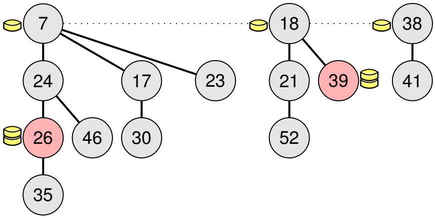

5.2 Fibonacci Heaps (Analysis)

Frank Stajano Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Outline

Glimpse at the Analysis

Amortized Analysis

Bounding the Maximum Degree

Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$
- DECREASE-KEY: actual $\mathcal{O}(\text{\# cuts}) \leq \mathcal{O}(\text{marks}(H))$

Amortized Analysis via Potential Method

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- DECREASE-KEY: actual $\mathcal{O}(\#\text{ cuts}) \leq \mathcal{O}(\text{marks}(H))$

$$\boxed{\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)}$$

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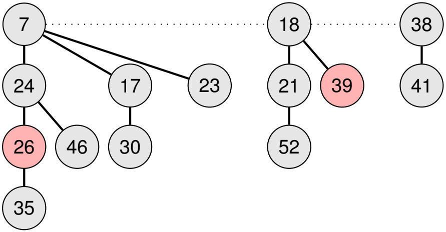

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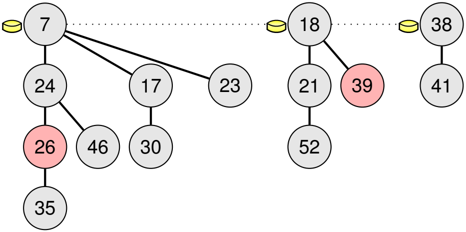

Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$
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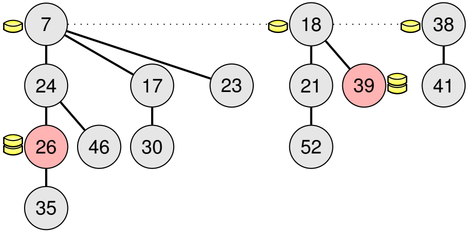

Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$ amortized $\mathcal{O}(d(n))$
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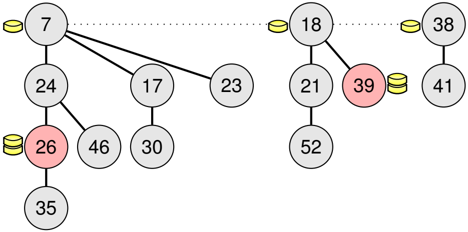
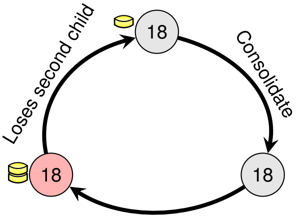
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$



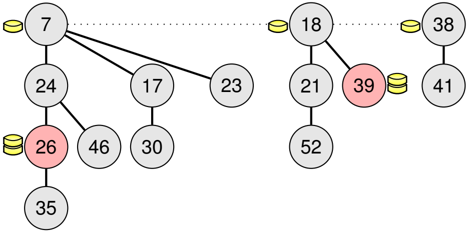
Lifecycle of a node



Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$ ✓
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$ amortized $\mathcal{O}(d(n))$?
- DECREASE-KEY: actual $\mathcal{O}(\#\text{ cuts}) \leq \mathcal{O}(\text{marks}(H))$ amortized $\mathcal{O}(1)$?

$$\boxed{\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)}$$



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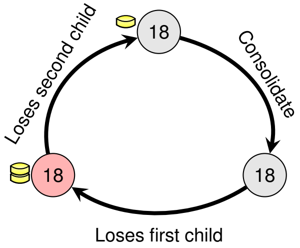

Outline

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Amortized Analysis of DECREASE-KEY

Actual Cost

- DECREASE-KEY: $\mathcal{O}(x + 1)$, where x is the number of cuts.

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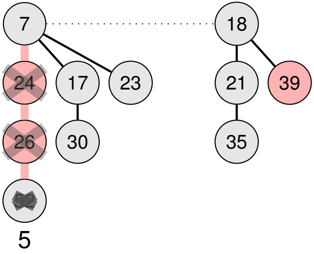
Change in Potential



Amortized Analysis of DECREASE-KEY

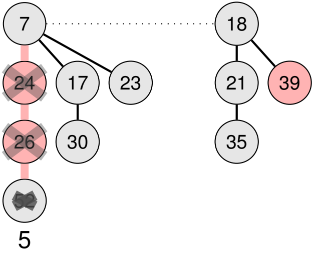
Actual Cost

- DECREASE-KEY: $\mathcal{O}(x + 1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{trees}(H') =$



Amortized Analysis of DECREASE-KEY

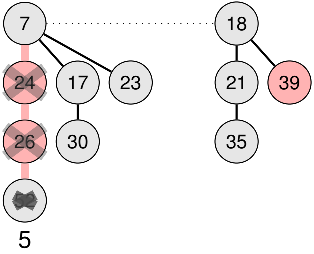
Actual Cost

- DECREASE-KEY: $\mathcal{O}(x + 1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$



Amortized Analysis of DECREASE-KEY

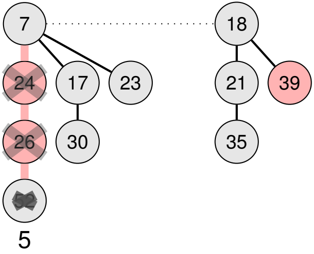
Actual Cost

- DECREASE-KEY: $\mathcal{O}(x + 1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq$



