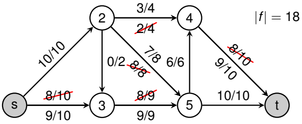
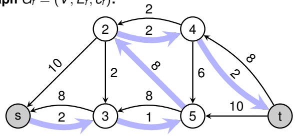
Graph $G = (V, E, c)$**:**



Residual Graph $G_f = (V, E_f, c_f)$**:**



6.3: Maximum Flows

Frank Stajano Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Announcements

- Deadline for Microchallenge 7 today!
- There is a list of errata for the slides on the webpage
- There might be a little bit of time in the last lecture to revisit one of the previous topics and briefly discuss some data structure/algorithm/proof etc. which may or may not have been covered in previous lectures.
  If you have any suggestion, please send an email today.

Outline

Analysis of Ford-Fulkerson

Matchings in Bipartite Graphs

Introduction and Line Intersection

Max-Flow Min-Cut Theorem

Theorem

The value of the max-flow is equal to the capacity of the min-cut, that is

$$\max_{f} |f| = \min_{S,T \subseteq V} \text{cap}(S, T).$$

Analysis of Ford-Fulkerson

```
def FordFulkerson(G)
    initialize flow to 0 on all edges
    while an augmenting path in G_f can be found:
        push as much extra flow as possible through it
```

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Lemma

If all capacities $c(u, v)$ are integral, then the flow at every iteration of Ford-Fulkerson is integral.

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Lemma

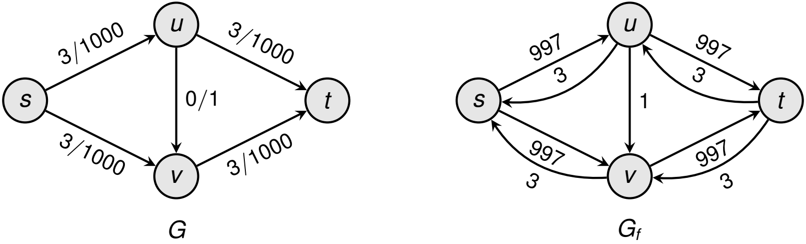
If all capacities $c(u, v)$ are integral, then the flow at every iteration of Ford-Fulkerson is integral.

Theorem

For integral capacities $c(u, v)$, Ford-Fulkerson terminates after $V \cdot C$ *iterations*, where $C := \max_{u,v} c(u, v)$ and returns the maximum flow.

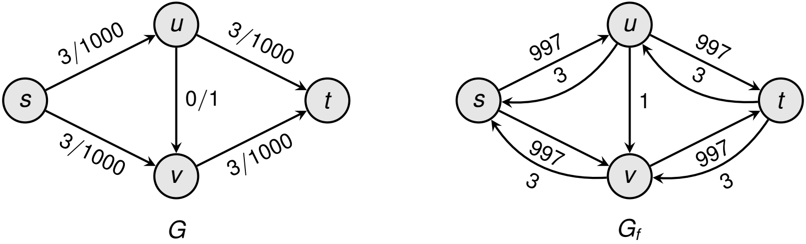
at the time of termination, no augmenting path
$\Rightarrow$ Ford-Fulkerson returns maxflow (Key Lemma)

Slow Convergence of Ford-Fulkerson (Figure 26.7)



Number of iterations is at least $C := \max_{u,v} c(u, v)$!

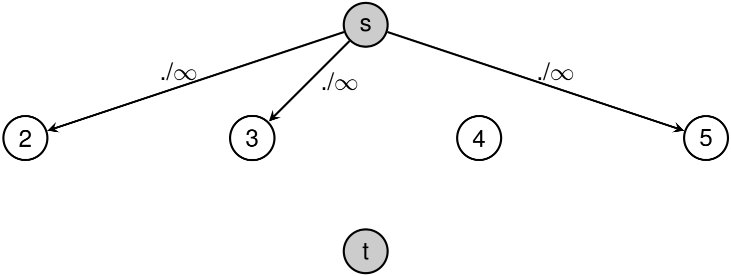
Slow Convergence of Ford-Fulkerson (Figure 26.7)



Number of iterations is at least $C := \max_{u,v} c(u,v)$!

For irrational capacities, Ford-Fulkerson may even fail to terminate!

Non-Termination of Ford-Fulkerson for Irrational Capacities



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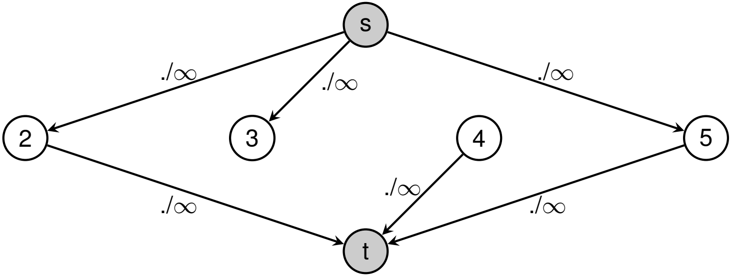

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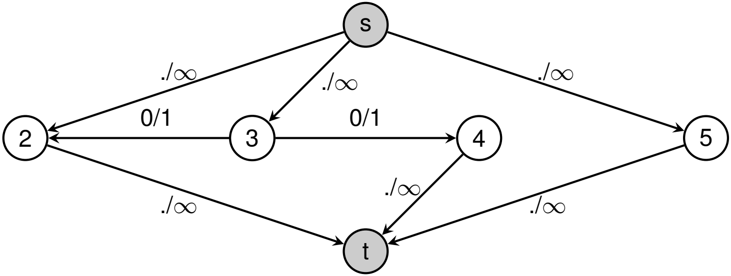

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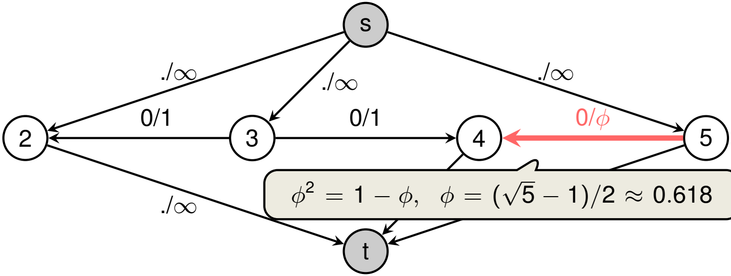
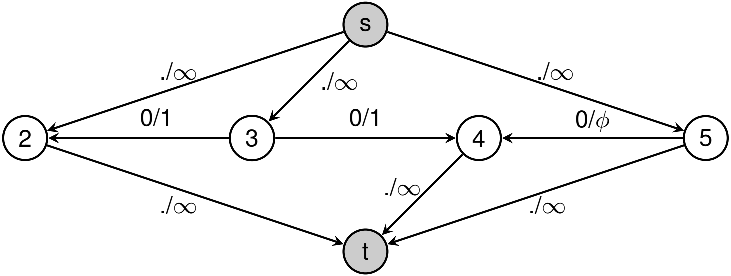


$$\phi^2 = 1 - \phi, \quad \phi = (\sqrt{5}-1)/2 \approx 0.618$$

Non-Termination of Ford-Fulkerson for Irrational Capacities



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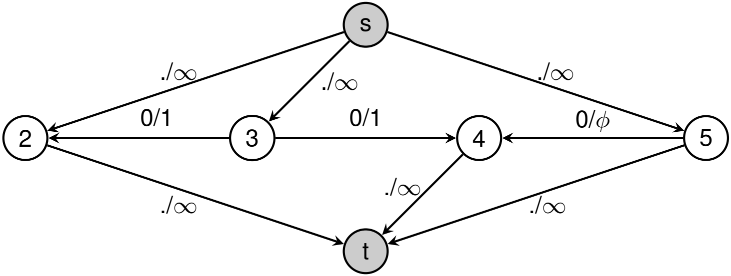

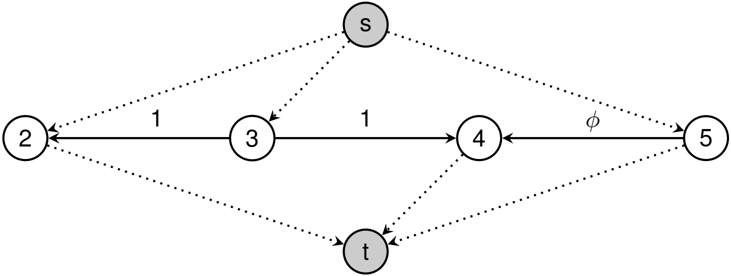

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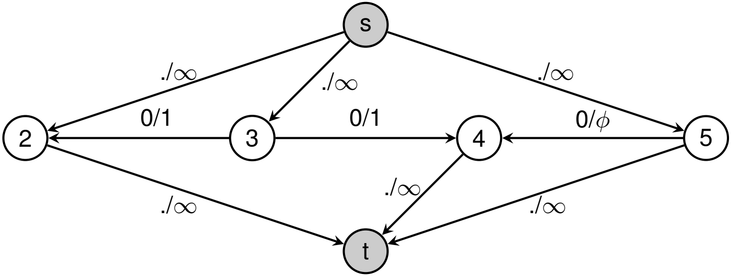
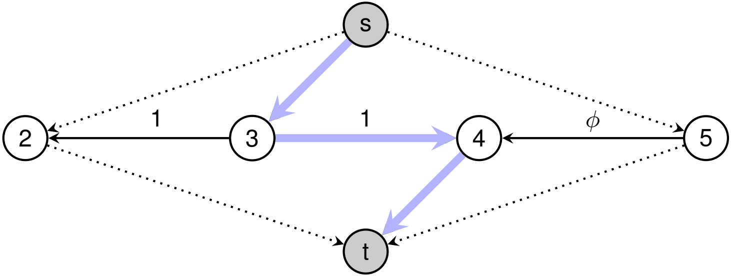


Iteration: 1, $|f| = 0$



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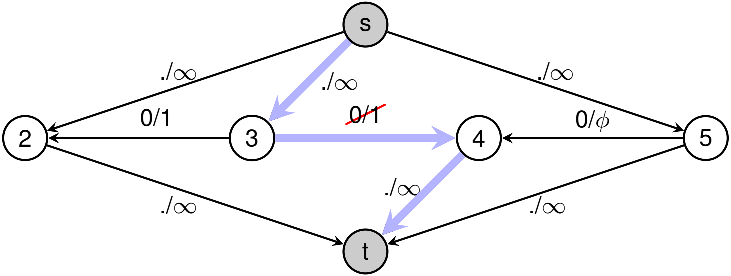

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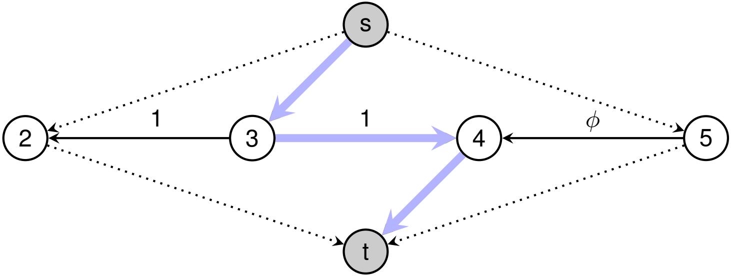

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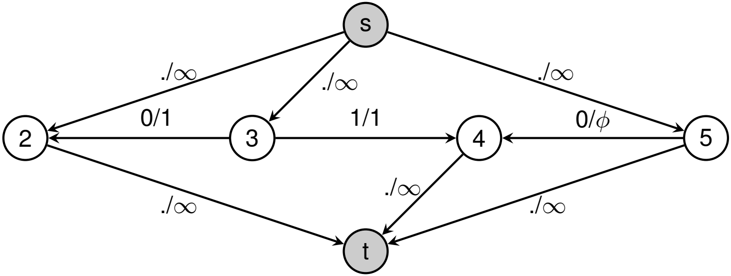
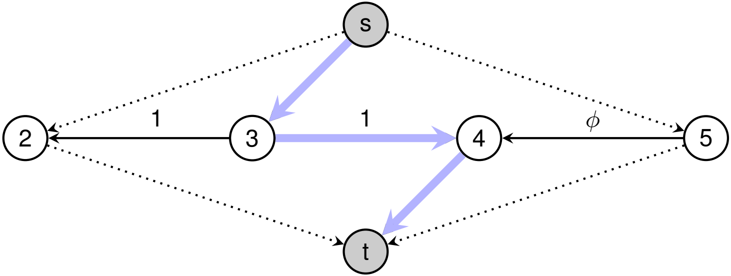


Iteration: 1, $|f| = 1$



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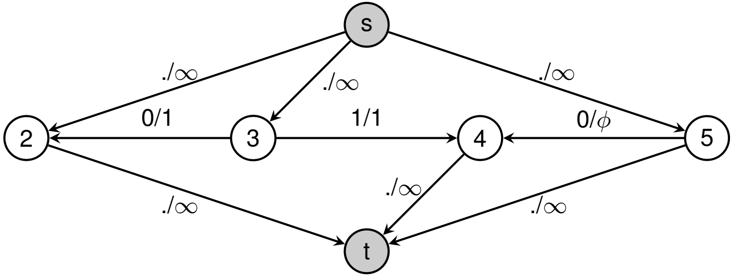

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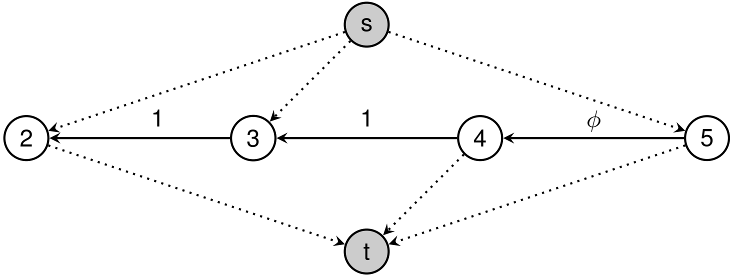

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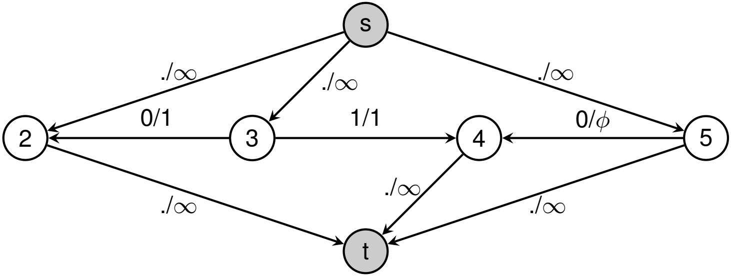
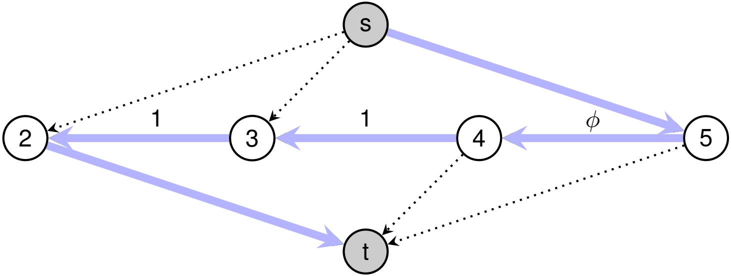


Iteration: 2, $|f| = 1$



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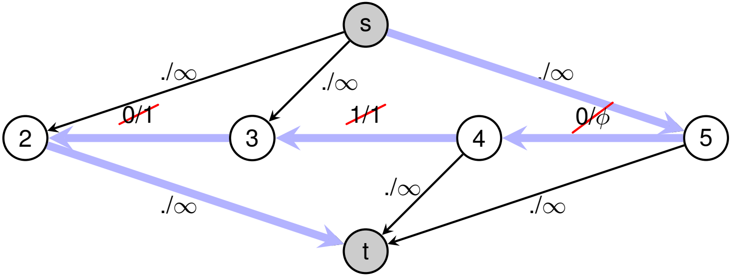

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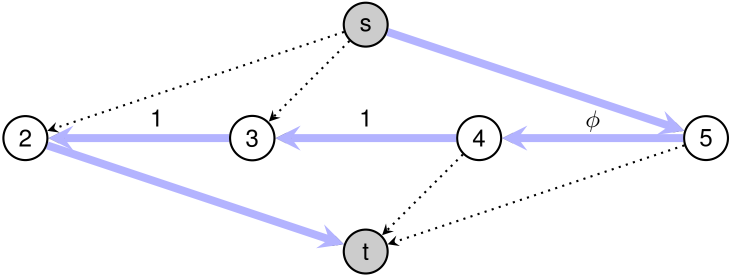

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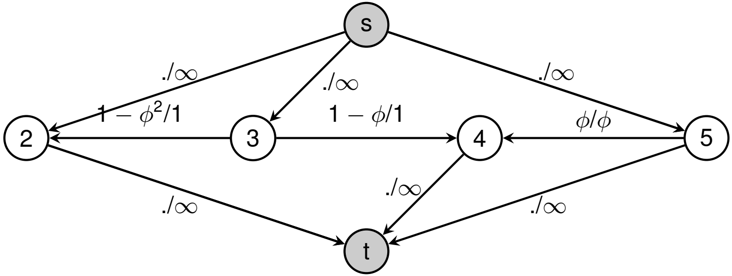
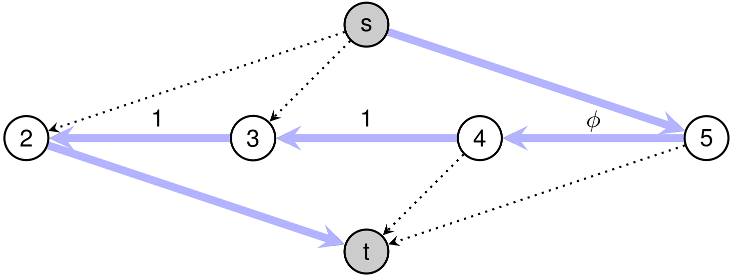


Iteration: 2, $|f| = 1 + \phi$



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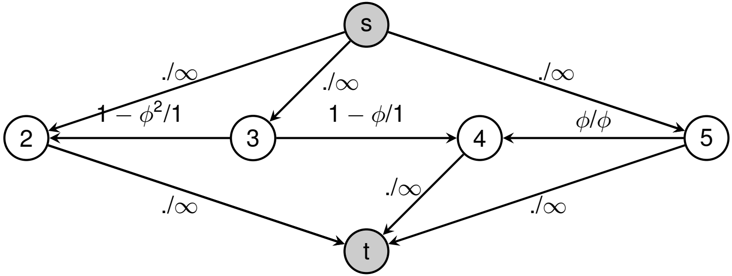

