V. Approximation Algorithms via Exact Algorithms

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UNIVERSITY OF CAMBRIDGE

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

The Subset-Sum Problem

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- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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This problem is NP-hard

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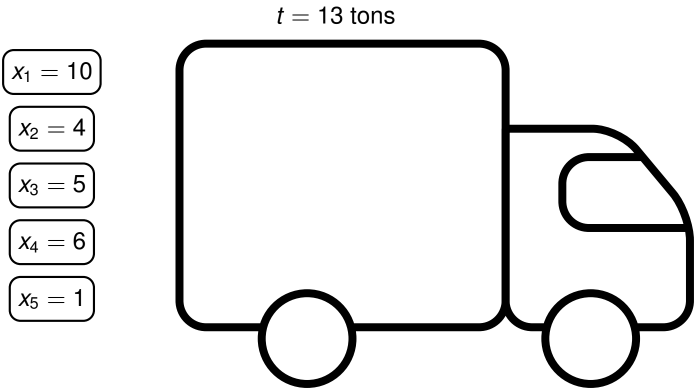

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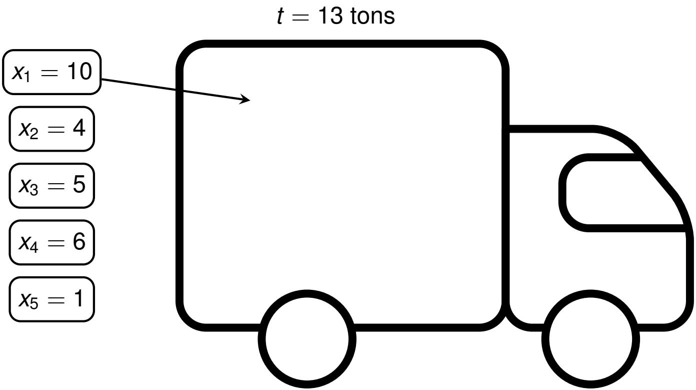

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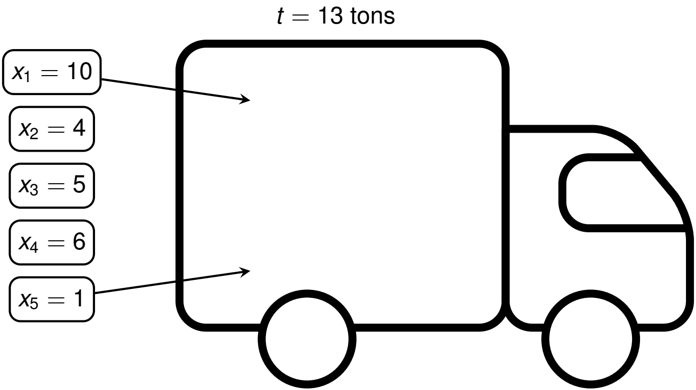

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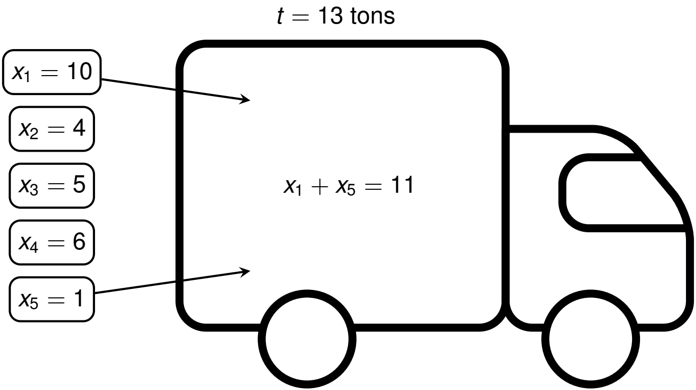