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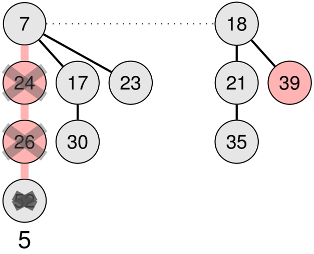
Actual Cost

- DECREASE-KEY: $\mathcal{O}(x+1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq \text{marks}(H) - x + 2$



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Actual Cost

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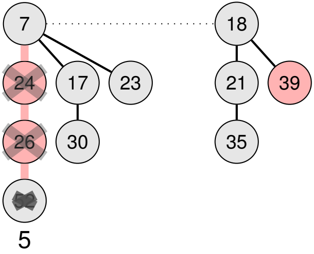
- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq \text{marks}(H) - x + 2$

$\Rightarrow \Delta\Phi \leq x + 2 \cdot (-x + 2) = 4 - x.$



Amortized Analysis of DECREASE-KEY

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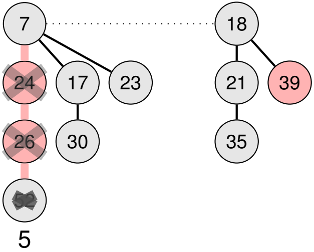


Amortized Cost

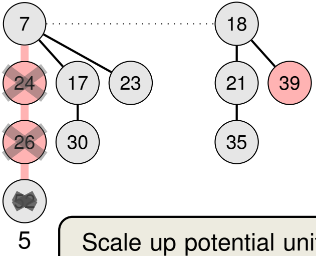
$$\widetilde{c}_i = c_i + \Delta\Phi$$

Amortized Analysis of Decrease-Key

Actual Cost

- Decrease-Key: $\mathcal{O}(x+1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$



Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq \text{marks}(H) - x + 2$

$\Rightarrow$ $\Delta\Phi \leq x + 2 \cdot (-x+2) = 4 - x.$

Scale up potential units

Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(x+1) + 4 - x$$

Amortized Analysis of DECREASE-KEY

Actual Cost

- DECREASE-KEY: $\mathcal{O}(x + 1)$, where x is the number of cuts.

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

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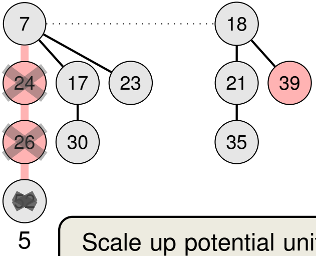


Scale up potential units

Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(x + 1) + 4 - x = \mathcal{O}(1)$$

Amortized Analysis of DECREASE-KEY

Actual Cost

- DECREASE-KEY: $\mathcal{O}(x+1)$, where x is the number of cuts.

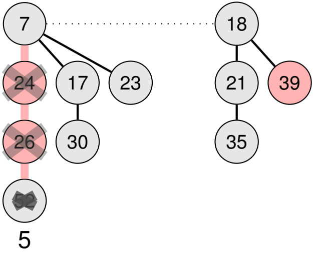
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

First Coin $\rightsquigarrow$ pays cut
Second Coin $\rightsquigarrow$ increase of trees(H)

Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq \text{marks}(H) - x + 2$

$\Rightarrow \Delta\Phi \leq x + 2 \cdot (-x + 2) = 4 - x.$



Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(x+1) + 4 - x = \mathcal{O}(1)$$

Amortized Analysis of EXTRACT-MIN

Actual Cost

- EXTRACT-MIN: $\mathcal{O}(\text{trees}(H) + d(n))$

Amortized Analysis of EXTRACT-MIN

Actual Cost

- EXTRACT-MIN: $\mathcal{O}(\text{trees}(H) + d(n))$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

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$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

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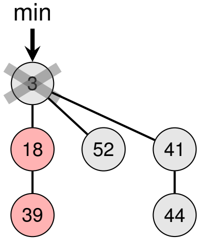
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H')$? $\text{marks}(H)$



Amortized Analysis of Extract-Min

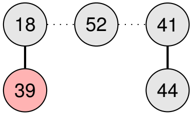
Actual Cost

- Extract-Min: $\mathcal{O}(\text{trees}(H) + d(n))$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H')$? $\text{marks}(H)$



Amortized Analysis of EXTRACT-MIN

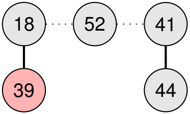
Actual Cost

- EXTRACT-MIN: $\mathcal{O}(\text{trees}(H) + d(n))$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$



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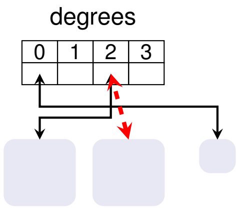
Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq$



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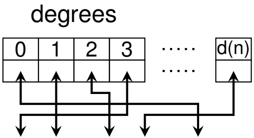
- $\text{trees}(H') \leq$



Amortized Analysis of EXTRACT-MIN

Actual Cost

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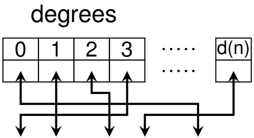
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq d(n) + 1$



Amortized Analysis of EXTRACT-MIN

Actual Cost

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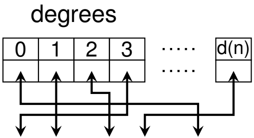
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq \ d(n) + 1$

$\Rightarrow \ \Delta\Phi \leq d(n) + 1 - \text{trees}(H)$



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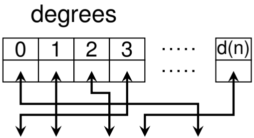