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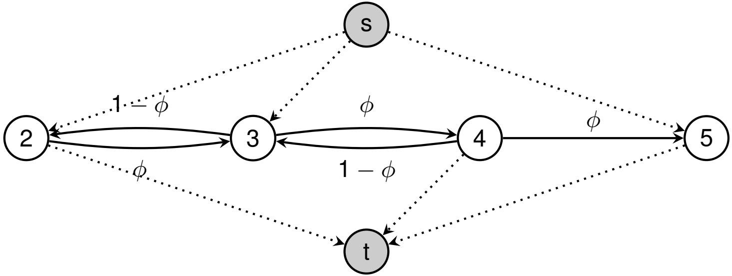

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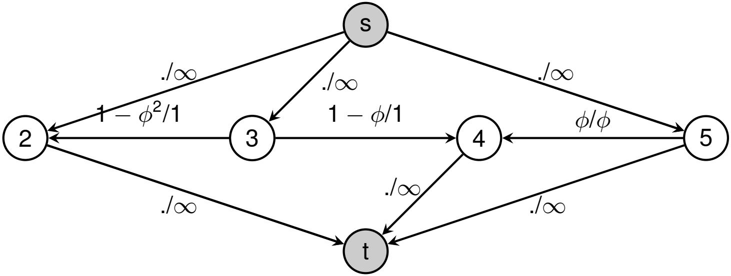
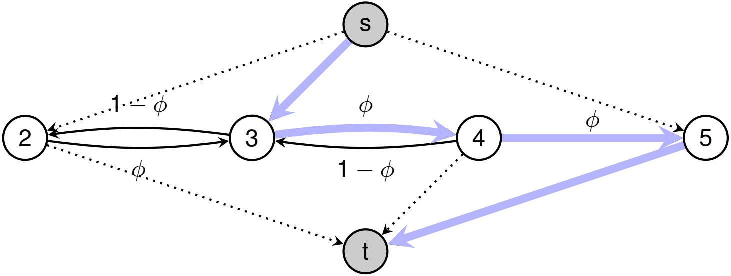


Iteration: 3, $|f| = 1 + \phi$



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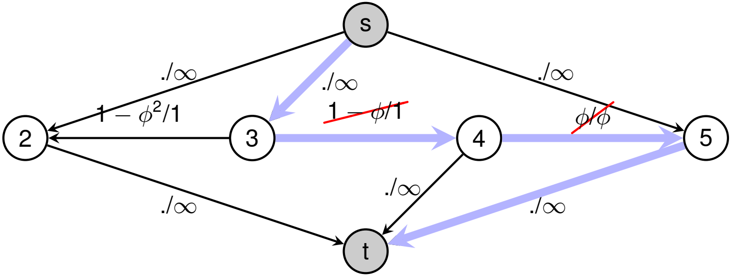

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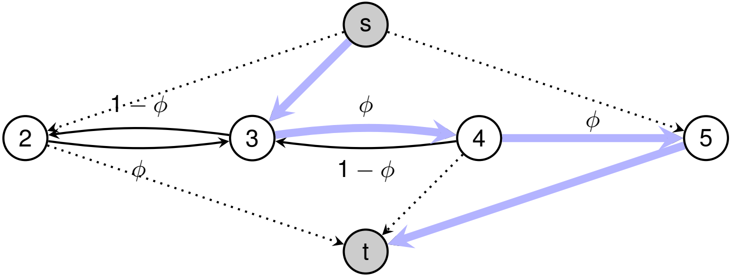

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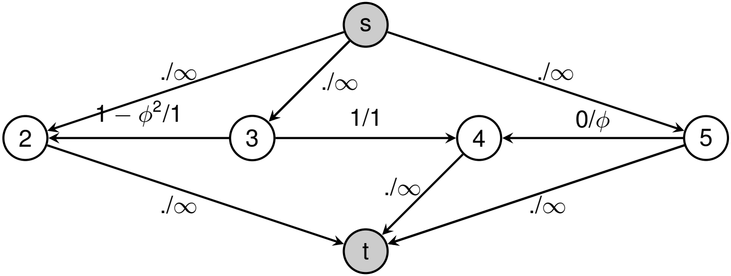
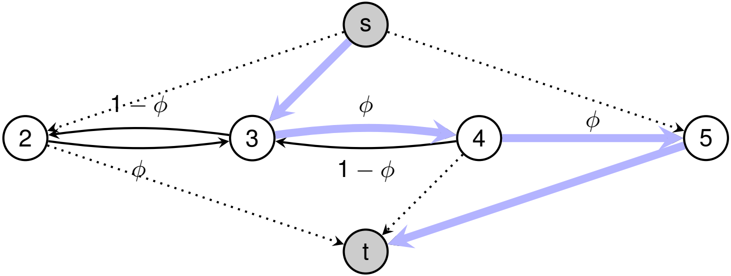


Iteration: 3, $|f| = 1 + 2 \cdot \phi$



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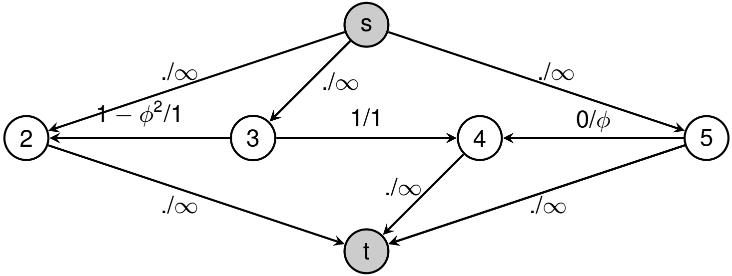

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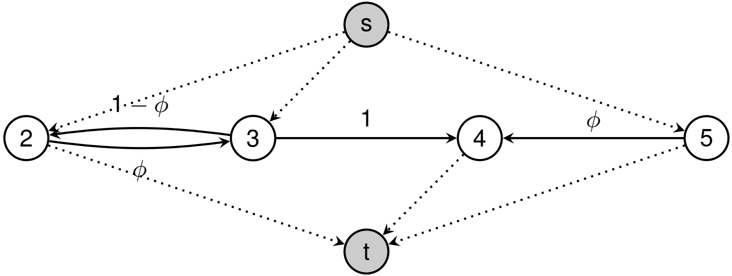

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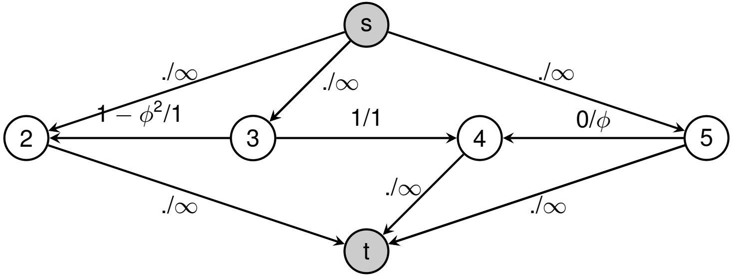
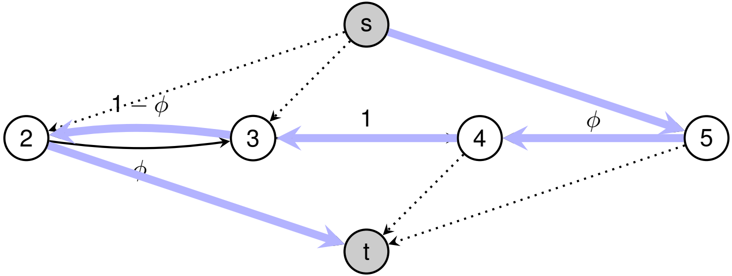


Iteration: 4, $|f| = 1 + 2 \cdot \phi$



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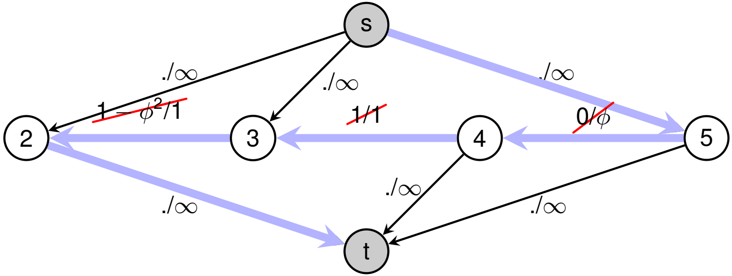

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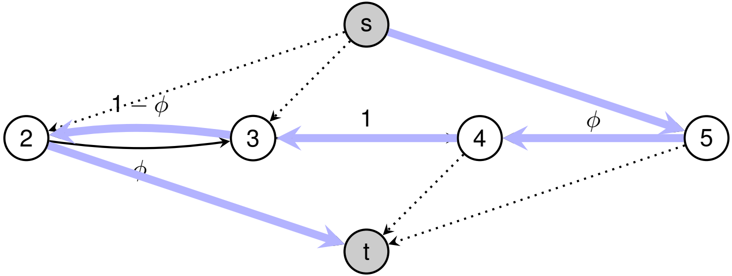

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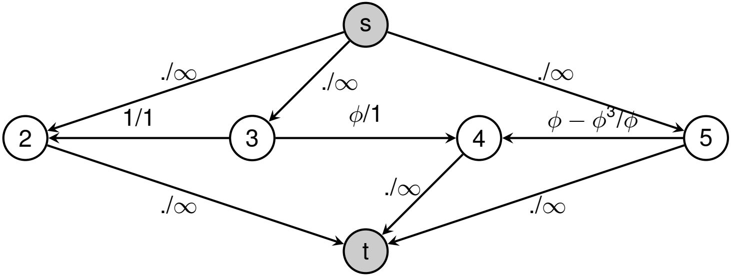
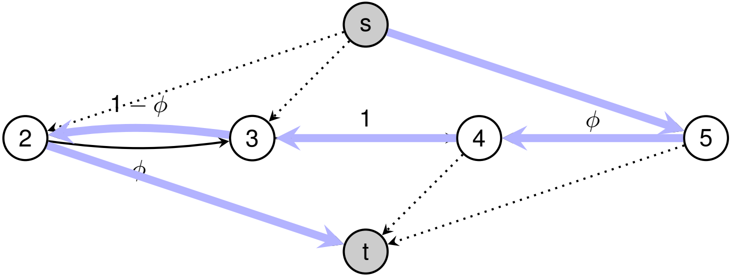


Iteration: 4, $|f| = 1 + 2 \cdot \phi + \phi^2$



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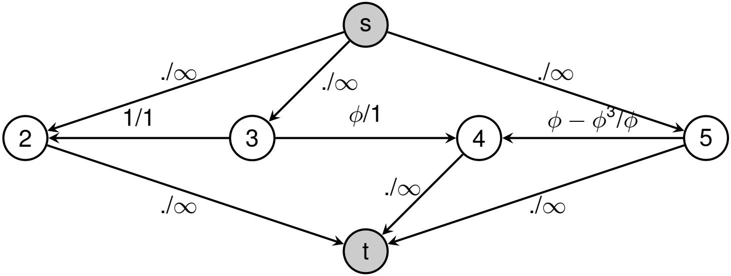

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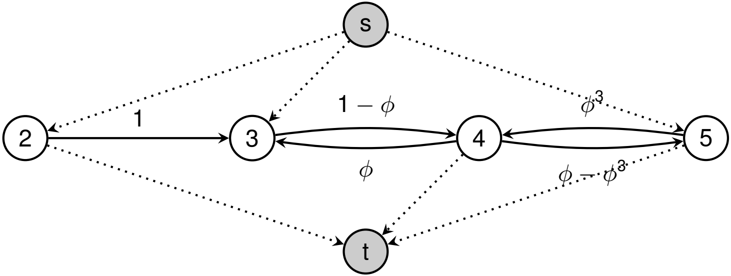

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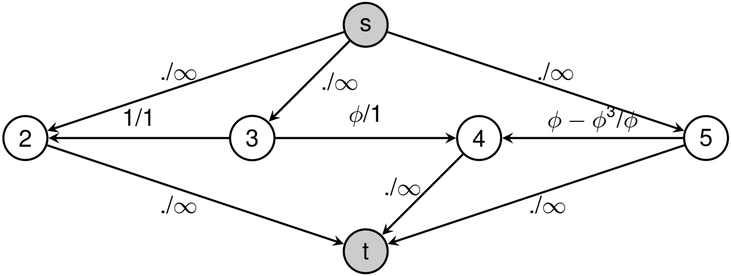
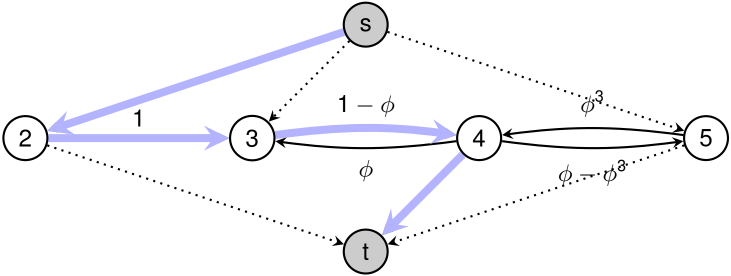


Iteration: 5, $|f| = 1 + 2 \cdot \phi + \phi^2$



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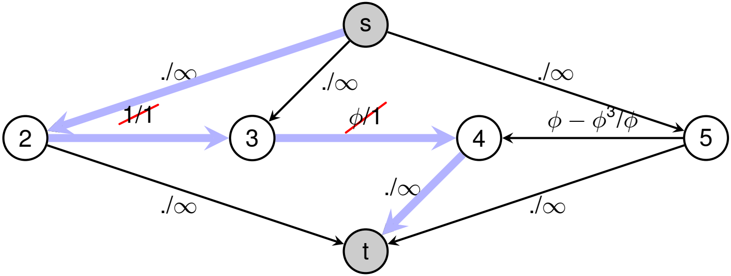

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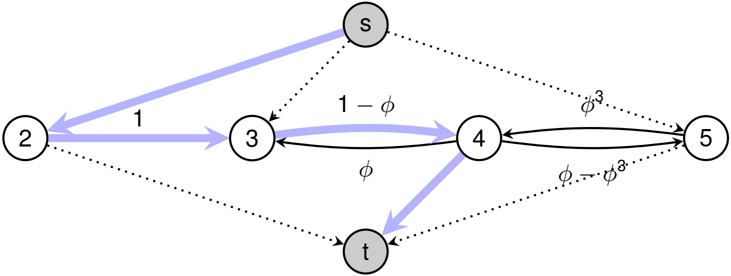

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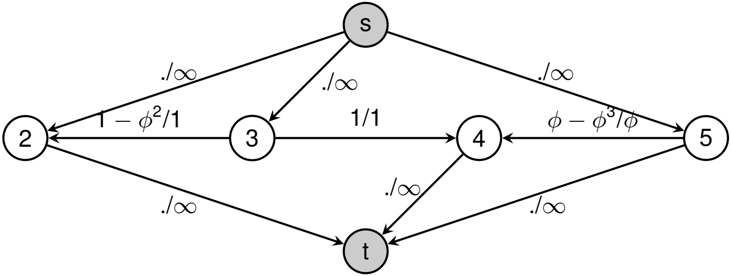
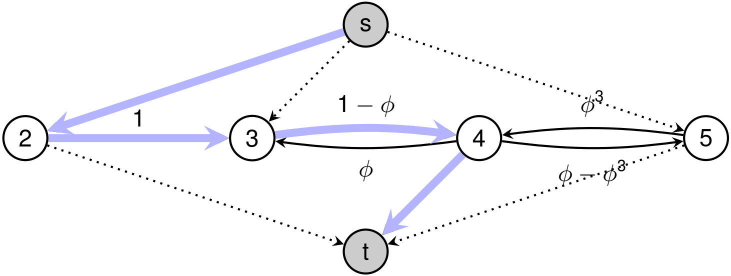


Iteration: 5, $|f| = 1 + 2 \cdot \phi + 2 \cdot \phi^2$



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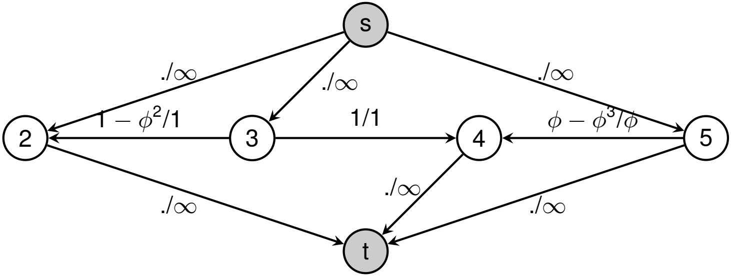

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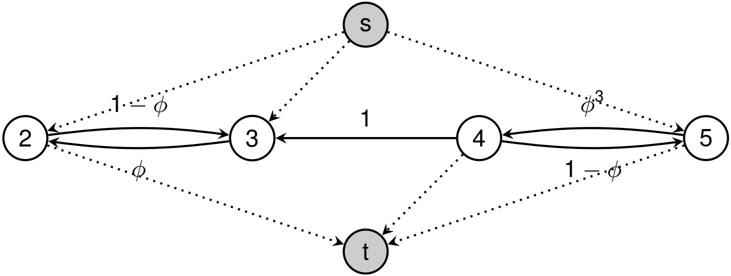
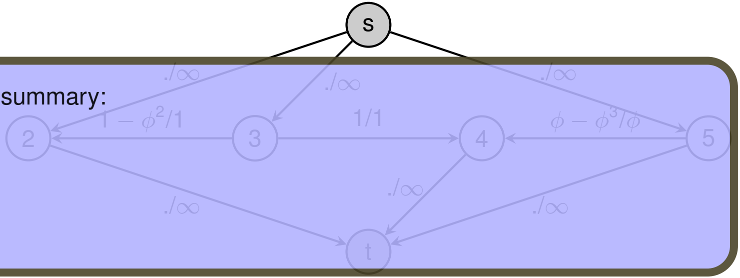
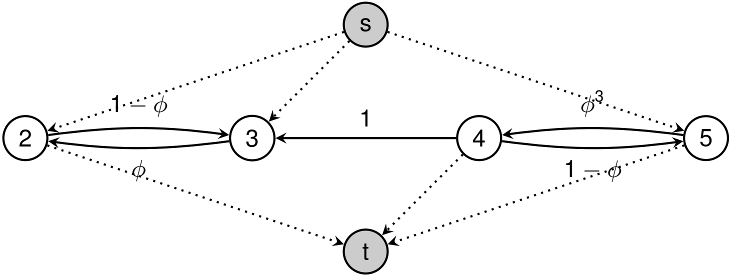


Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:





