Definition. A partial function f is partial recursive ($f \in \mathbf{PR}$) if it can be built up in finitely many steps from the basic functions by use of the operations of composition, primitive recursion and minimization.

The members of $\mathbf{PR}$ that are total are called recursive functions.

Fact: there are recursive functions that are not primitive recursive.

Examples of recursive definitions

$$\begin{cases} f_1(0) & \equiv 0 \\ f_1(x+1) & \equiv f_1(x) + (x+1) \end{cases}$$

$f_1(x) =$ sum of $0, 1, 2, \ldots, x$

$$\begin{cases} f_2(0) & \equiv 0 \\ f_2(1) & \equiv 1 \\ f_2(x+2) & \equiv f_2(x) + f_2(x+1) \end{cases}$$

$f_2(x) = x$th Fibonacci number

$f_2 \in$ PRIM even though this is not a primitive recursive definition

Ackermann's function

There is a (unique) function $ack \in \mathbb{N}^2 \to \mathbb{N}$ satisfying

$$\begin{aligned} ack(0, x_2) &= x_2 + 1 \\ ack(x_1 + 1, 0) &= ack(x_1, 1) \\ ack(x_1 + 1, x_2 + 1) &= ack(x_1, ack(x_1 + 1, x_2)) \end{aligned}$$

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- ack is computable, hence recursive [proof: exercise].

```
        OCaml version 4.00.1

# let rec ack (x : int)(y : int) : int =
  match x ,y with
      0 , y -> y+1
    | x , 0 -> ack (x-1) 1
    | x ,y -> ack (x-1) (ack x (y-1));;
val ack : int -> int -> int = <fun>
# ack 0 0;;
- : int = 1
# ack 1 1;;
- : int = 3
# ack 2 2;;
- : int = 7
# ack 3 3;;
- : int = 61
# ack 4 4;;
Stack overflow during evaluation (looping recursion?).
#
```

Ackermann's function

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- ack is computable, hence recursive [proof: exercise].
- **Fact:** ack grows faster than any primitive recursive function $f \in \mathbb{N}^2 \to \mathbb{N}$:
  $\exists N_f \; \forall x_1, x_2 > N_f \; (f(x_1, x_2) < ack(x_1, x_2))$.
  Hence ack is not primitive recursive.

In fact, writing a_x for $ack(x, -) \in \mathbb{N} \to \mathbb{N}$, one has

$a_{x+1}(y) = (a_x \circ \cdots \circ a_x)(1)$ ← this is an e.g. of a prim. rec. definition "of higher type"

(compose y times)

Lambda calculus

Notions of computability

- Church (1936): λ-calculus
- Turing (1936): Turing machines.

Turing showed that the two very different approaches determine the same class of computable functions. Hence:

Church-Turing Thesis. Every algorithm [in intuitive sense of Lect. 1] can be realized as a Turing machine.

Notation for function definitions in mathematical discourse:

NAMED

"let f be the function $f(x) = x^2 + x + 1$ .... [f] ..."

ANONYMOUS

"the function $x \mapsto x^2 + x + 1$ ..."

"the function $\lambda x . x^2 + x + 1$ ..."

↑ LAMBDA NOTATION

λ-Terms, M

are built up from a given, countable collection of

- variables $x, y, z, \ldots$

by two operations for forming λ-terms:

- λ-abstraction: $(\lambda x.M)$
  (where x is a variable and M is a λ-term)
- application: $(M\,M')$
  (where M and M' are λ-terms).

Some random examples of λ-terms:

$$x \quad (\lambda x.x) \quad ((\lambda y.(x\,y))x) \quad (\lambda y.((\lambda y.(x\,y))x))$$

λ-Terms, M

Notational conventions:

- $(\lambda x_1\, x_2 \ldots x_n.M)$ means $(\lambda x_1.(\lambda x_2 \ldots (\lambda x_n.M) \ldots))$
- $(M_1\, M_2 \ldots M_n)$ means $(\ldots (M_1\, M_2) \ldots M_n)$ (i.e. application is left-associative)
- drop outermost parentheses and those enclosing the body of a λ-abstraction. E.g. write $(\lambda x.(x(\lambda y.(y\, x))))$ as $\lambda x.x(\lambda y.y\, x)$.
- $x \# M$ means that the variable x does not occur anywhere in the λ-term M.