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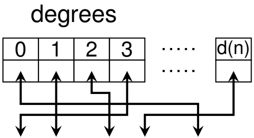


Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(\text{trees(H)} + d(n)) + d(n) + 1 - \text{trees(H)}$$

Amortized Analysis of EXTRACT-MIN

Actual Cost

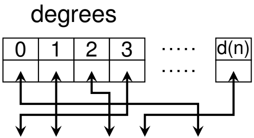
- EXTRACT-MIN: $\mathcal{O}(\text{trees}(H) + d(n))$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq d(n) + 1$

$\Rightarrow$ $\Delta\Phi \leq d(n) + 1 - \text{trees}(H)$



Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(\text{trees(H)} + d(n)) + d(n) + 1 - \text{trees(H)} = \mathcal{O}(d(n))$$

Actual Cost

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- $\text{marks}(H') \leq \text{marks}(H)$
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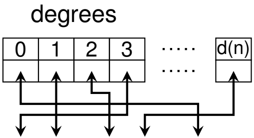

Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(\text{trees(H)} + d(n)) + d(n) + 1 - \text{trees(H)} = \mathcal{O}(d(n))$$

How to bound $d(n)$?

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Bounding the Maximum Degree

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Binomial Heap

Every tree is a binomial tree $\Rightarrow d(n) \leq \log_2 n$.

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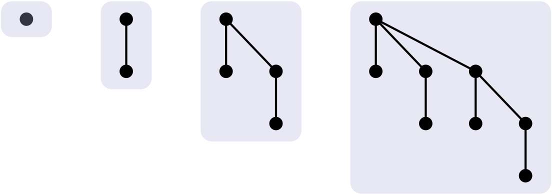

Bounding the Maximum Degree

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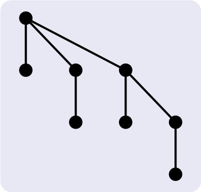
Every tree is a binomial tree $\Rightarrow d(n) \leq \log_2 n$.



$d = 3, n = 2^3$

Bounding the Maximum Degree

Binomial Heap

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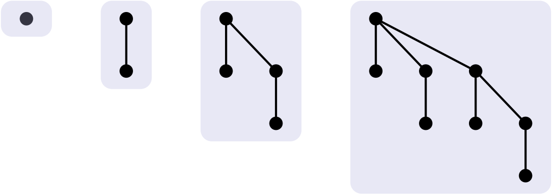

Bounding the Maximum Degree

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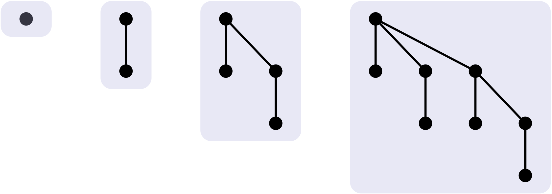

Fibonacci Heap

Not all trees are binomial trees, but still $d(n) \leq \log_\varphi n$, where $\varphi \approx 1.62$.

Lower Bounding Degrees of Children

$$d(n) \leq \log_{\varphi} n$$

Lower Bounding Degrees of Children

We will prove a stronger statement:
Any tree with degree k contains at least φ^k nodes.

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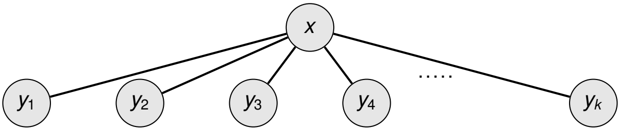
- Consider any node x of degree k (not necessarily a root) at the final state

Lower Bounding Degrees of Children

We will prove a stronger statement:
Any tree with degree k contains at least φ^k nodes.

$$d(n) \leq \log_{\varphi} n$$

- Consider any node x of degree k (not necessarily a root) at the final state
- Let $y_1, y_2, \ldots, y_k$ be the children in the order of attachment



Lower Bounding Degrees of Children

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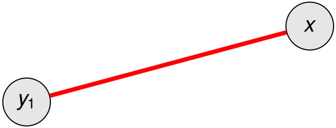

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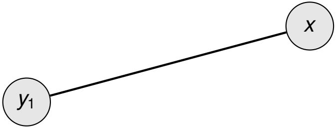

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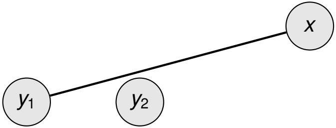

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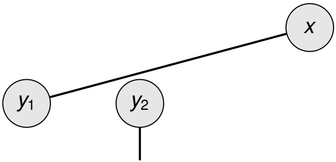

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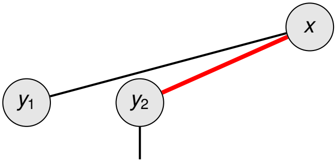

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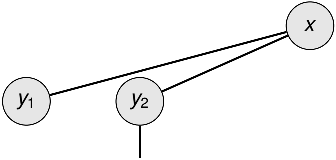

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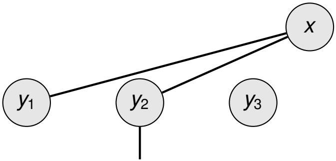

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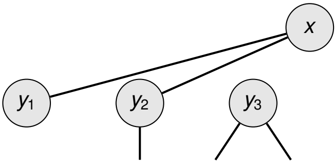

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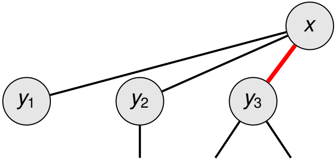

