_3$

ANTISYMMETRIC $\qquad s \preceq s' \;\&\; s' \preceq s$

$\qquad \Longrightarrow s = s'$

For $f \in D$, graph$(f) = \{ (x, fx) \mid fx$ is defined $\}$

$$D \stackrel{\text{def}}{=} (State \rightharpoonup State)$$

Has even more structure.

- **Partial order $\sqsubseteq$ on D:**

$w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

iff the graph of w is included in the graph of w'.

Hence $\sqsubseteq$ is a partial order.

- **Least element $\bot \in D$ w.r.t. $\sqsubseteq$:**

$\bot$ = totally undefined partial function

= partial function with empty graph

(satisfies $\bot \sqsubseteq w$, for all $w \in D$).

$D = (\text{State} \rightharpoonup \text{State})$

$f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$

$\rightsquigarrow$ limit $f_\infty \in D$

$\bigcup_n \underline{\text{graph}}(f_n)$ is a partial function

and we define f_∞ to be the partial function with this graph: $\underline{\text{graph}}(f_\infty) = \bigcup_n \underline{\text{graph}}(f_n)$.

Topic 2

Least Fixed Points

$$(P, \sqsubseteq_P) \xrightarrow[f]{} (Q, \sqsubseteq_Q)$$

$f : P \to Q$ is monotone $\overset{\text{def}}{\iff} \forall x \sqsubseteq_P y.\ f(x) \sqsubseteq_Q f(y)$.

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

$D = (States \to States)$

NB:

$f_{[\![B]\!],[\![C]\!]} : D \to D$ is monotone.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x \quad .$$

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.