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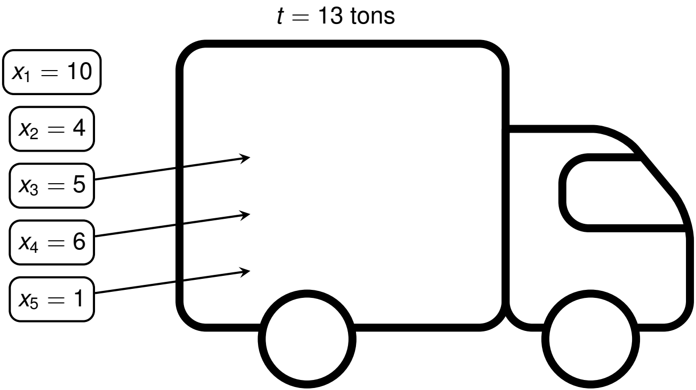

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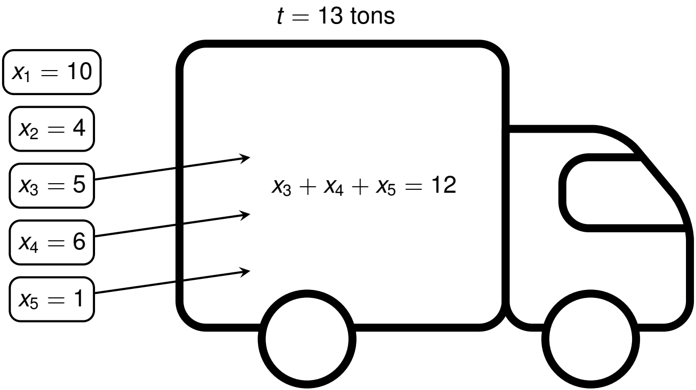

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

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EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
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    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$ $S + x := \{s + x : s \in S\}$
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Returns the merged list (in sorted order and without duplicates)

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implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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Example:

- $S = \{1, 4, 5\}$, $t = 10$

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$5 = 0 + 5 = 1 + 4$

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can be shown by induction on n

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Better runtime if t and/or $|L_i|$ are small.

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$$\frac{y}{1+\delta} \leq z \leq y.$$

"approximate representative"

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let m be the length of L
L' = ⟨y_1⟩
last = y_1
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maintained inductively

Trims list in time $\Theta(m)$, if L is given in sorted order.

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$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, \overset{\substack{\text{last} \\ \downarrow}}{20}, 21, \underset{\substack{\uparrow \\ \text{i}}}{22}, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
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    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
    if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
        append y_i onto the end of L'
        $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i =$ TRIM$(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

The FPTAS

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      L_i = TRIM(L_i, ε/2n)
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      remove from L_i every element that is greater than t
return the largest element in L_n
```

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

- We must bound the inaccuracy introduced by repeated trimming

The FPTAS

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
      L_i = TRIM(L_i, [ε/2n])
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Repeated application of TRIM to make sure L_i's remain short.

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
      remove from L_i every element that is greater than t
return the largest element in L_n
```

proper choice of δ meets these conflicting goals

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\boxed{\epsilon = 0.4}$

$\rightarrow$ z^* should be at least $\frac{1}{1.4}$ times the optimum

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      L_i = TRIM(L_i, ε/2n)
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101\rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104\rangle$
- line 5: $L_1 = \langle 0, 104\rangle$
- line 6: $L_1 = \langle 0, 104\rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206\rangle$
- line 5: $L_2 = \langle 0, 102, 206\rangle$
- line 6: $L_2 = \langle 0, 102, 206\rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407\rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407\rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303\rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404\rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404\rangle$
- line 6: $L_4 = \langle 0, 101, 201, \boxed{302}\rangle \rightarrow$ output

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = \boxed{0.4}$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

much better than $1+\varepsilon$-approximation!

Returned solution $z^* = 302$, which is $\boxed{2\%}$ within the optimum $307 = 104 + 102 + 101$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

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all elements in the trimmed lists are solutions

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$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y$$

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List after trimming

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- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{\boxed{(1+\epsilon/(2n))^i}} \leq z \leq y$$

Can be shown by induction on i

$i=1$: clear

$i=2$: $y = x_1 + x_2$, $x_1 \in L_1 \Rightarrow \exists z_1 \in L_1' : z_1 \geq \frac{x_1}{(1+\delta)}$

$z_1 + x_2 \in L_2 \Rightarrow \exists z \in L_2' : z \geq \frac{z_1 + x_2}{(1+\delta)}$