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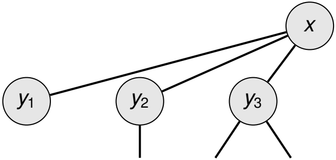

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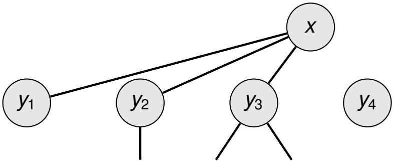

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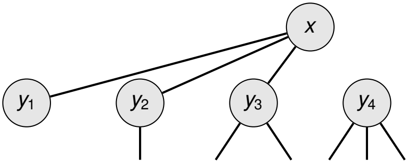

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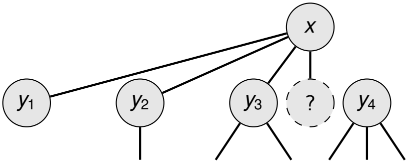

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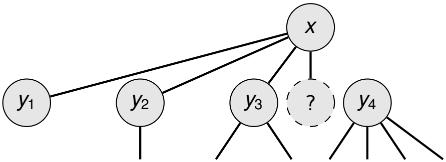

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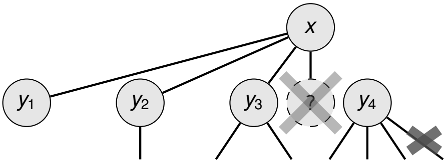

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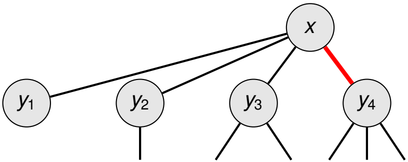

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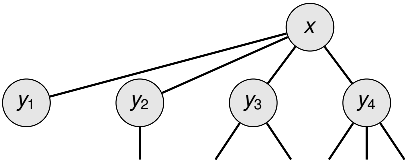

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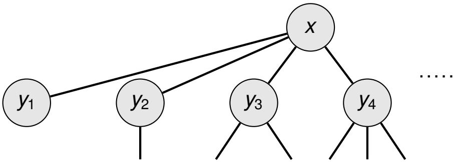

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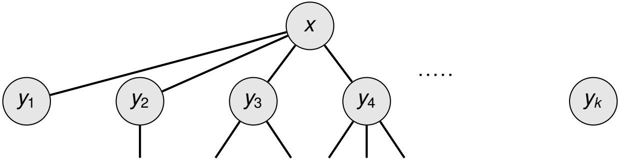

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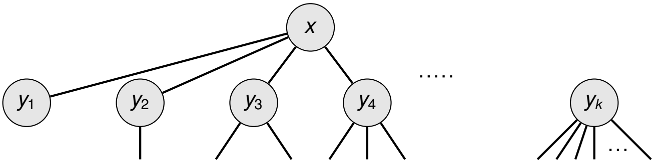

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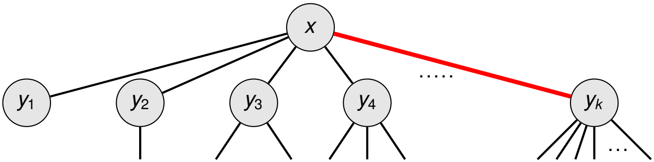

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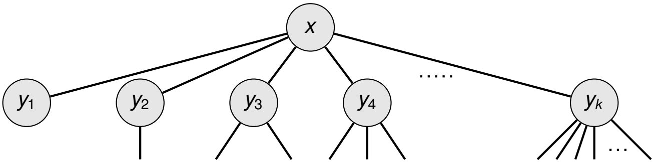

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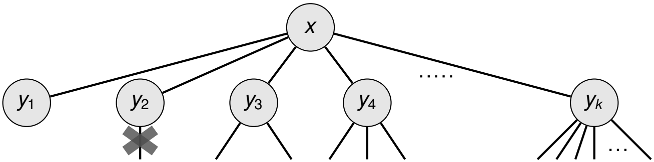

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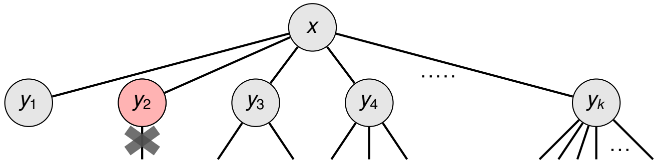

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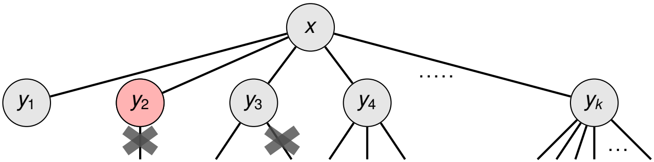

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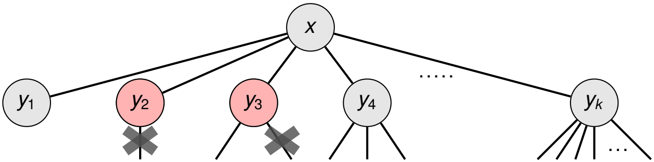

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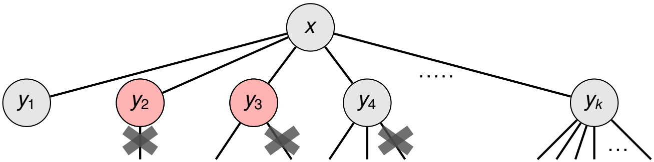