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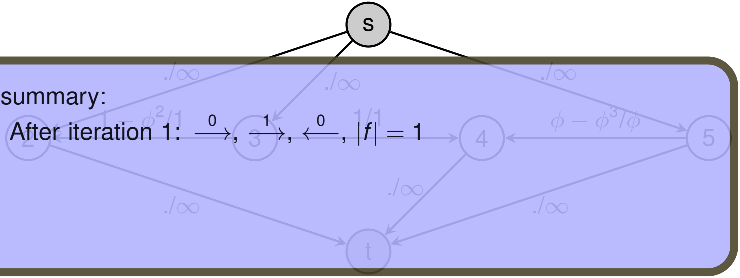
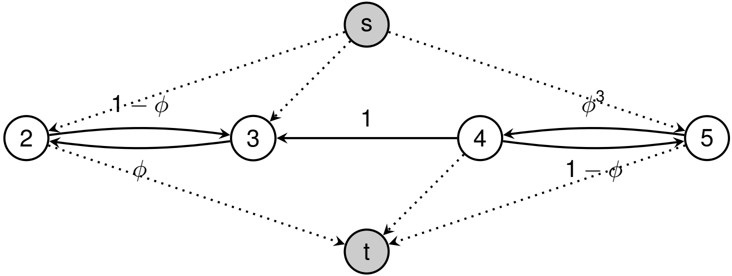


In summary:

- After iteration 1: $\xrightarrow{0}$, $\xrightarrow{1}$, $\xleftarrow{0}$, $|f| = 1$



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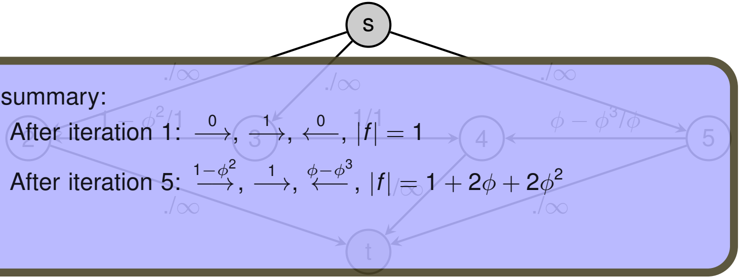
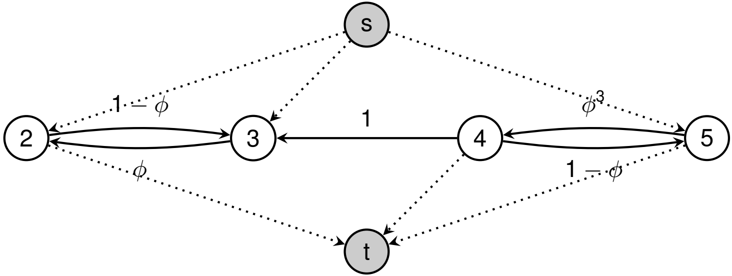


In summary:

- After iteration 1: $\xrightarrow{0}$, $\xrightarrow{1}$, $\xleftarrow{0}$, $|f| = 1$
- After iteration 5: $\xrightarrow{1-\phi^2}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^3}$, $|f| = 1 + 2\phi + 2\phi^2$



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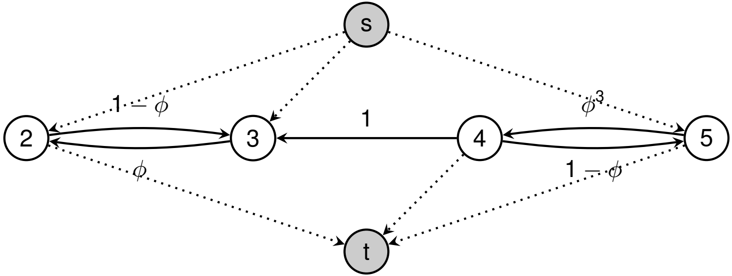
- After iteration 5: $\xrightarrow{1-\phi^2}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^3}$, $|f| = 1 + 2\phi + 2\phi^2$
- After iteration 9: $\xrightarrow{1-\phi^4}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^5}$, $|f| = 1 + 2\phi + 2\phi^2 + 2\phi^3 + 2\phi^4$



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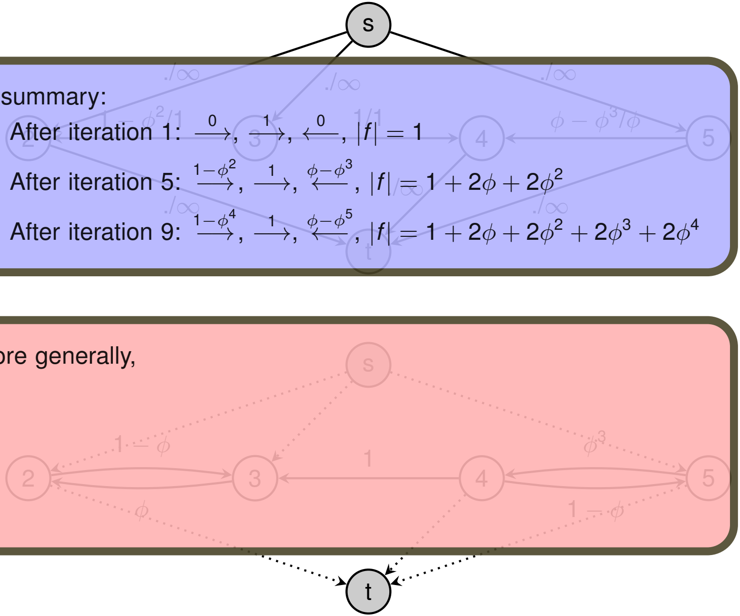

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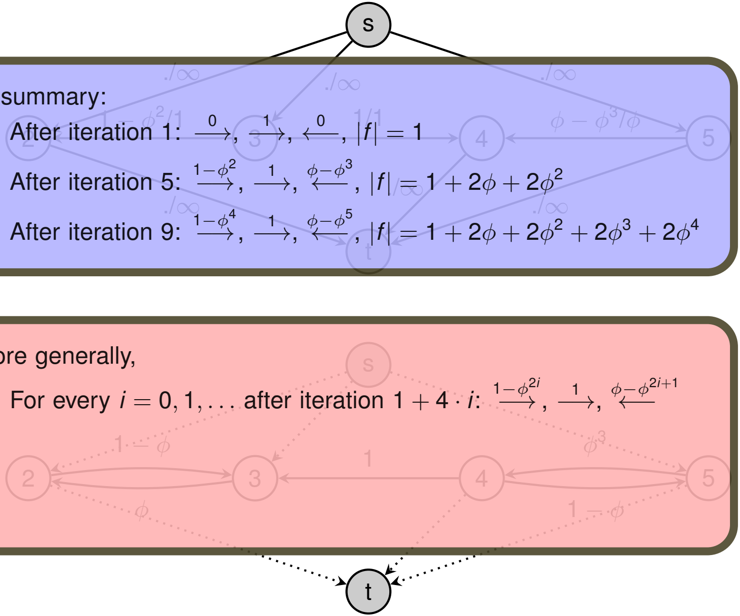
More generally,

Non-Termination of Ford-Fulkerson for Irrational Capacities



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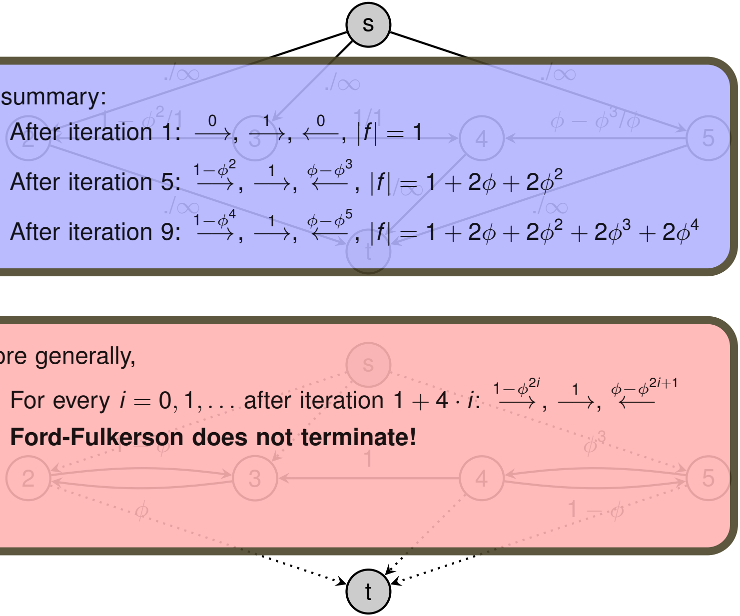

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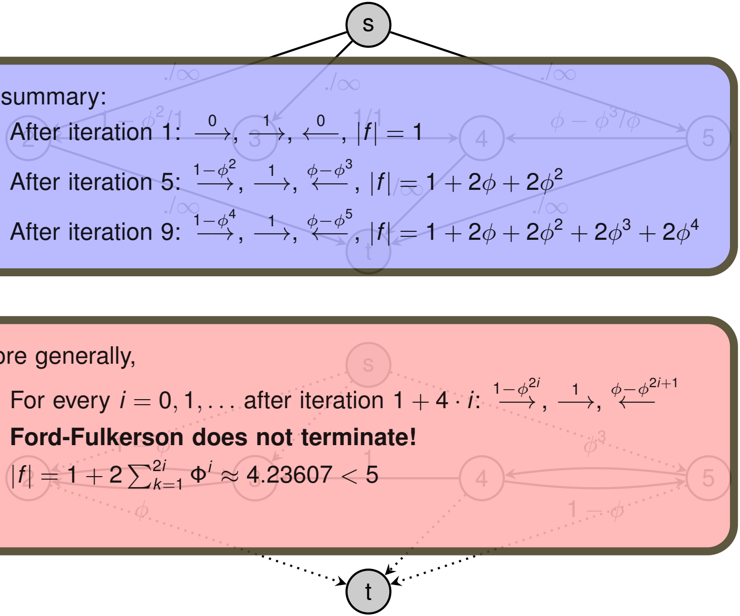

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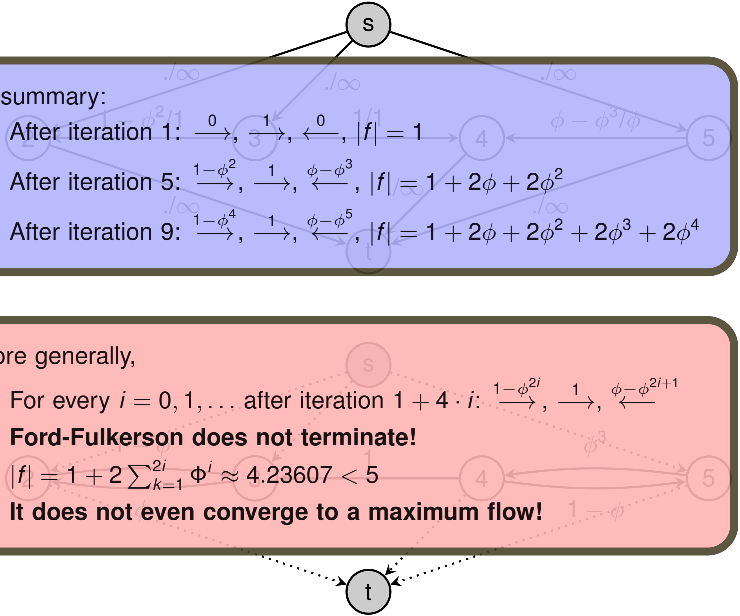

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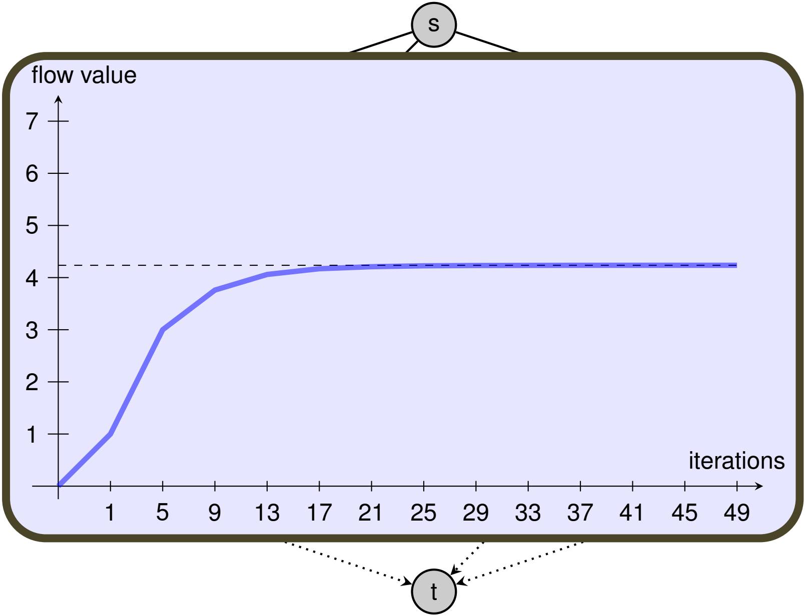
- **It does not even converge to a maximum flow!**

Non-Termination of Ford-Fulkerson for Irrational Capacities



Summary and Outlook

Ford-Fulkerson Method

- works only for integral (rational) capacities

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- Idea: Find an augmenting path with high capacity
- Consider subgraph of G_f consisting of edges (u, v) with $c_f(u, v) > \Delta$
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- Runtime: $O(E^2 \cdot \log C)$

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Edmonds-Karp Algorithm

- Idea: Find the shortest augmenting path in G_f
- Runtime: $O(E^2 \cdot V)$

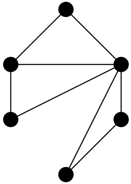
Outline

Application: Maximum-Bipartite-Matching Problem

Matching

A matching is a subset $M \subseteq E$ such that for all $v \in V$, at most one edge of M is incident to v.

Application: Maximum-Bipartite-Matching Problem



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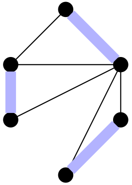

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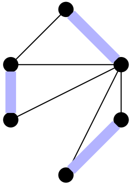

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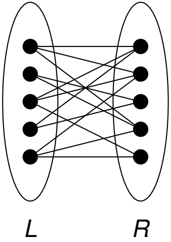

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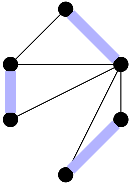

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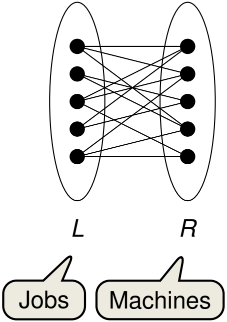

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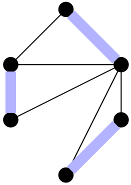

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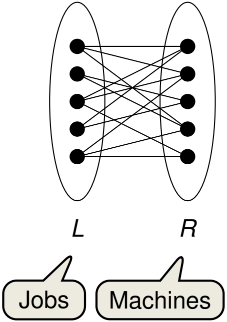

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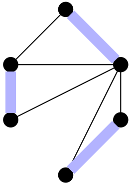

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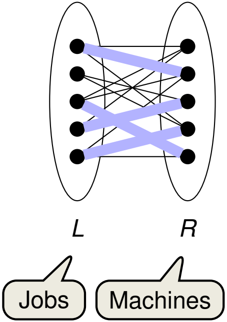

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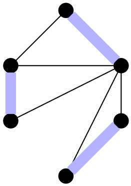

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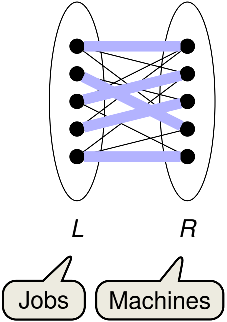

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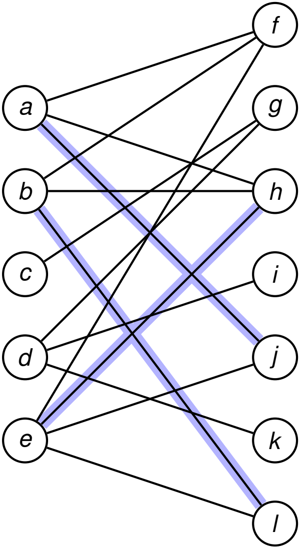
Matchings in Bipartite Graphs via Maximum Flows



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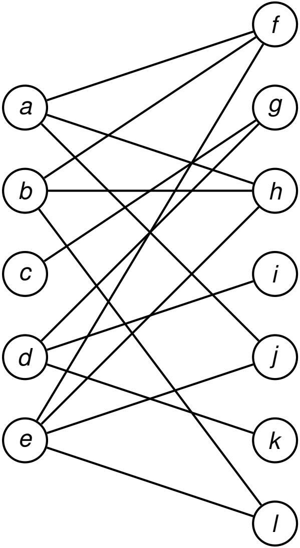

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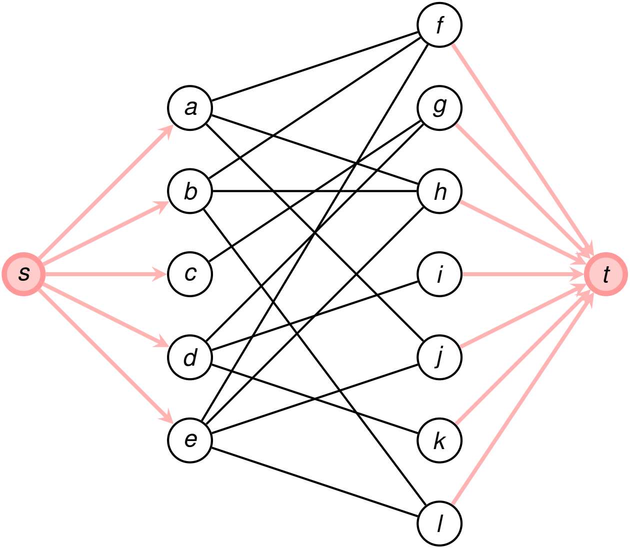