Free and bound variables

In $\lambda x.M$, we call x the bound variable and M the body of the λ-abstraction.

An occurrence of x in a λ-term M is called

- binding if in between λ and $.$
  (e.g. $(\lambda x.y\,x)\,x$)
- bound if in the body of a binding occurrence of x
  (e.g. $(\lambda x.y\,x)\,x$)
- free if neither binding nor bound
  (e.g. $(\lambda x.y\,x)x$).

Free and bound variables

Sets of free and bound variables:

$$
\begin{aligned}
FV(x) &= \{x\} \\
FV(\lambda x.M) &= FV(M) - \{x\} \\
FV(M\,N) &= FV(M) \cup FV(N) \\
BV(x) &= \emptyset \\
BV(\lambda x.M) &= BV(M) \cup \{x\} \\
BV(M\,N) &= BV(M) \cup BV(N)
\end{aligned}
$$

E.g. $FV((\lambda x.\,y x) x) = \{x, y\}$

$BV((\lambda x.\,y x) x) = \{x\}$

Free and bound variables

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$$
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BV(x) &= \emptyset \\
BV(\lambda x.M) &= BV(M) \cup \{x\} \\
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\end{aligned}
$$

If $FV(M) = \emptyset$, M is called a closed term, or combinator.

E.g. $FV(\lambda y.\lambda x.(\lambda x.y x) x) = \emptyset$

α-Equivalence $M =_\alpha M'$

$\lambda x.M$ is intended to represent the function f such that

$f(x) = M$ for all x.

So the name of the bound variable is immaterial: if $M' = M\{x'/x\}$ is the result of taking M and changing all occurrences of x to some variable $x' \# M$, then $\lambda x.M$ and $\lambda x'.M'$ both represent the same function.

For example, $\lambda x.x$ and $\lambda y.y$ represent the same function (the identity function).

α-Equivalence $M =_\alpha M'$

is the binary relation inductively generated by the rules:

$$\frac{}{x =_\alpha x} \qquad \frac{z \# (M\,N) \qquad M\{z/x\} =_\alpha N\{z/y\}}{\lambda x.M =_\alpha \lambda y.N}$$

$$\frac{M =_\alpha M' \qquad N =_\alpha N'}{M\,N =_\alpha M'\,N'}$$

where $M\{z/x\}$ is M with all occurrences of x replaced by z.

α-Equivalence $M =_\alpha M'$

For example:

$$
\begin{array}{rc}
 & \lambda x.(\lambda x x'.x)\, x' =_\alpha \lambda y.(\lambda x\, x'.x) x' \\
\text{because} & (\lambda z\, x'.z) x' =_\alpha (\lambda x\, x'.x) x' \\
\text{because} & \lambda z\, x'.z =_\alpha \lambda x\, x'.x \text{ and } x' =_\alpha x' \\
\text{because} & \lambda x'.u =_\alpha \lambda x'.u \text{ and } x' =_\alpha x' \\
\text{because} & u =_\alpha u \text{ and } x' =_\alpha x'.
\end{array}
$$

α-Equivalence $M =_\alpha M'$

Fact: $=_\alpha$ is an equivalence relation (reflexive, symmetric and transitive).

We do not care about the particular names of bound variables, just about the distinctions between them. So α-equivalence classes of λ-terms are more important than λ-terms themselves.

- Textbooks (and these lectures) suppress any notation for α-equivalence classes and refer to an equivalence class via a representative λ-term (look for phrases like "we identify terms up to α-equivalence" or "we work up to α-equivalence").
- For implementations and computer-assisted reasoning, there are various devices for picking canonical representatives of α-equivalence classes (e.g. de Bruijn indexes, graphical representations, ...).