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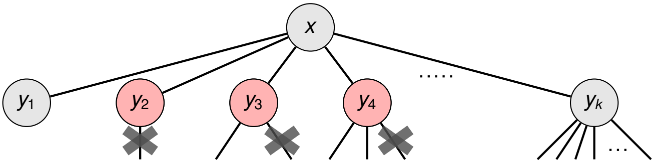

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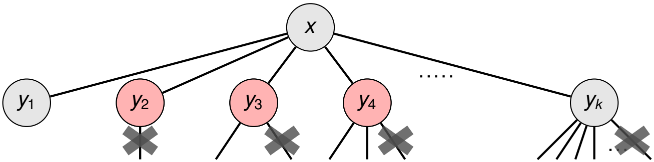

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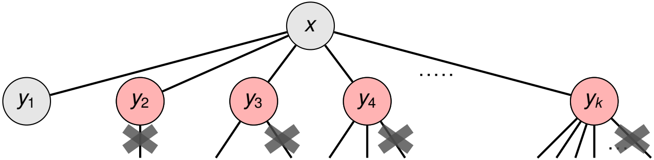

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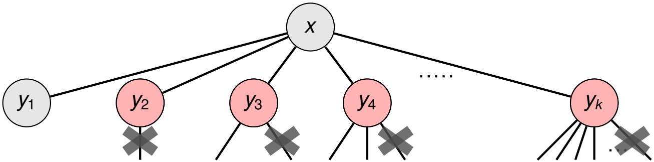

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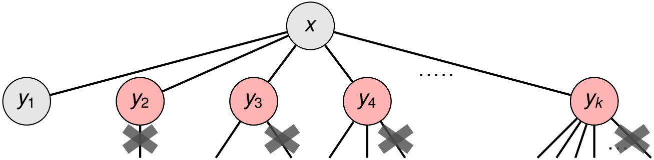
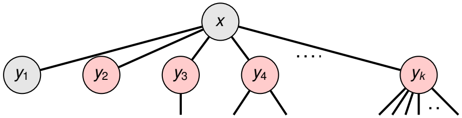
$\Rightarrow$ $\boxed{\forall 1 \leq i \leq k: \quad d_i \geq i - 2}$



From Degrees to Minimum Subtree Sizes



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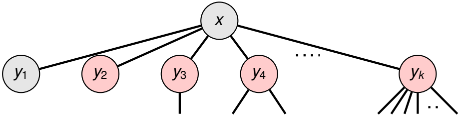

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Definition

Let $N(k)$ be the minimum possible number of nodes of a subtree rooted at a node of degree k.

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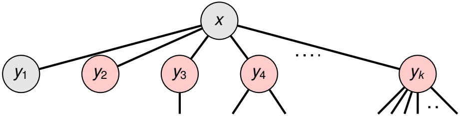

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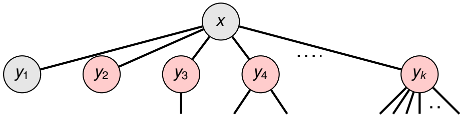

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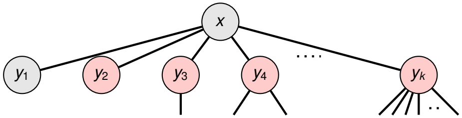

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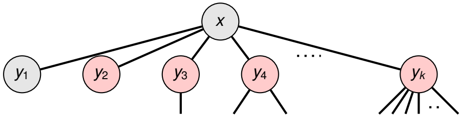

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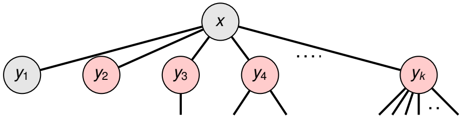

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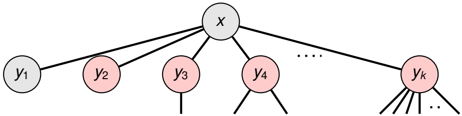

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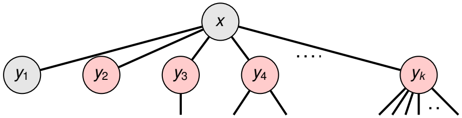

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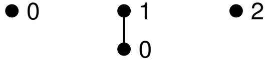
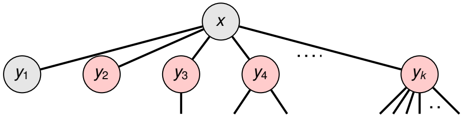


From Degrees to Minimum Subtree Sizes



$$\forall 1 \leq i \leq k: \quad d_i \geq i - 2$$

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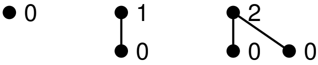

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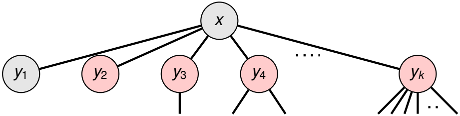

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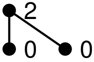
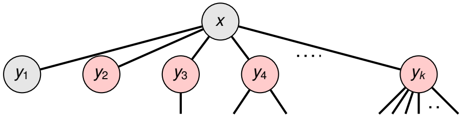


$N(3)$

From Degrees to Minimum Subtree Sizes



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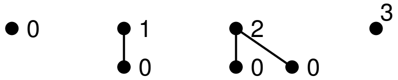

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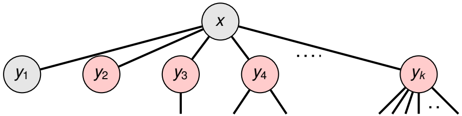