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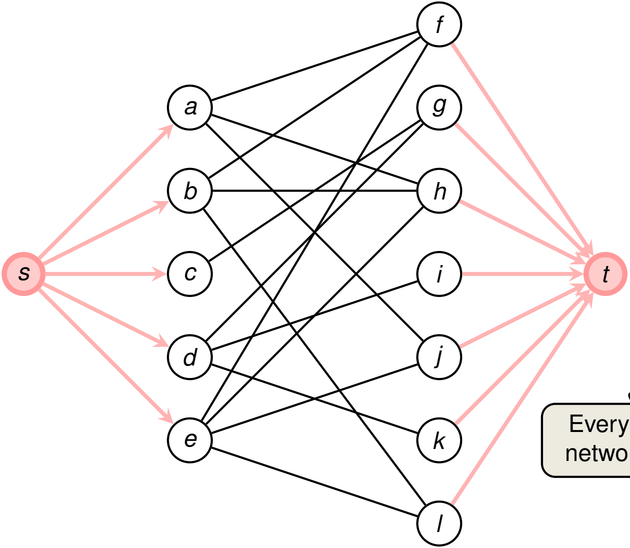
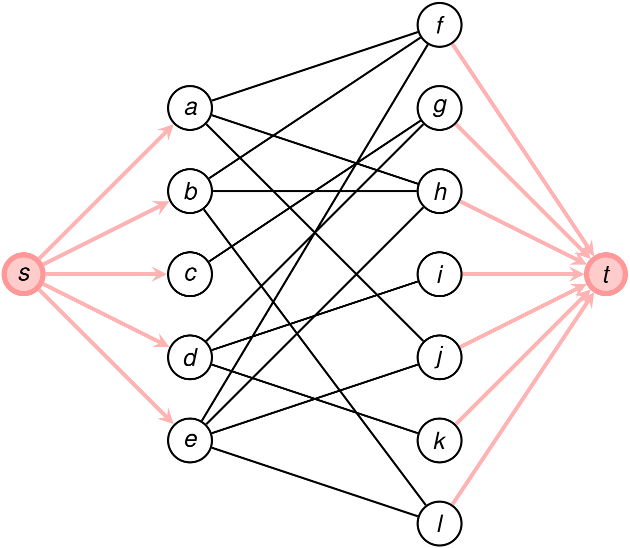


Every edge in the flow network has capacity 1

Matchings in Bipartite Graphs via Maximum Flows



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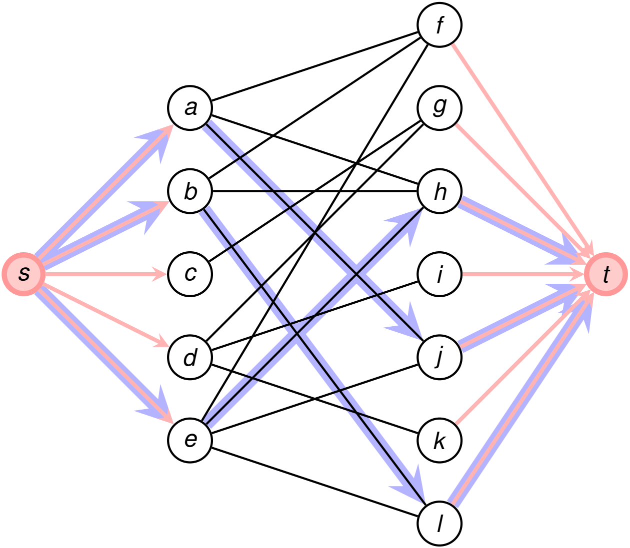

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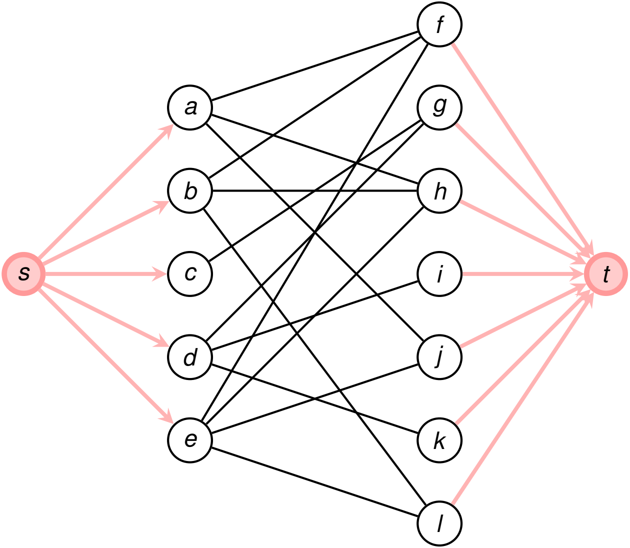

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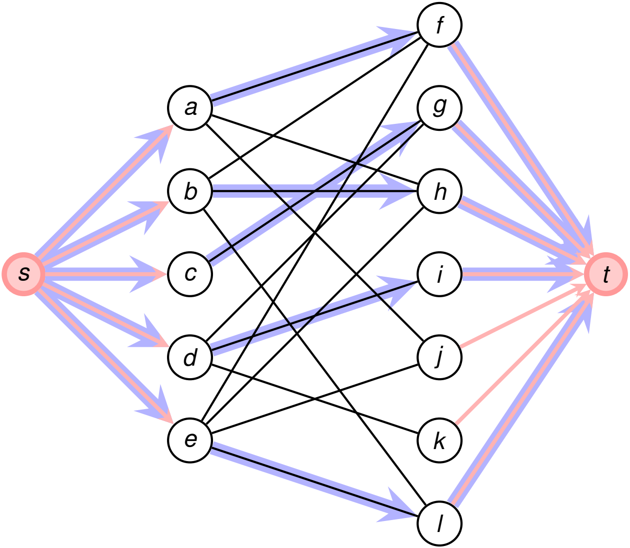
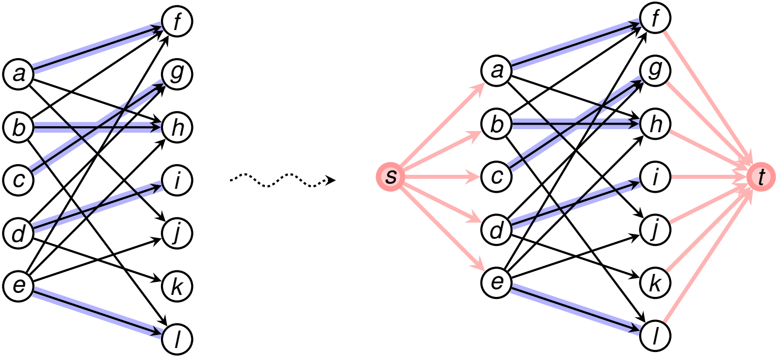


Correspondence between Maximum Matchings and Max Flow

Theorem (Corollary 26.11)

The cardinality of a maximum matching M in a bipartite graph G equals the value of a maximum flow f in the corresponding flow network $\widetilde{G}$.

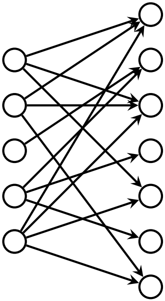


Graph G

Graph $\widetilde{G}$

From Matching to Flow

- Given a maximum matching of cardinality k



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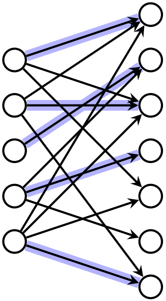

Graph G

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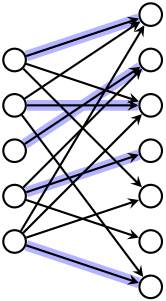
- Given a maximum matching of cardinality k
- Consider flow f that sends one unit along each each of k paths



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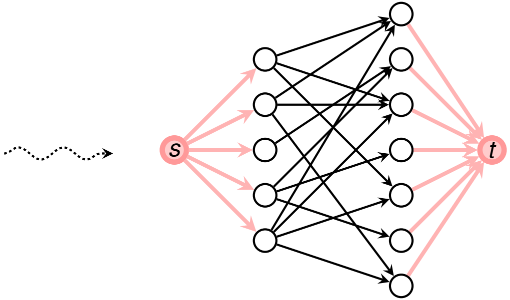

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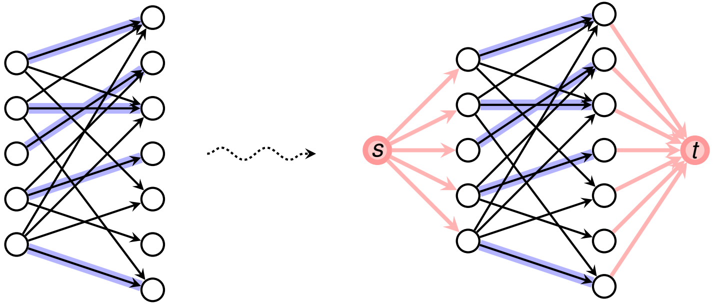

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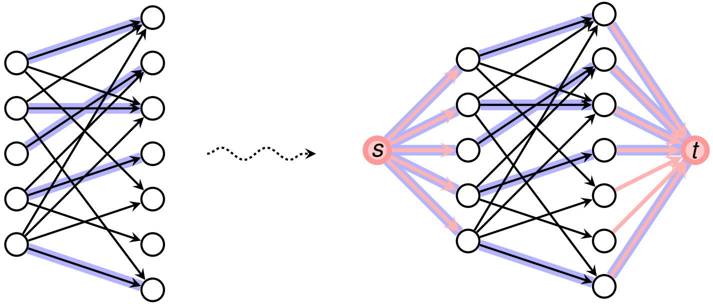

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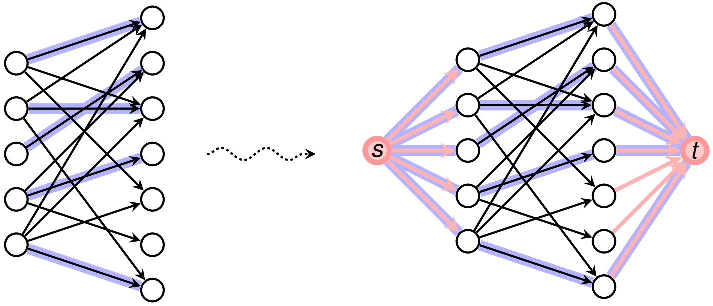
$\Rightarrow$ f is a flow and has value k



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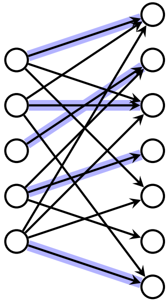
max cardinality matching $\leq$ value of maxflow



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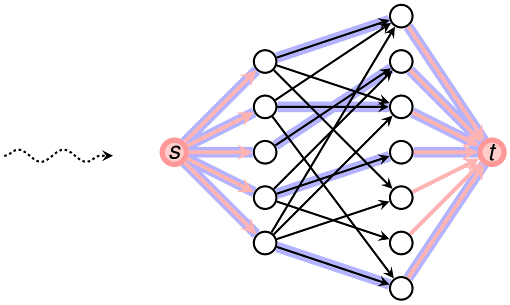

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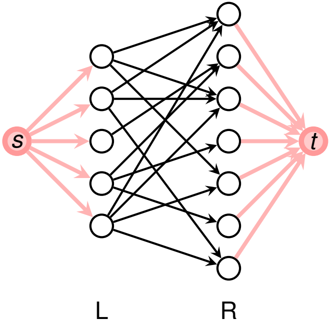

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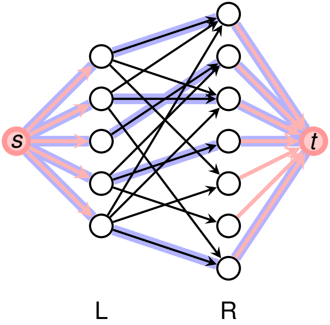

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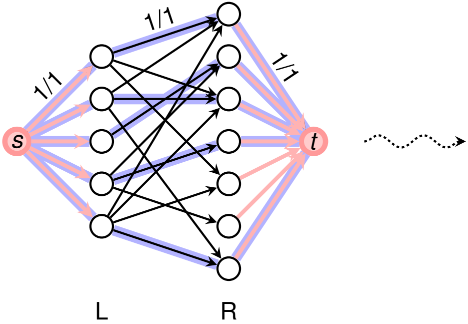
- Integrality Theorem $\Rightarrow f(u,v) \in \{0,1\}$ and k integral



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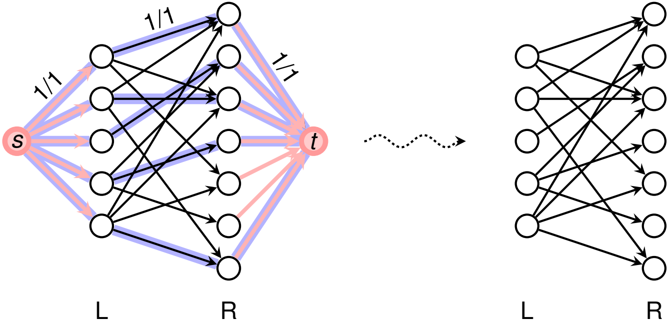

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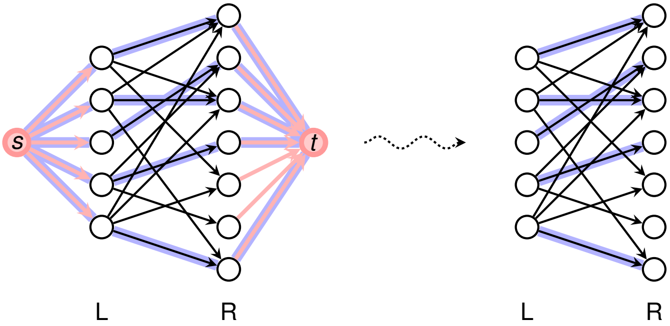

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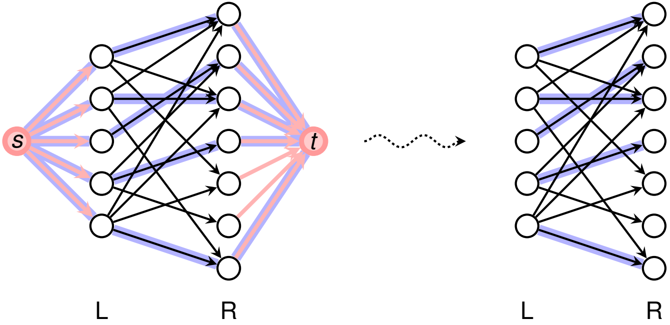

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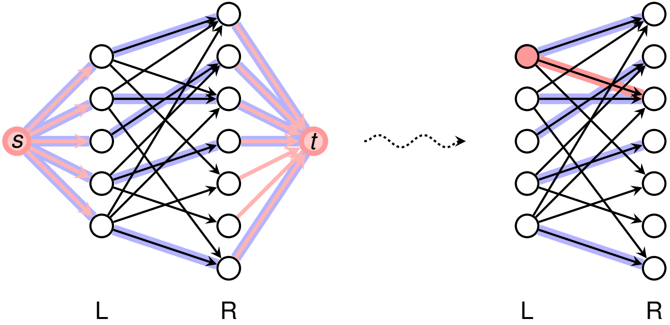

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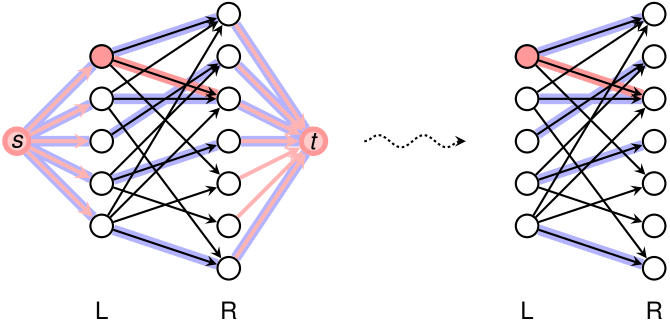

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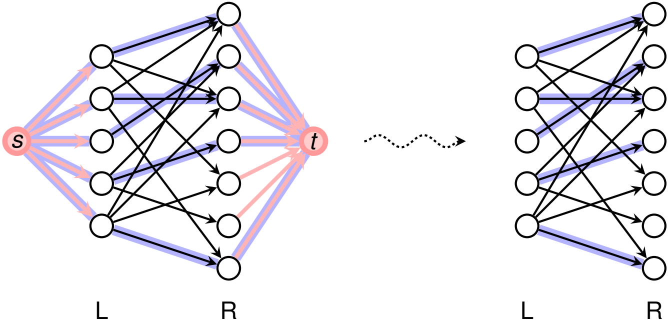

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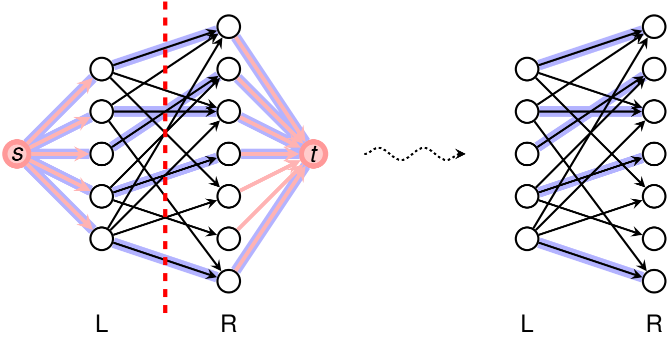
b) Flow Conservation $\Rightarrow$ every node in R receives at most one unit

c) Cut $(L \cup \{s\}, R \cup \{t\})$



From Flow to Matching

- Let f be a maximum flow in $\widetilde{G}$ of value k
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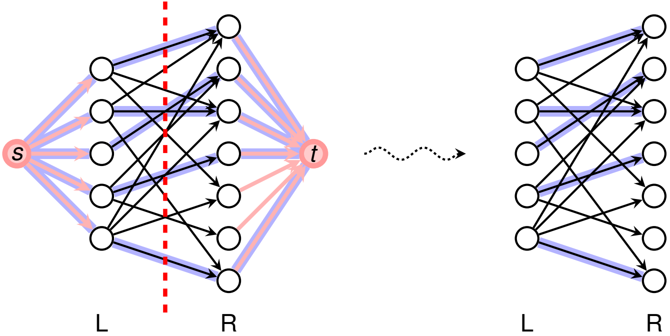
c) Cut $(L \cup \{s\}, R \cup \{t\})$ $\Rightarrow$ net flow is k



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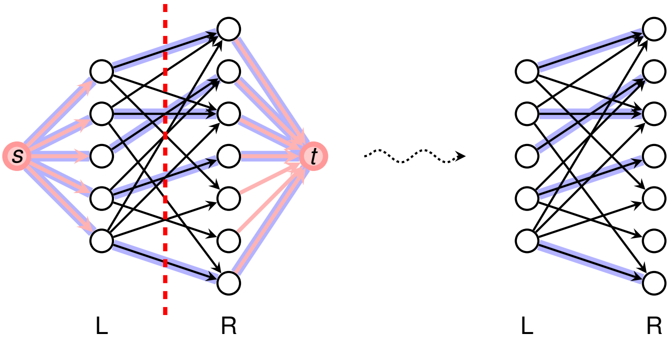

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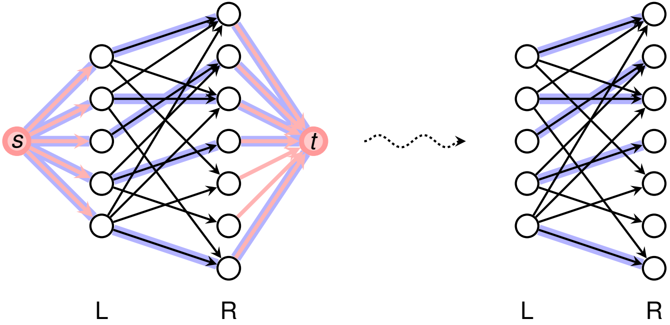

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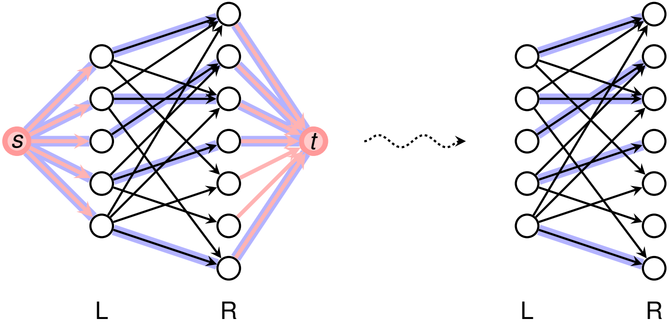
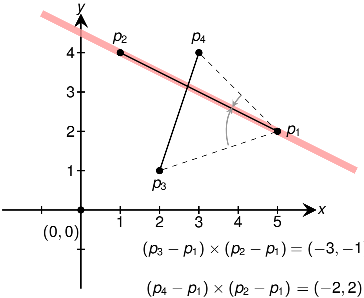
value of maxflow $\leq$ max cardinality matching





$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1) = (-2, 2) \times (-4, 2) = 4$$

7: Geometric Algorithms

Frank Stajano Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Outline

Analysis of Ford-Fulkerson

Matchings in Bipartite Graphs

Introduction and Line Intersection

Introduction

Computational Geometry

- Branch that studies algorithms for geometric problems

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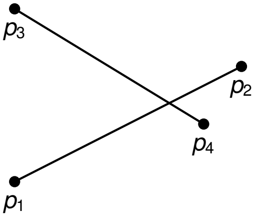

Do these lines intersect?