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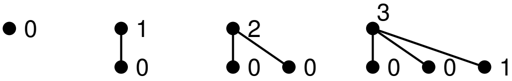

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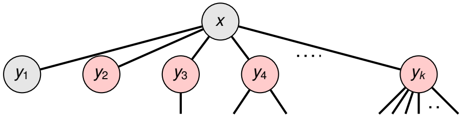

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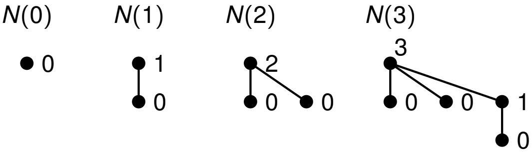

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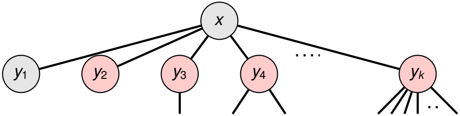

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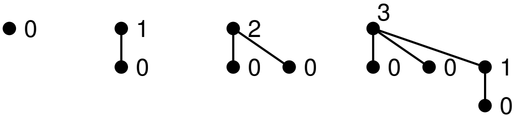

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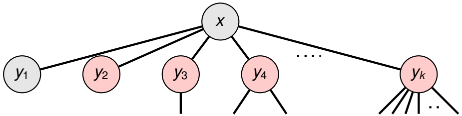

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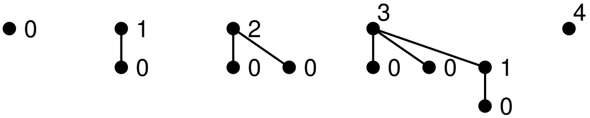

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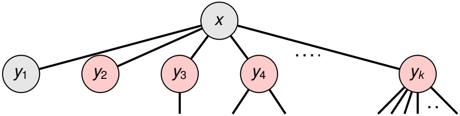

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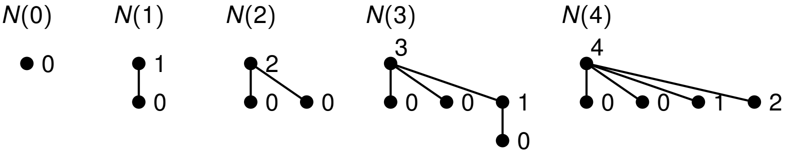

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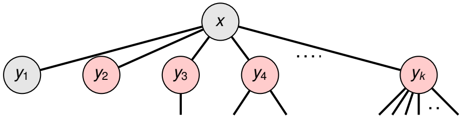

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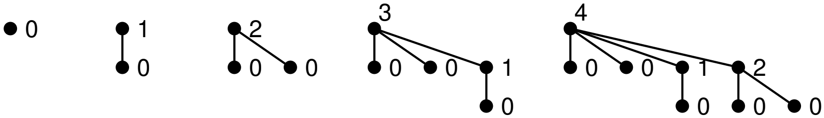

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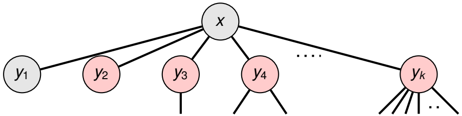

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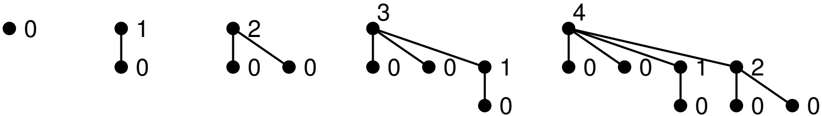

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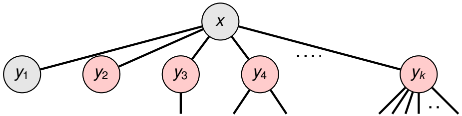

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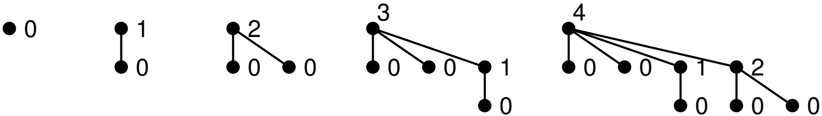

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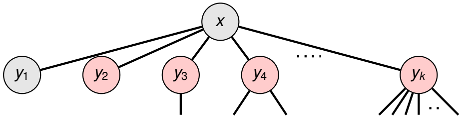

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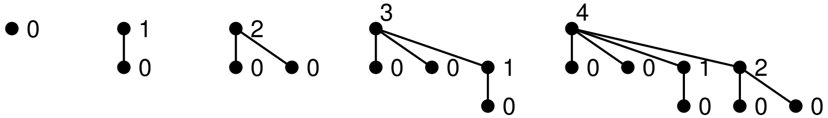

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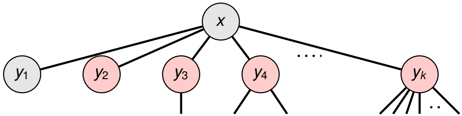

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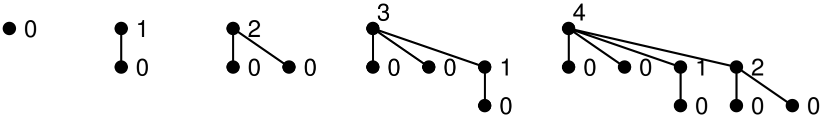