Introduction

Computational Geometry

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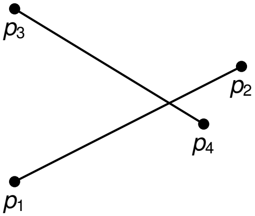
Applications

- computer graphics
- computer vision
- textile layout
- VLSI design

⋮



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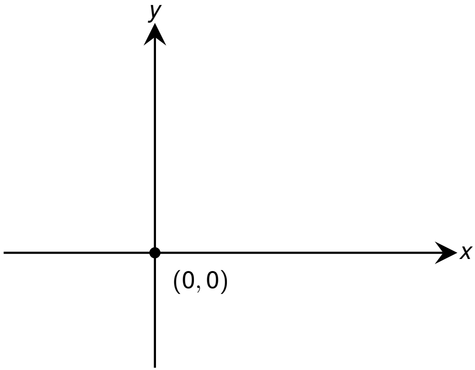
Cross Product (Area)



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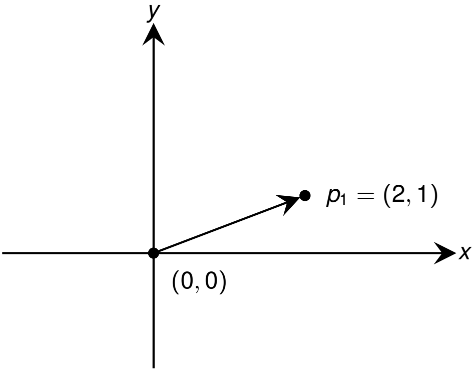

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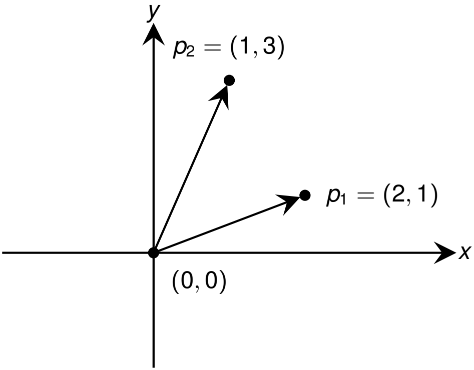

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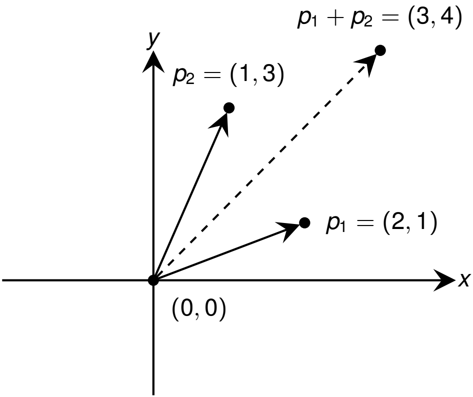

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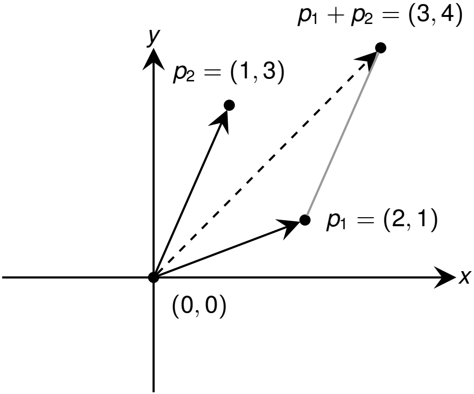

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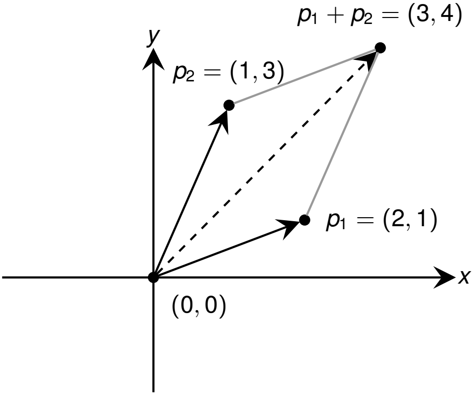

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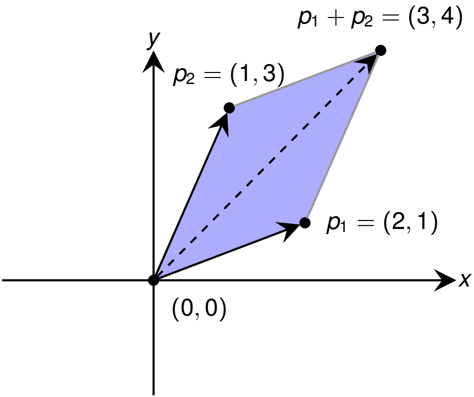

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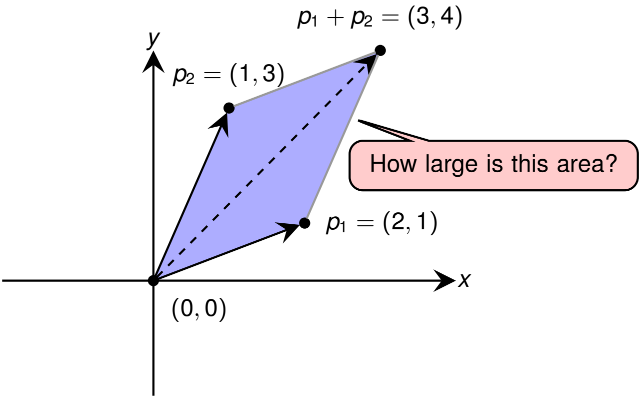

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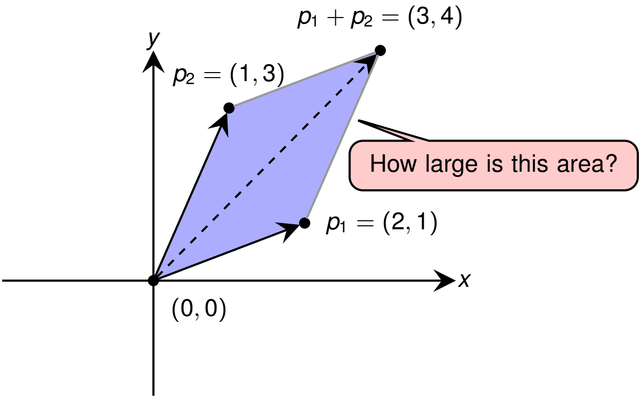
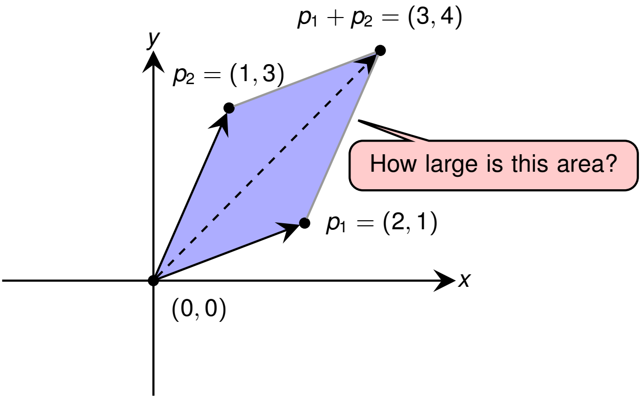


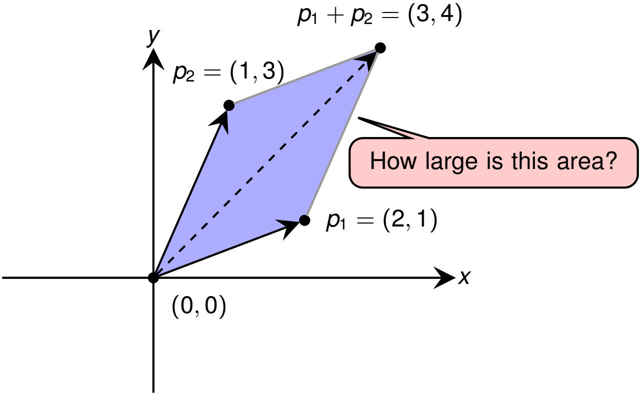
$p_1 \times p_2$

Cross Product (Area)



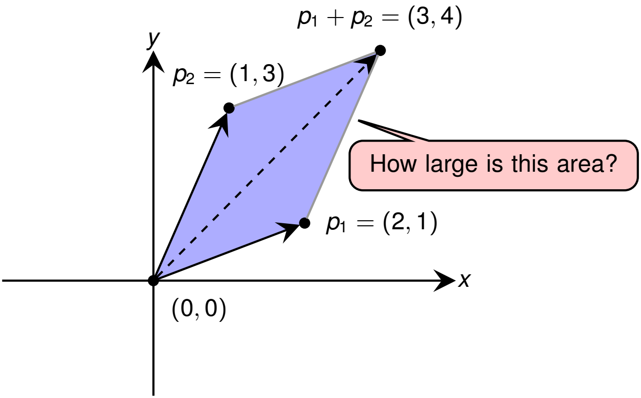
$$p_1 \times p_2 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Cross Product (Area)



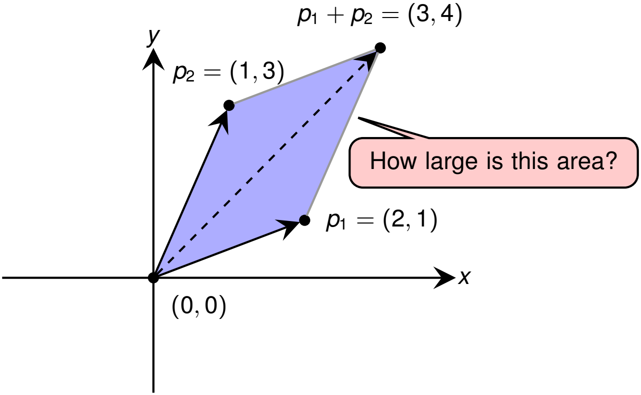
$$p_1 \times p_2 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

Cross Product (Area)



$$p_1 \times p_2 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 = 2 \cdot 3 - 1 \cdot 1$$

Cross Product (Area)



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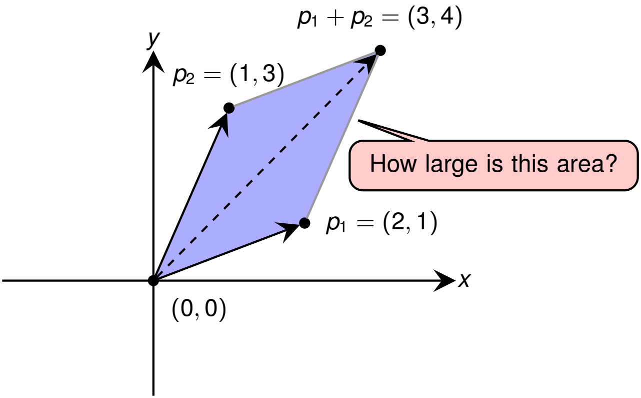

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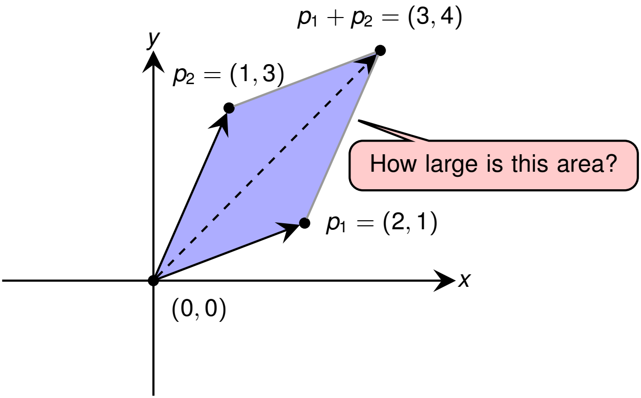
$p_2 \times p_1$

Cross Product (Area)



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$$p_2 \times p_1 = y_1 x_2 - y_2 x_1$$

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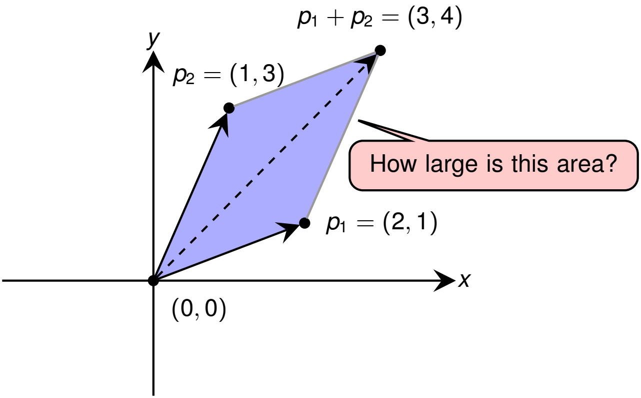

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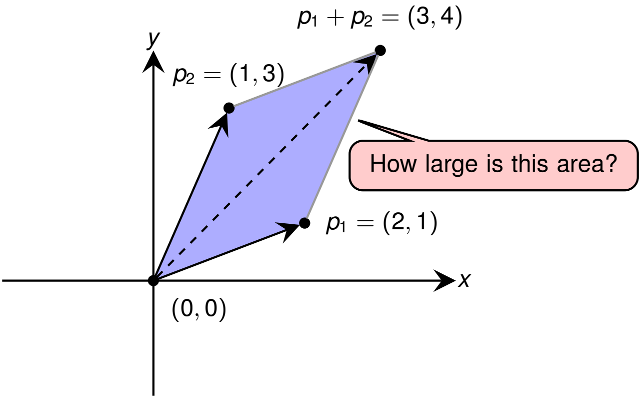

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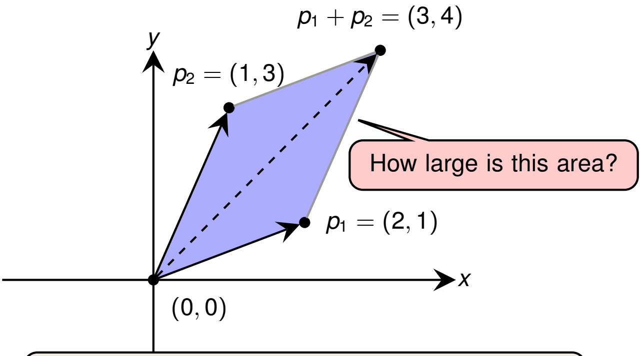

Alternatively, one could take the dot-product (but not used here): $p_1 \cdot p_2 = \|p_1\| \cdot \|p_2\| \cdot \cos(\phi)$.

$$p_1 \times p_2 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 = 2 \cdot 3 - 1 \cdot 1 = 5$$

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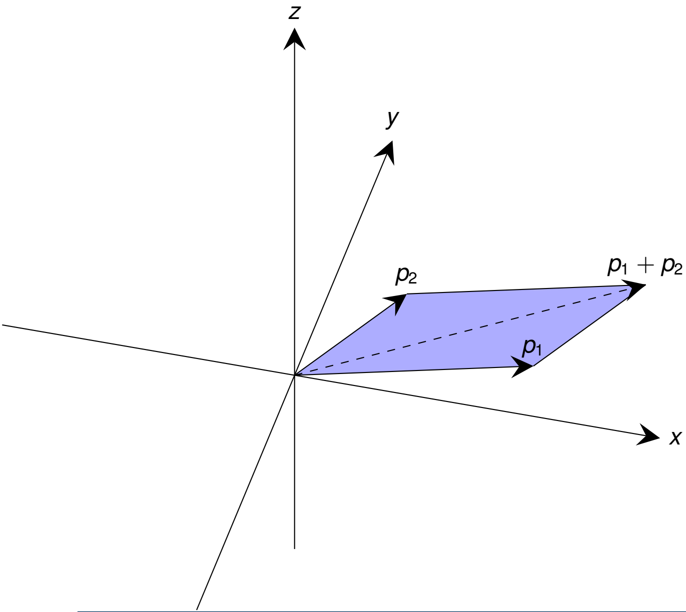
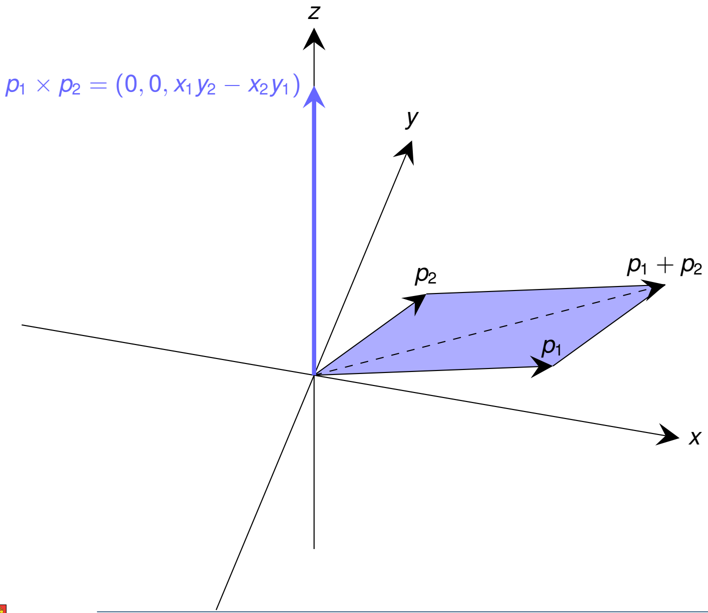
Cross Product in 3D



Cross Product in 3D



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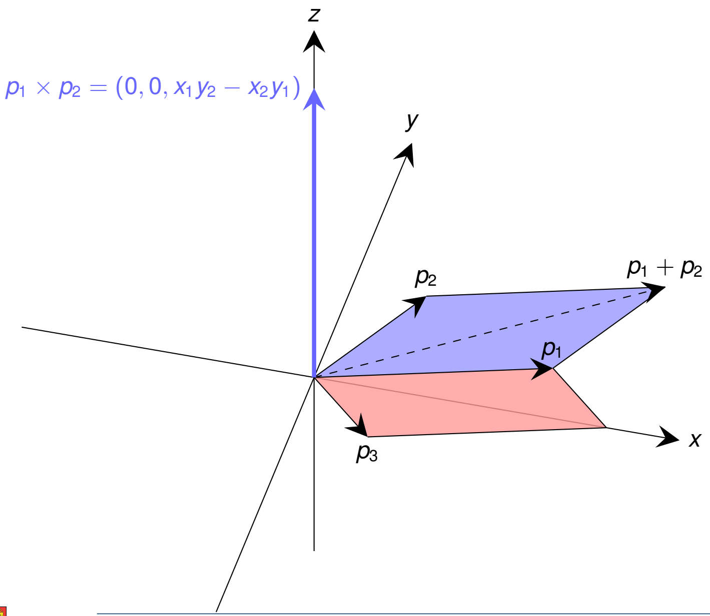

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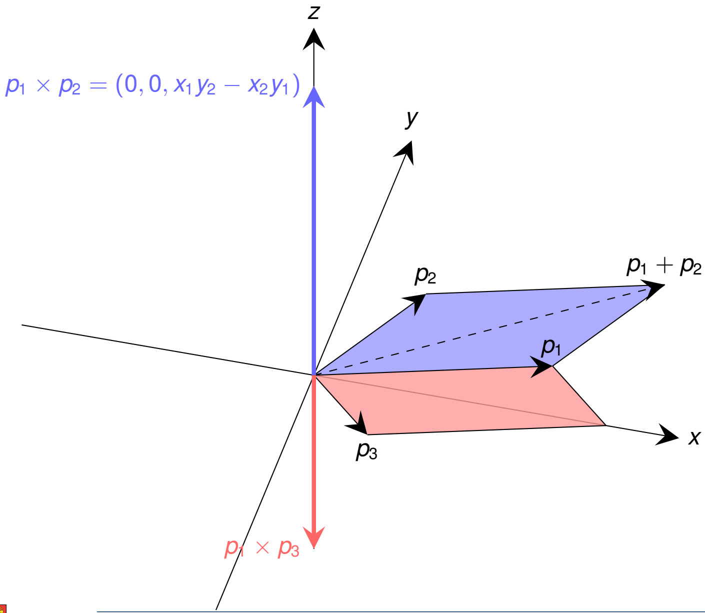

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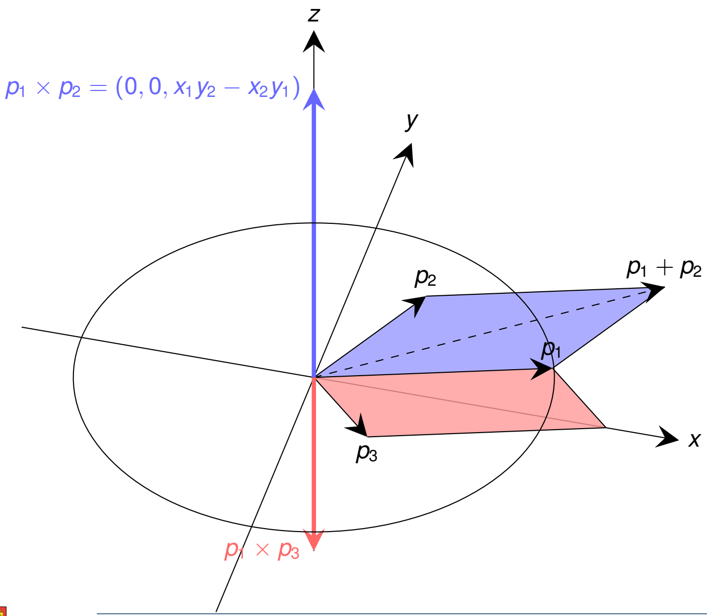

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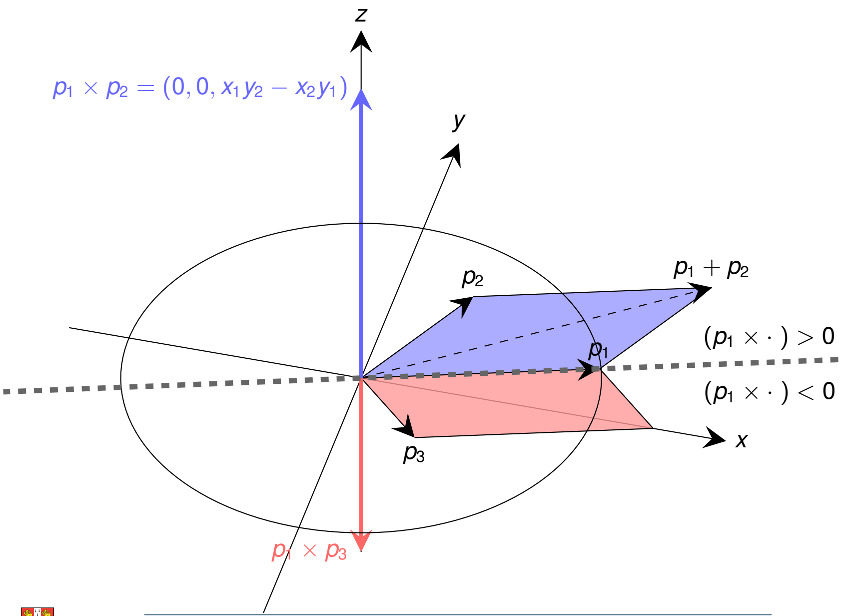

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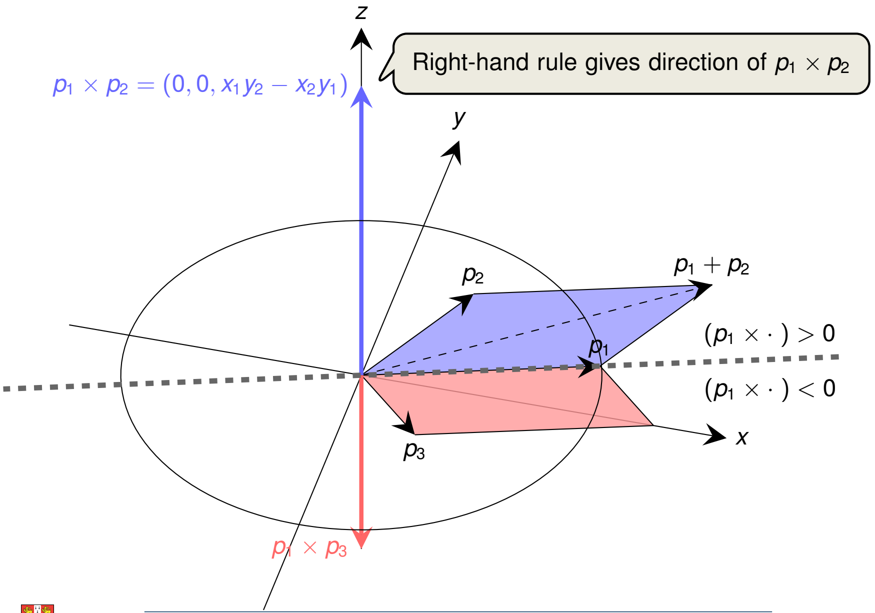
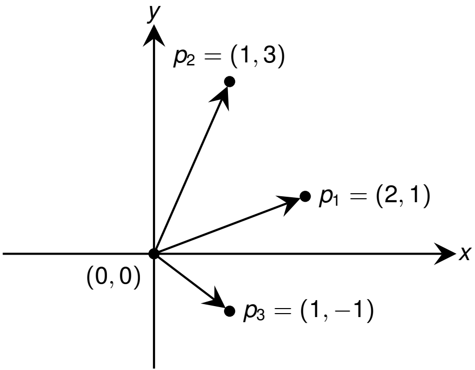


Using Cross product to determine Turns (1/2)



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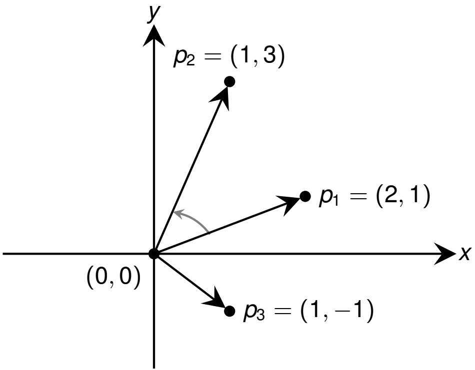

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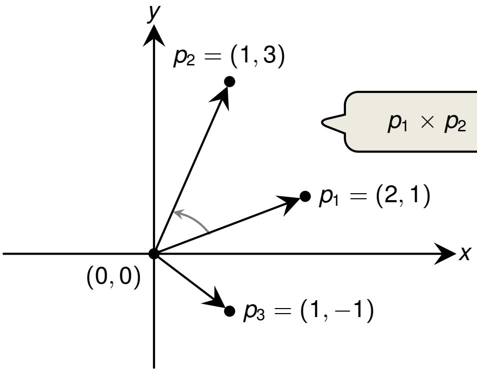

$p_1 \times p_2 > 0$: left (counterclockwise) turn

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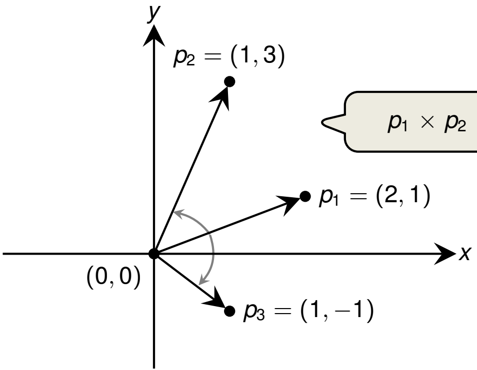

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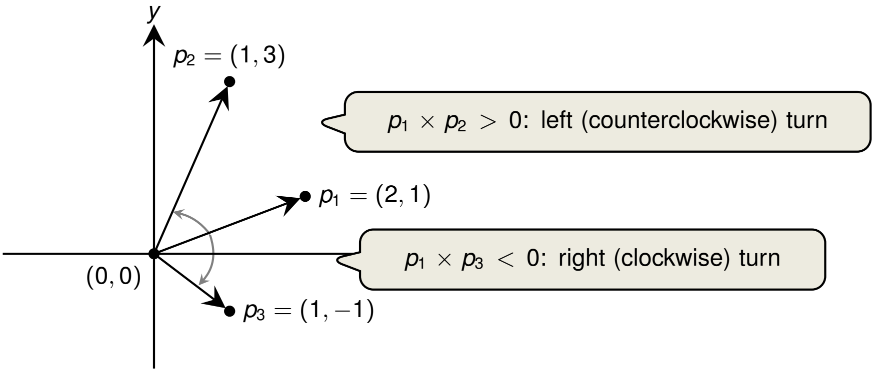

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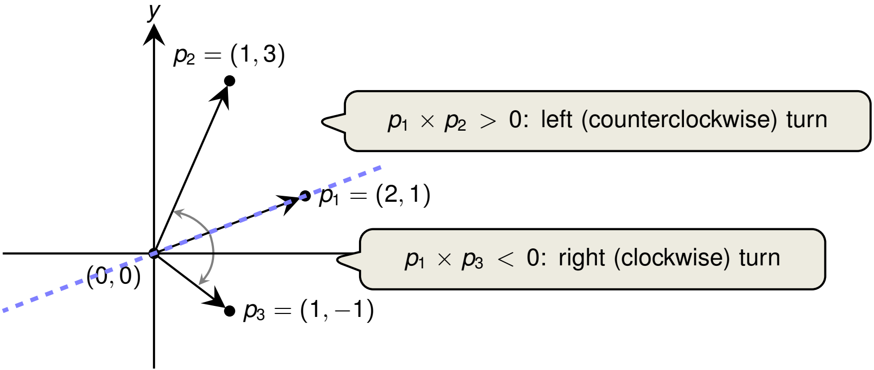

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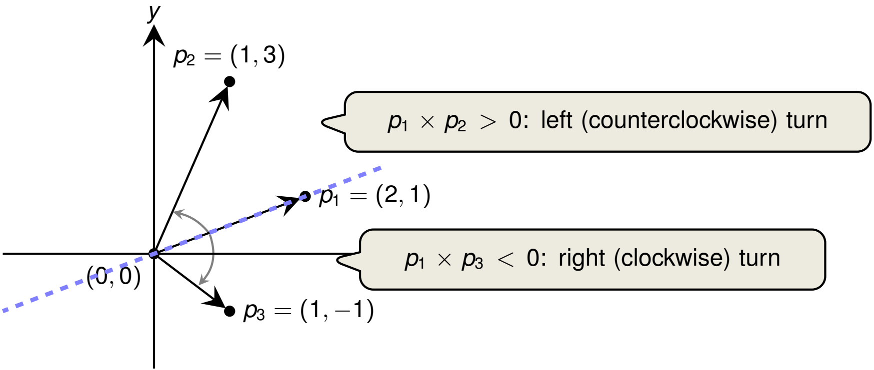
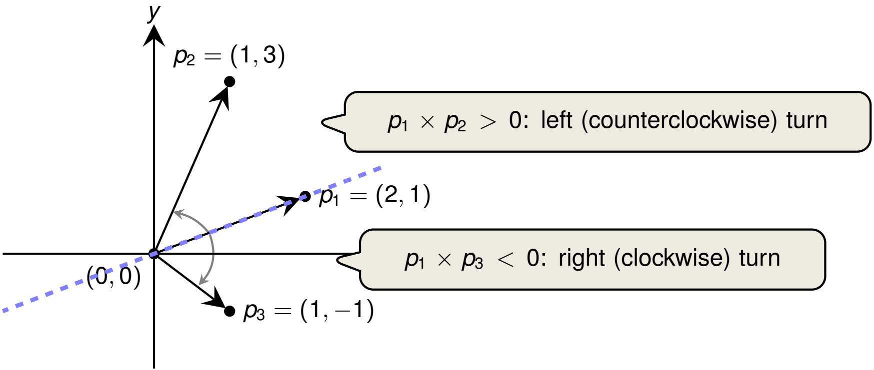


Sign of cross product determines turn!

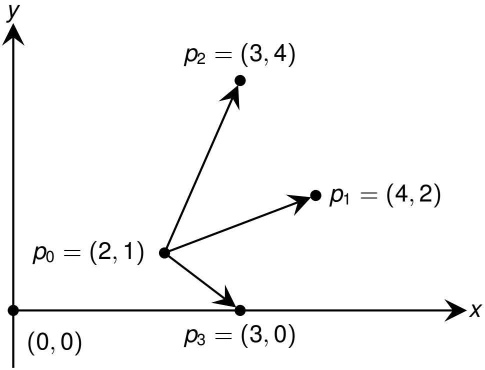
Using Cross product to determine Turns (1/2)



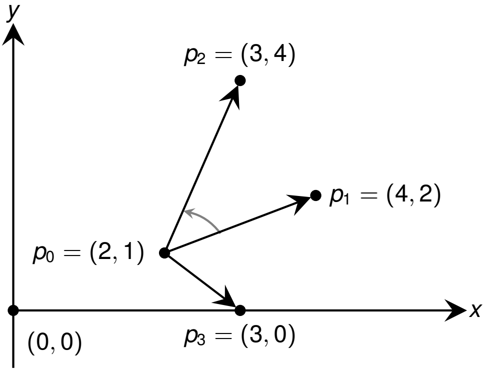
Sign of cross product determines turn!