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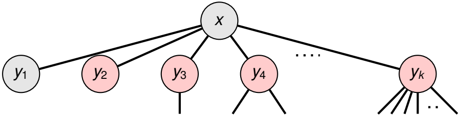

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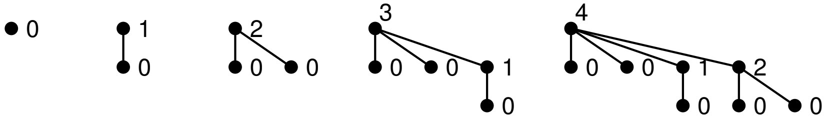

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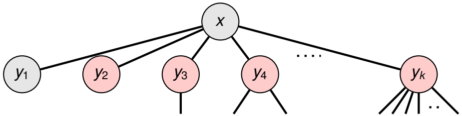

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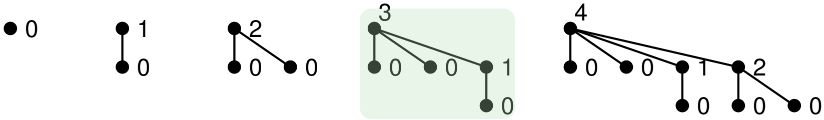

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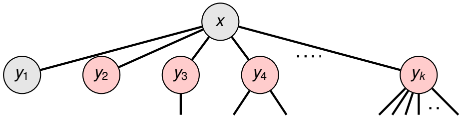

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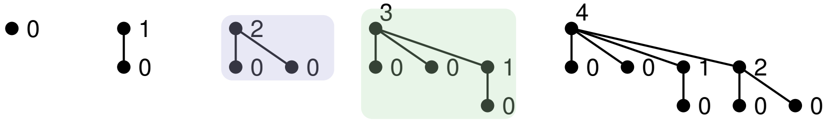

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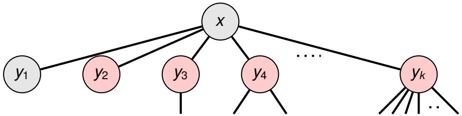

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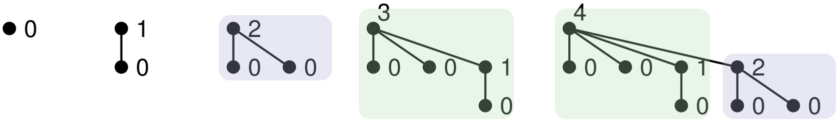

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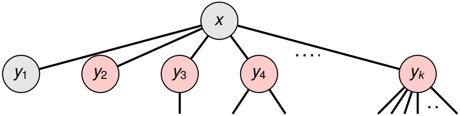

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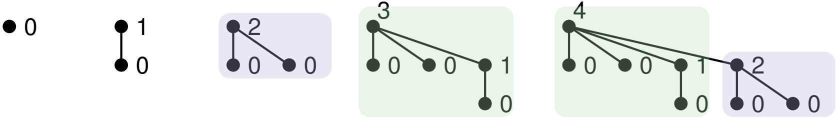

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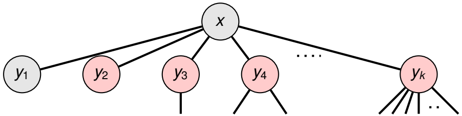

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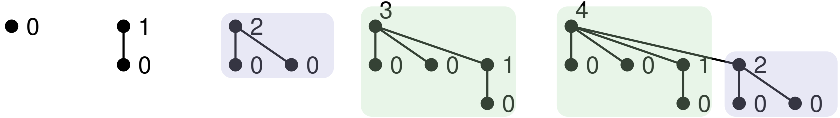

From Minimum Subtree Sizes to Fibonacci Numbers

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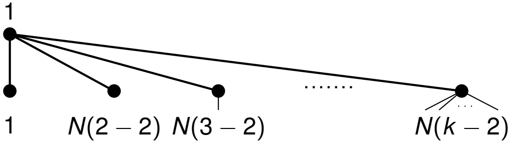

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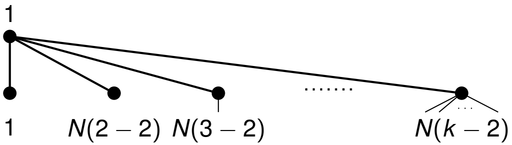

$$\begin{aligned}
N(k) &= 1 + 1 + N(2-2) + N(3-2) + \cdots + N(k-2) \\
&= 1 + 1 + \sum_{\ell=0}^{k-2} N(\ell) \\
&= 1 + 1 + \sum_{\ell=0}^{k-3} N(\ell) + N(k-2) \\
&= N(k-1) + N(k-2) \\
&= F(k+1) + F(k) = F(k+2) \qquad \square
\end{aligned}$$

Exponential Growth of Fibonacci Numbers

Lemma 19.3

For all integers $k \geq 0$, the $(k+2)$nd Fib. number satisfies $F(k+2) \geq \varphi^k$, where $\varphi = (1+\sqrt{5})/2 = 1.61803\ldots$.

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$$\begin{aligned} F(k+2) &= F(k+1) + F(k) \\ &\geq \varphi^{k-1} + \varphi^{k-2} && \text{(by the inductive hypothesis)} \\ &= \varphi^{k-2} \cdot (\varphi + 1) \end{aligned}$$

Exponential Growth of Fibonacci Numbers

Lemma 19.3

For all integers $k \geq 0$, the $(k+2)$nd Fib. number satisfies $F(k+2) \geq \varphi^k$, where $\varphi = (1+\sqrt{5})/2 = 1.61803\ldots$.