Cross product equals zero iff vectors are colinear

Using Cross product to determine Turns (2/2)



Using Cross product to determine Turns (2/2)



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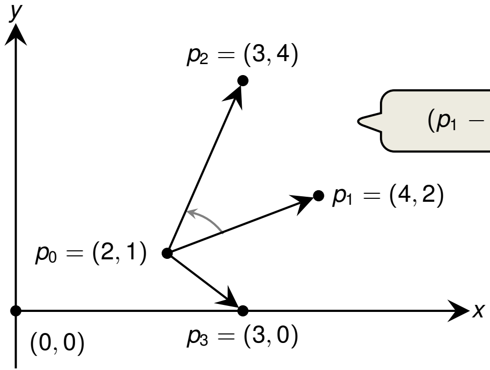
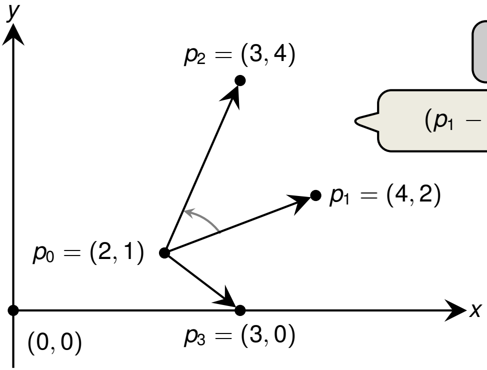


$(p_1 - p_0) \times (p_2 - p_0) > 0$: left turn

Using Cross product to determine Turns (2/2)



$(2,1) \times (1,3) = 5$

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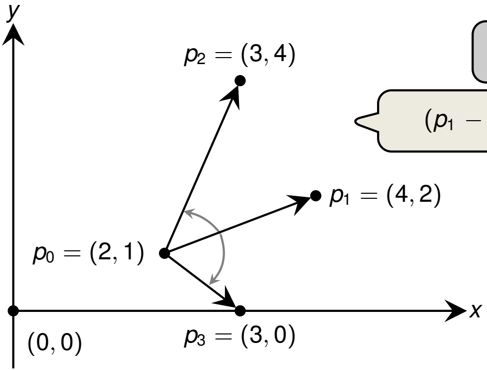

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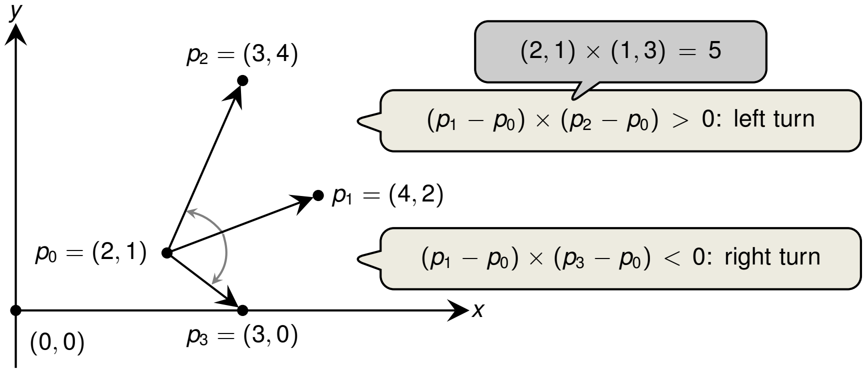

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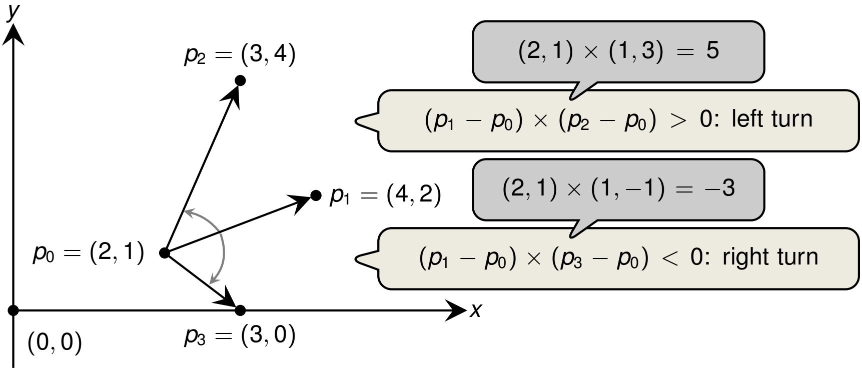

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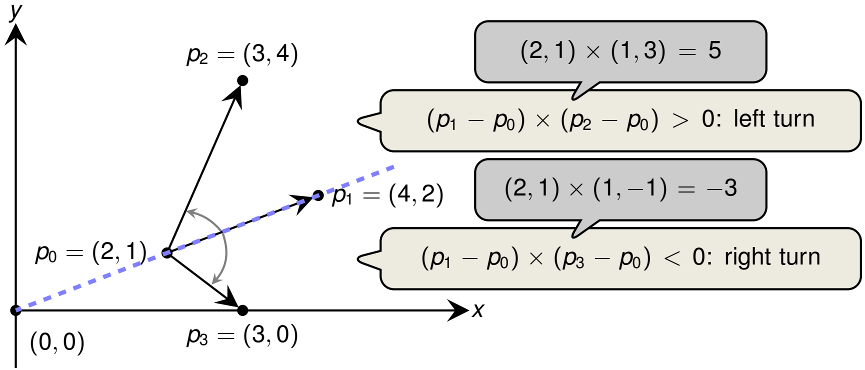
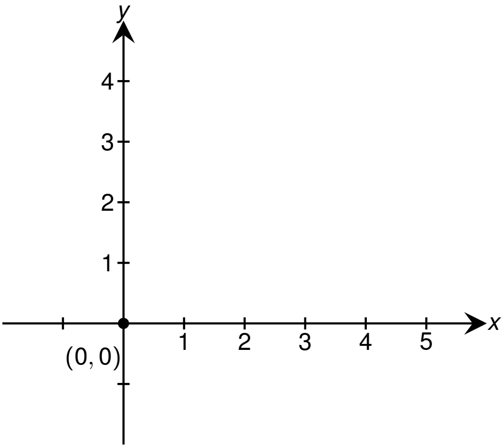


Solving Line Intersection (without Trigonometry and Division!)



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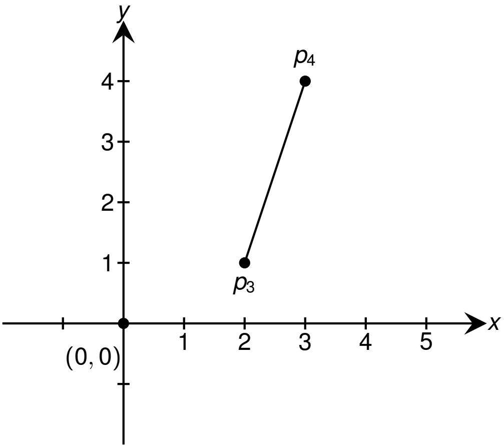

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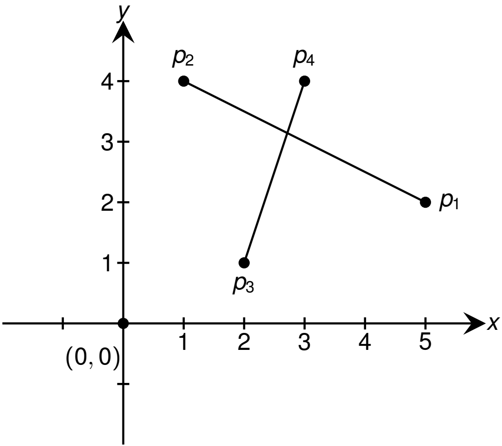

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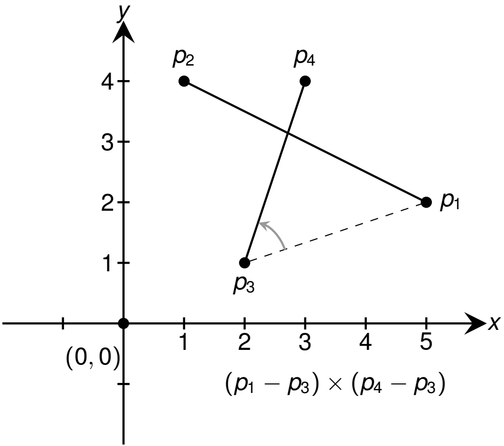
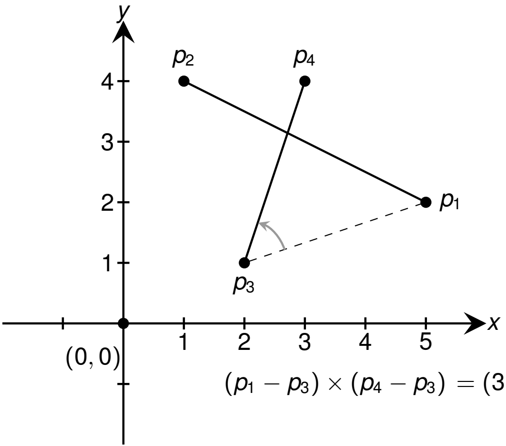


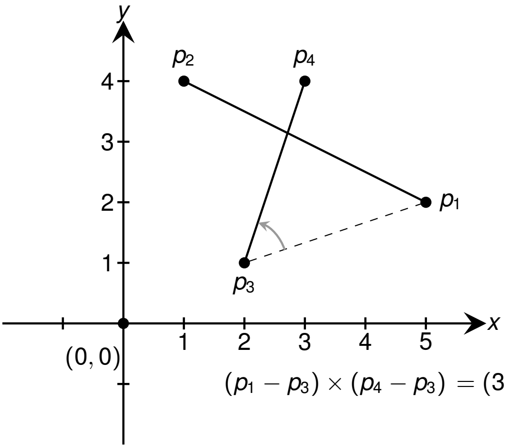
$$(p_1 - p_3) \times (p_4 - p_3)$$

Solving Line Intersection (without Trigonometry and Division!)



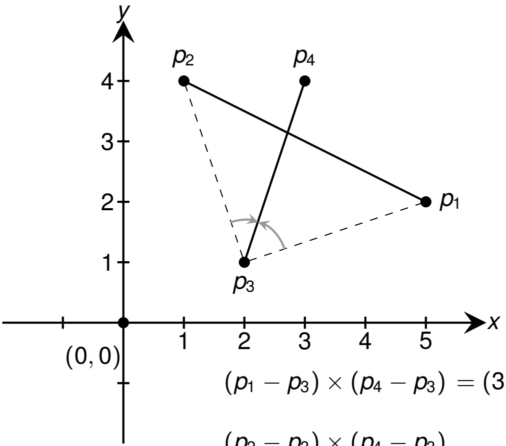
$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3)$$

Solving Line Intersection (without Trigonometry and Division!)



$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

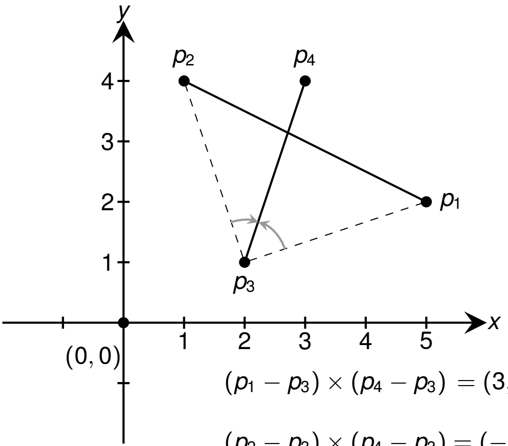
Solving Line Intersection (without Trigonometry and Division!)



$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

$$(p_2 - p_3) \times (p_4 - p_3)$$

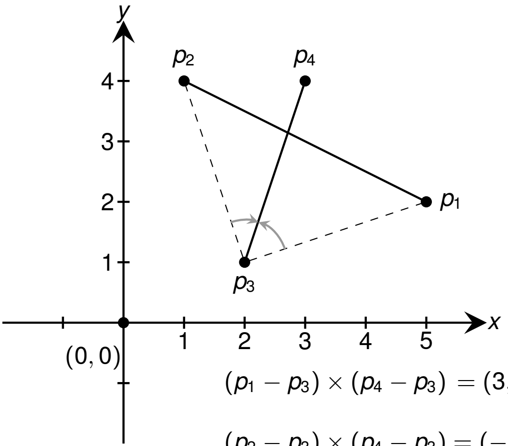
Solving Line Intersection (without Trigonometry and Division!)



$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

$$(p_2 - p_3) \times (p_4 - p_3) = (-1, 3) \times (1, 3)$$

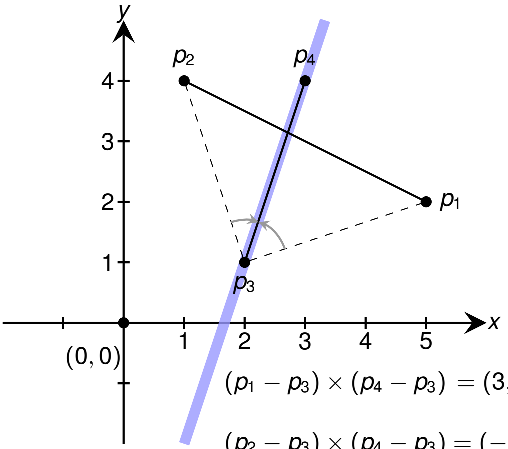
Solving Line Intersection (without Trigonometry and Division!)



$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

$$(p_2 - p_3) \times (p_4 - p_3) = (-1, 3) \times (1, 3) = -6$$

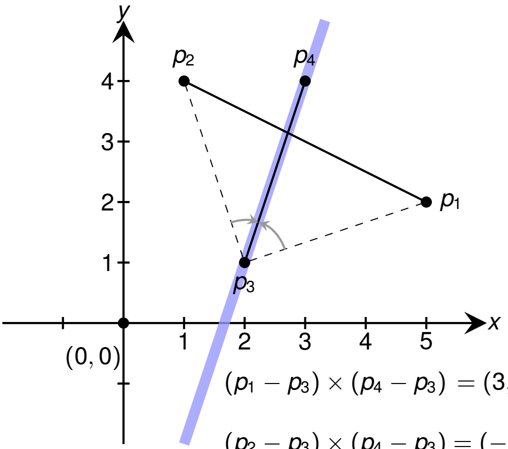
Solving Line Intersection (without Trigonometry and Division!)



$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

$$(p_2 - p_3) \times (p_4 - p_3) = (-1, 3) \times (1, 3) = -6$$

Solving Line Intersection (without Trigonometry and Division!)

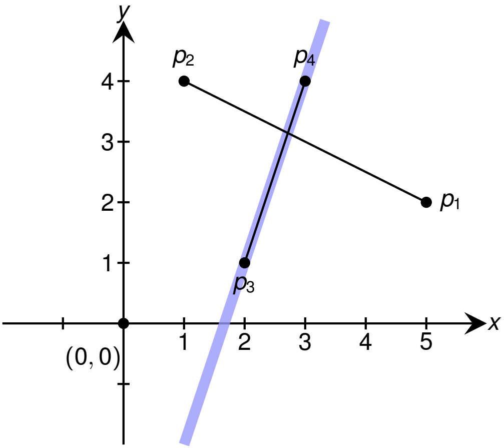


$$(p_1 - p_3) \times (p_4 - p_3) = (3, 1) \times (1, 3) = 8$$

$$(p_2 - p_3) \times (p_4 - p_3) = (-1, 3) \times (1, 3) = -6$$

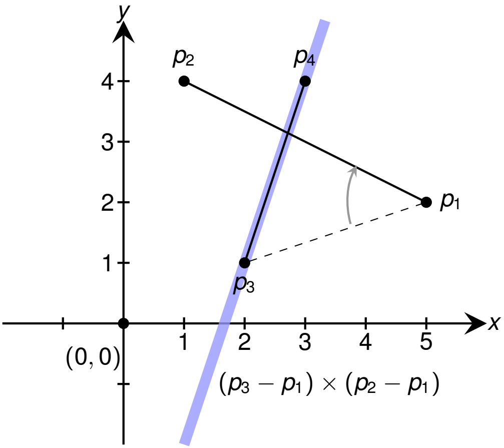
Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)



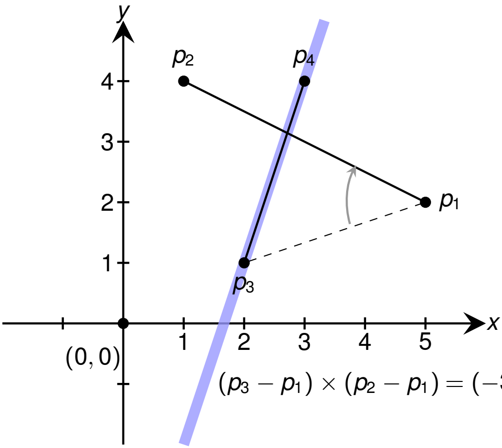
Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)



Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

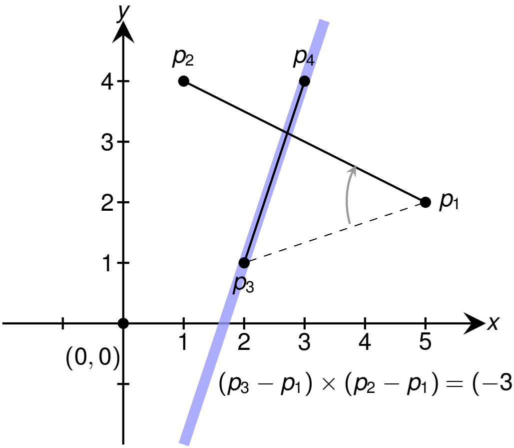
Solving Line Intersection (without Trigonometry and Division!)



$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2)$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

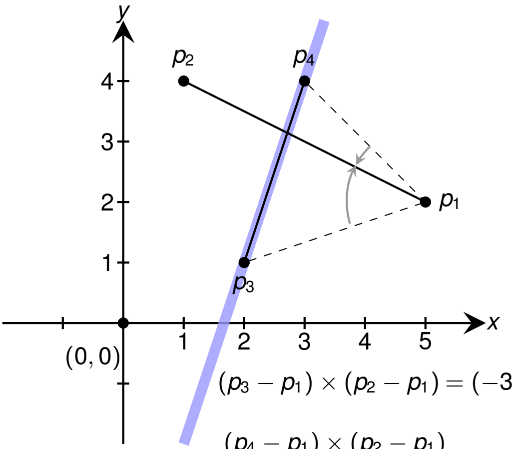
Solving Line Intersection (without Trigonometry and Division!)



$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

Opposite signs $\Rightarrow$ $\overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)

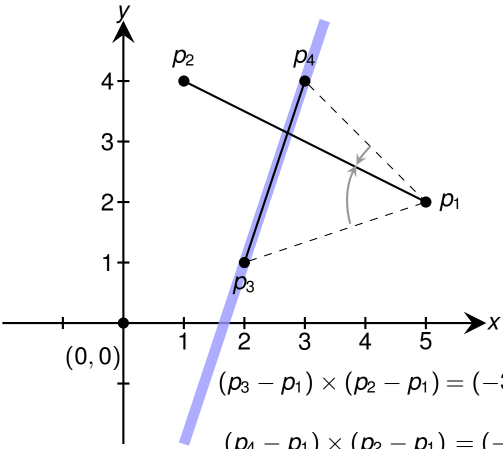


$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1)$$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)

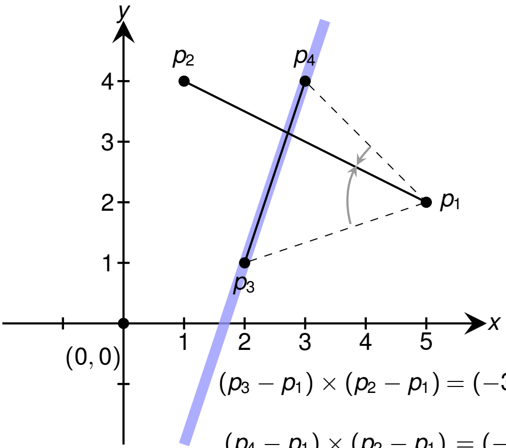


$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1) = (-2, 2) \times (-4, 2)$$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)

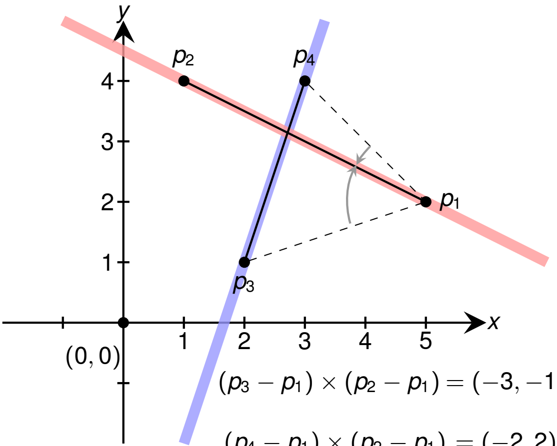


$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1) = (-2, 2) \times (-4, 2) = 4$$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Solving Line Intersection (without Trigonometry and Division!)