$\varphi^2 = \varphi + 1$

Fibonacci Numbers grow at least exponentially fast in k.

Proof by induction on k:

- Base $k = 0$: $F(2) = 1$ and $\varphi^0 = 1$ ✓
- Base $k = 1$: $F(3) = 2$ and $\varphi^1 \approx 1.619 < 2$ ✓
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&= \varphi^{k-2} \cdot (\varphi + 1) \\
&= \varphi^{k-2} \cdot \varphi^2 && (\varphi^2 = \varphi + 1) \\
&= \varphi^k && \square
\end{aligned}$$

Putting the Pieces Together

Amortized Analysis

- INSERT: amortized cost $\mathcal{O}(1)$
- EXTRACT-MIN amortized cost $\mathcal{O}(d(n))$
- DECREASE-KEY amortized cost $\mathcal{O}(1)$

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$N(k)$

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$$N(k) = F(k+2)$$

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& n \geq N(k) = F(k+2) \geq \varphi^k \\
\Rightarrow \quad & \log_\varphi n \geq k
\end{aligned}
$$

Putting the Pieces Together

Amortized Analysis

- INSERT: amortized cost $\mathcal{O}(1)$
- EXTRACT-MIN amortized cost $\cancel{\mathcal{O}(d(n))}$ $\mathcal{O}(\log n)$
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$$n \geq N(k) = F(k+2) \geq \varphi^k$$

$$\Rightarrow \qquad \log_\varphi n \geq k$$

What if we don't have marked nodes?

- INSERT: actual $\mathcal{O}(1)$
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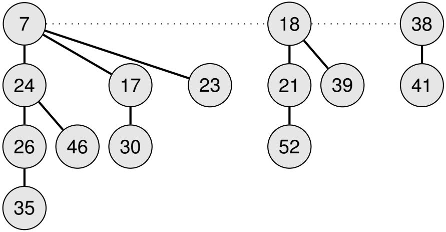

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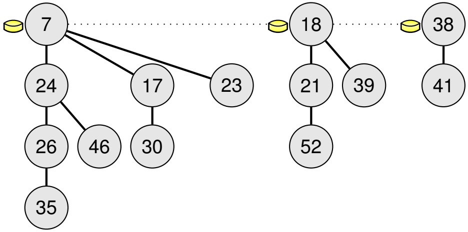

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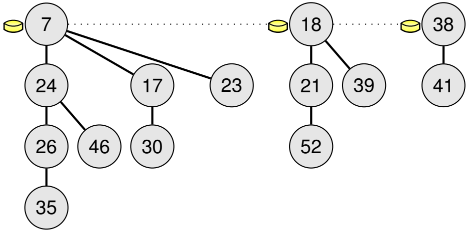
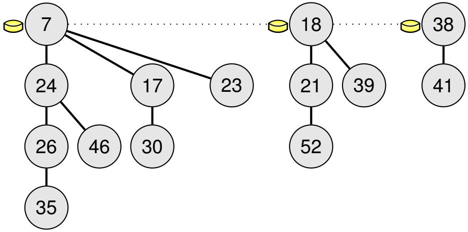


What if we don't have marked nodes?

- INSERT: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$ amortized $\mathcal{O}(d(n)) \neq \mathcal{O}(\log n)$
- DECREASE-KEY: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$

$$\Phi(H) = \text{trees}(H)$$



Summary

Operation	Linked list	Binary heap	Binomial heap	Fibon. heap
MAKE-HEAP	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
<u>INSERT</u>	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
MINIMUM	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
<u>EXTRACT-MIN</u>	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$
UNION	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
<u>DECREASE-KEY</u>	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
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Can we perform EXTRACT-MIN in $o(\log n)$?

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Can we perform EXTRACT-MIN in $o(\log n)$?

If this was possible, then there would be a sorting algorithm with runtime $o(n \log n)$!

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Crucial for many applications including shortest paths and minimum spanning trees!

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DELETE = DECREASE-KEY + EXTRACT-MIN

EXTRACT-MIN = MIN + DELETE

Recent Studies of Fibonacci Heaps

- Fibonacci Numbers were discovered >800 years ago
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$\Rightarrow$ marked bit is not redundant!

Outlook: A More Efficient Priority Queue for fixed Universe

Operation	Fibonacci heap amortized cost	Van Emde Boas Tree actual cost
INSERT	$\mathcal{O}(1)$	$\mathcal{O}(\log\log u)$
MINIMUM	$\mathcal{O}(1)$	$\mathcal{O}(1)$
EXTRACT-MIN	$\mathcal{O}(\log n)$	$\mathcal{O}(\log\log u)$
MERGE/UNION	$\mathcal{O}(1)$	-
DECREASE-KEY	$\mathcal{O}(1)$	$\mathcal{O}(\log\log u)$
DELETE	$\mathcal{O}(\log n)$	$\mathcal{O}(\log\log u)$
SUCC	-	$\mathcal{O}(\log\log u)$
PRED	-	$\mathcal{O}(\log\log u)$
MAXIMUM	-	$\mathcal{O}(1)$

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DECREASE-KEY	$\mathcal{O}(1)$	$\mathcal{O}(\log\log u)$
DELETE	$\mathcal{O}(\log n)$	$\mathcal{O}(\log\log u)$
SUCC	-	$\mathcal{O}(\log\log u)$
PRED	-	$\mathcal{O}(\log\log u)$
MAXIMUM	-	$\mathcal{O}(1)$

all this requires key values to be in a universe of size u!