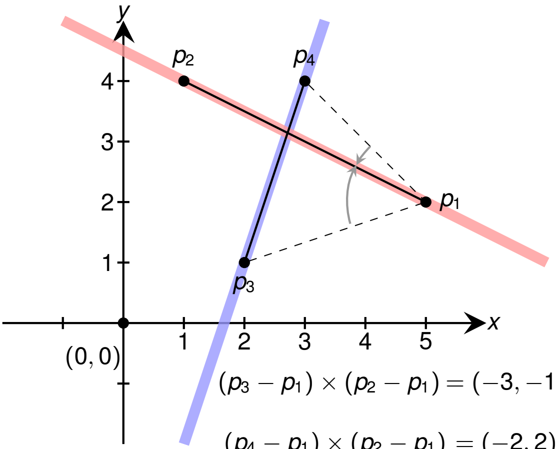
$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1) = (-2, 2) \times (-4, 2) = 4$$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs $\Rightarrow \overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

Solving Line Intersection (without Trigonometry and Division!)



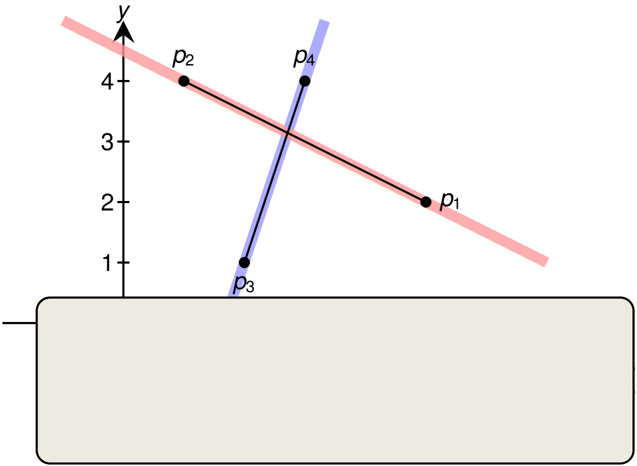
$$(p_3 - p_1) \times (p_2 - p_1) = (-3, -1) \times (-4, 2) = -10$$

$$(p_4 - p_1) \times (p_2 - p_1) = (-2, 2) \times (-4, 2) = 4$$

Opposite signs $\Rightarrow \overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs $\Rightarrow \overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

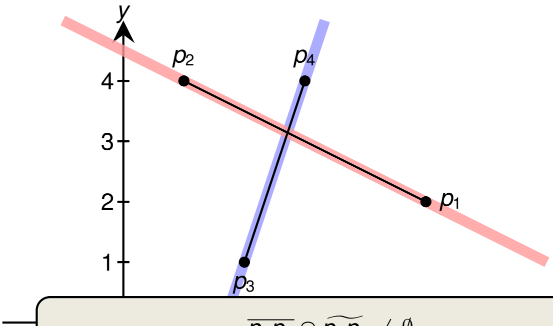
Solving Line Intersection (without Trigonometry and Division!)



Opposite signs ⇒ $\overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs ⇒ $\overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

Solving Line Intersection (without Trigonometry and Division!)

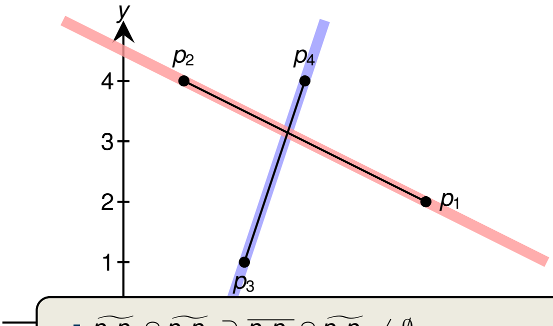


$$\overline{p_1 p_2} \cap \widetilde{p_3 p_4} \neq \emptyset$$
$$\widetilde{p_1 p_2} \cap \overline{p_3 p_4} \neq \emptyset$$

Opposite signs $\Rightarrow$ $\overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs $\Rightarrow$ $\overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

Solving Line Intersection (without Trigonometry and Division!)

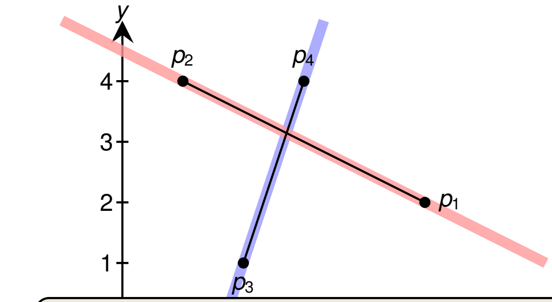


- $\widetilde{p_1 p_2} \cap \widetilde{p_3 p_4} \supseteq \overline{p_1 p_2} \cap \widetilde{p_3 p_4} \neq \emptyset$
- $\widetilde{p_1 p_2} \cap \widetilde{p_3 p_4} \supseteq \widetilde{p_1 p_2} \cap \overline{p_3 p_4} \neq \emptyset$

Opposite signs $\Rightarrow$ $\overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs $\Rightarrow$ $\overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

Solving Line Intersection (without Trigonometry and Division!)

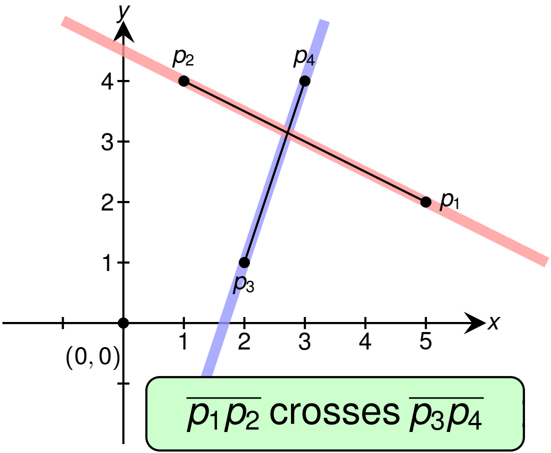


- $\widetilde{p_1p_2} \cap \widetilde{p_3p_4} \supseteq \overline{p_1p_2} \cap \widetilde{p_3p_4} \neq \emptyset$
- $\widetilde{p_1p_2} \cap \widetilde{p_3p_4} \supseteq \widetilde{p_1p_2} \cap \overline{p_3p_4} \neq \emptyset$
- Since $\widetilde{p_1p_2} \cap \widetilde{p_3p_4}$ consists of (at most) one point $\Rightarrow \overline{p_1p_2} \cap \overline{p_3p_4} \neq \emptyset$

Opposite signs $\Rightarrow \overline{p_1p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs $\Rightarrow \overline{p_3p_4}$ crosses (infinite) line through p_1 and p_2

Solving Line Intersection (without Trigonometry and Division!)

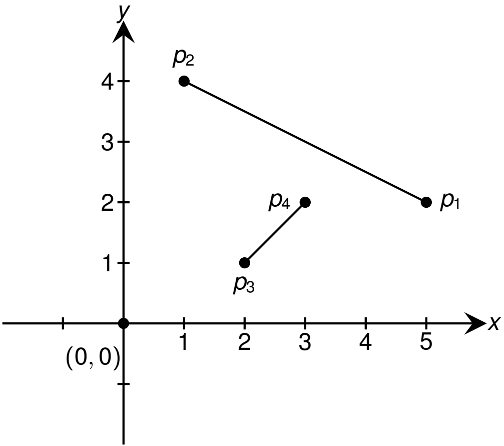


$\overline{p_1 p_2}$ crosses $\overline{p_3 p_4}$

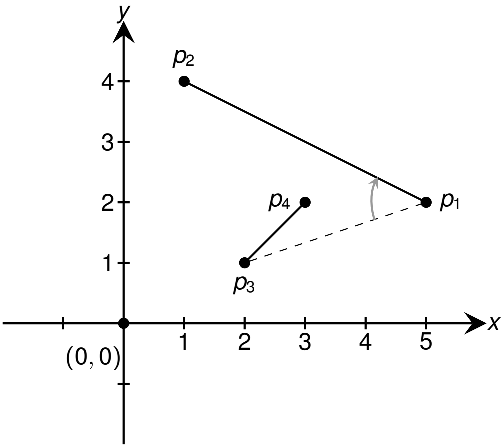
Opposite signs ⇒ $\overline{p_1 p_2}$ crosses (infinite) line through p_3 and p_4

Opposite signs ⇒ $\overline{p_3 p_4}$ crosses (infinite) line through p_1 and p_2

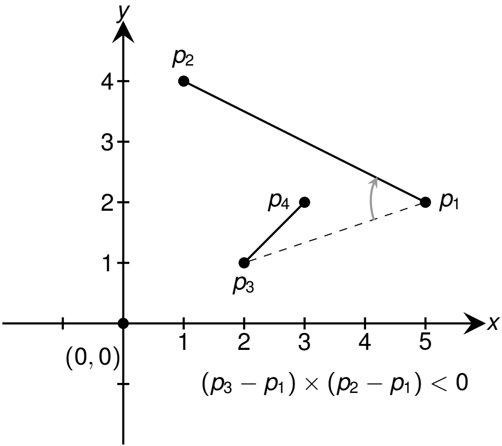
Solving Line Intersection (without Trigonometry and Division!)



Solving Line Intersection (without Trigonometry and Division!)

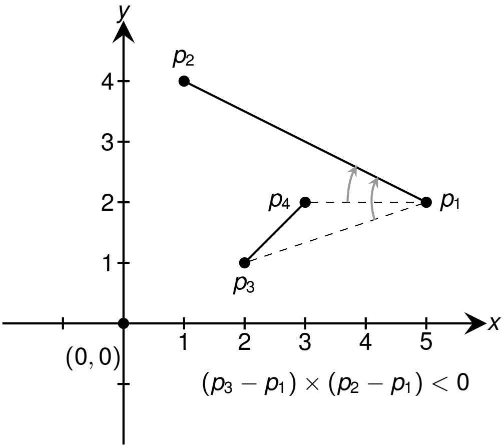


Solving Line Intersection (without Trigonometry and Division!)



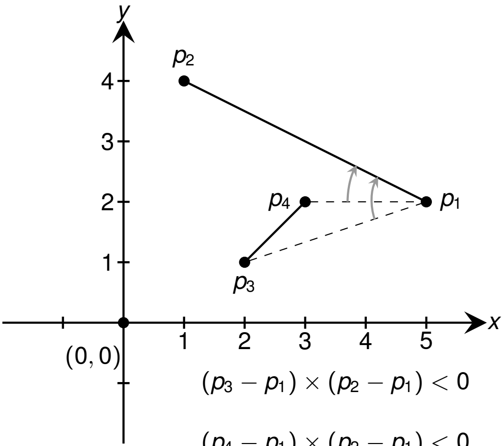
$$(p_3 - p_1) \times (p_2 - p_1) < 0$$

Solving Line Intersection (without Trigonometry and Division!)



$$(p_3 - p_1) \times (p_2 - p_1) < 0$$

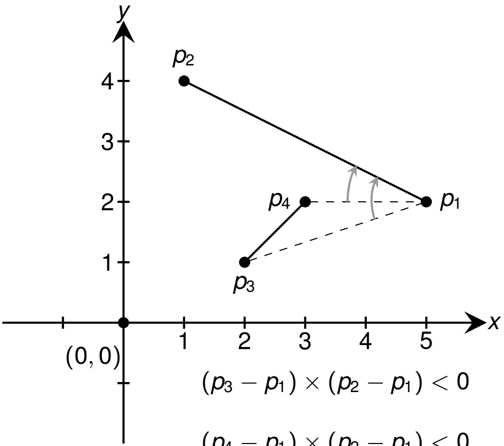
Solving Line Intersection (without Trigonometry and Division!)



$$(p_3 - p_1) \times (p_2 - p_1) < 0$$

$$(p_4 - p_1) \times (p_2 - p_1) < 0$$

Solving Line Intersection (without Trigonometry and Division!)

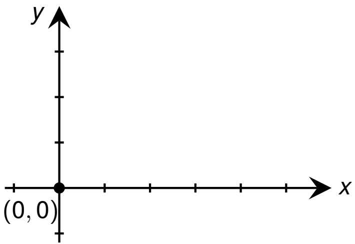


$$(p_3 - p_1) \times (p_2 - p_1) < 0$$

$$(p_4 - p_1) \times (p_2 - p_1) < 0$$

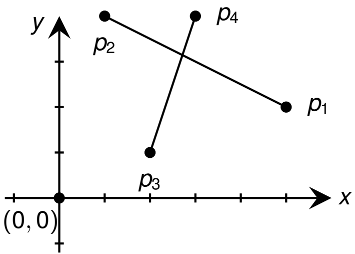
$\overline{p_1 p_2}$ does **not** cross $\overline{p_3 p_4}$

Solving Line Intersection



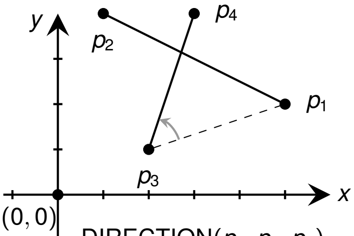

```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```

Solving Line Intersection



0: DIRECTION(p_i, p_j, p_k)
1: return $(p_k - p_i) \times (p_j - p_i)$

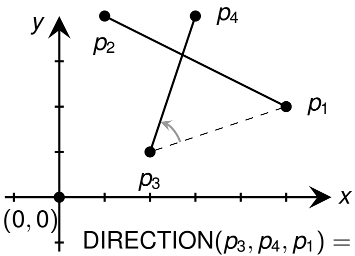
Solving Line Intersection



$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

DIRECTION(p_i, p_j, p_k)
  return $(p_k - p_i) \times (p_j - p_i)$

Solving Line Intersection

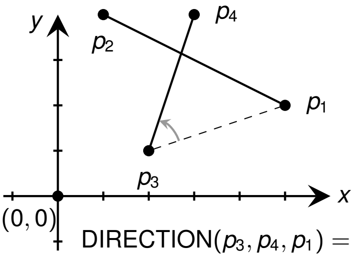


$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```

```
SEGMENTS-INTERSECT(p_i, p_j, p_k)
    d_1 = DIRECTION(p_3, p_4, p_1)
    d_2 = DIRECTION(p_3, p_4, p_2)
    d_3 = DIRECTION(p_1, p_2, p_3)
    d_4 = DIRECTION(p_1, p_2, p_4)
    If d_1 · d_2 < 0 and d_3 · d_4 < 0 return TRUE
        · · · (handle all degenerate cases)
```

Solving Line Intersection

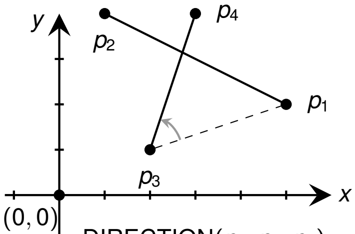


$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

DIRECTION(p_i, p_j, p_k)
    return $(p_k - p_i) \times (p_j - p_i)$

SEGMENTS-INTERSECT(p_i, p_j, p_k)
    $d_1 =$ DIRECTION(p_3, p_4, p_1)
    $d_2 =$ DIRECTION(p_3, p_4, p_2)
    $d_3 =$ DIRECTION(p_1, p_2, p_3)
    $d_4 =$ DIRECTION(p_1, p_2, p_4)
    If $d_1 \cdot d_2 < 0$ and $d_3 \cdot d_4 < 0$ return TRUE
        $\cdots$ (handle all degenerate cases)

Solving Line Intersection



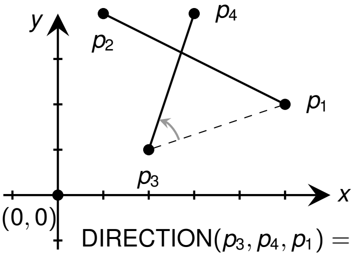
$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```

```
SEGMENTS-INTERSECT(p_i, p_j, p_k)
    d_1 = DIRECTION(p_3, p_4, p_1)
    d_2 = DIRECTION(p_3, p_4, p_2)
    d_3 = DIRECTION(p_1, p_2, p_3)
    d_4 = DIRECTION(p_1, p_2, p_4)
    If d_1 · d_2 < 0 and d_3 · d_4 < 0 return TRUE
        ··· (handle all degenerate cases)
```

In total 4 satisfying conditions!

Solving Line Intersection



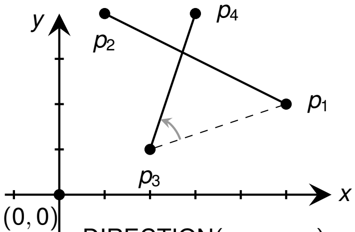
$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```

```
SEGMENTS-INTERSECT(p_i, p_j, p_k)
    d_1 = DIRECTION(p_3, p_4, p_1)
    d_2 = DIRECTION(p_3, p_4, p_2)
    d_3 = DIRECTION(p_1, p_2, p_3)
    d_4 = DIRECTION(p_1, p_2, p_4)
    If d_1 · d_2 < 0 and d_3 · d_4 < 0 return TRUE
        · · · (handle all degenerate cases)
```

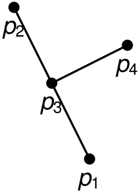
Lines could touch or be colinear

Solving Line Intersection



$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

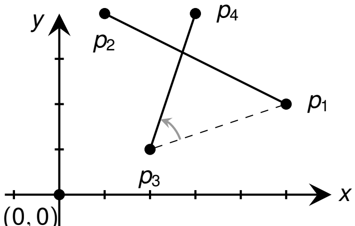
```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```




```
SEGMENTS-INTERSECT(p_i, p_j, p_k)
    d_1 = DIRECTION(p_3, p_4, p_1)
    d_2 = DIRECTION(p_3, p_4, p_2)
    d_3 = DIRECTION(p_1, p_2, p_3)
    d_4 = DIRECTION(p_1, p_2, p_4)
    If d_1 · d_2 < 0 and d_3 · d_4 < 0 return TRUE
        · · · (handle all degenerate cases)
```

Lines could touch or be colinear

Solving Line Intersection



$$\text{DIRECTION}(p_3, p_4, p_1) = (p_1 - p_3) \times (p_4 - p_3)$$

```
DIRECTION(p_i, p_j, p_k)
    return (p_k − p_i) × (p_j − p_i)
```

```
SEGMENTS-INTERSECT(p_i, p_j, p_k)
    d_1 = DIRECTION(p_3, p_4, p_1)
    d_2 = DIRECTION(p_3, p_4, p_2)
    d_3 = DIRECTION(p_1, p_2, p_3)
    d_4 = DIRECTION(p_1, p_2, p_4)
    If d_1 · d_2 < 0 and d_3 · d_4 < 0 return TRUE
        ··· (handle all degenerate cases)
```

Lines could touch or be colinear