$\Rightarrow z \geq \frac{x_1}{(1+\delta)^2} + \frac{x_2}{(1+\delta)} \geq \frac{x_1 + x_2}{(1+\delta)^2} = \frac{y}{(1+\delta)^2}$

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i=n

Can be shown by induction on i

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$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

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and now using the fact that $\left(1+\frac{\epsilon/2}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

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Taylor approximation of e

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$$\leq 1+\epsilon/2+(\epsilon/2)^2$$

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Taylor approximation of e

$$\leq 1 + \epsilon/2 + (\epsilon/2)^2 \leq 1 + \epsilon$$

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Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is polynomial in $|L_i|$)

iteration i runs in $O(|L_i|)$

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Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is polynomial in $|L_i|$)
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For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

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$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1+\epsilon/(2n))} + 2$$
$$\leq \frac{2n(1+\epsilon/(2n)) \ln t}{\epsilon} + 2$$

For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

$\Leftrightarrow x > 0 \quad \ln(x) \geq 1 - \frac{1}{x}$

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For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

$$\left(1+\frac{\varepsilon}{2n}\right) \leq \frac{3}{2}$$

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- This bound on $|L_i|$ is polynomial in the size of the input and in $1/\epsilon$.

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Need log(t) bits to represent t and n bits to represent S.

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i\colon x_i \in S'} x_i \leq t$.

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- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t

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A more general problem than Subset-Sum.

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("weighted" version of Subset-Sum)

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Theorem

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Algorithm very similar to APPROX-SUBSET-SUM.

Theorem

There is a FPTAS for the Knapsack problem.

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$

Parallel Machine Scheduling

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- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

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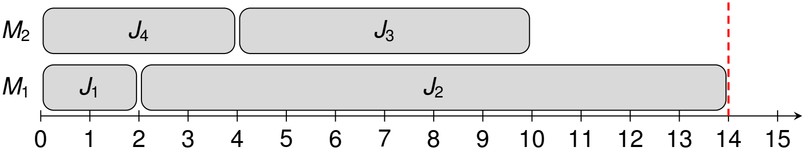

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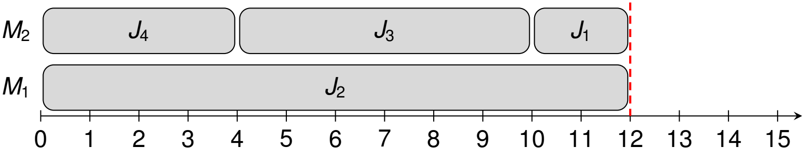

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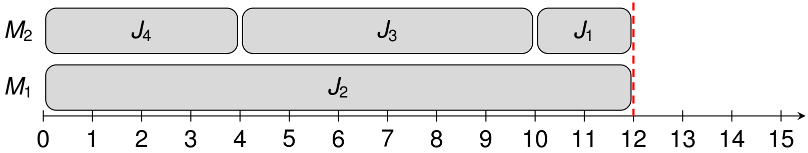
- J_4: $p_4 = 4$

For the analysis, it will be convenient to denote by C_i the completion time of a machine i.



NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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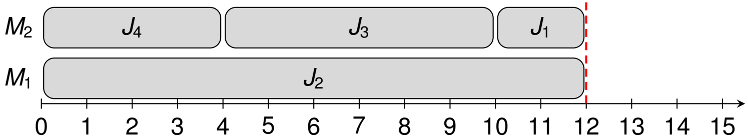

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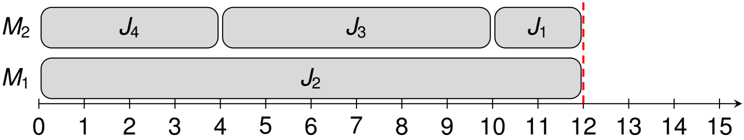

LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$

while there exists an unassigned job

    Schedule job on the machine with the least load

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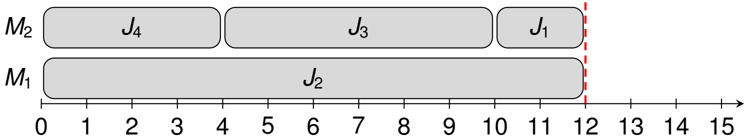

Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

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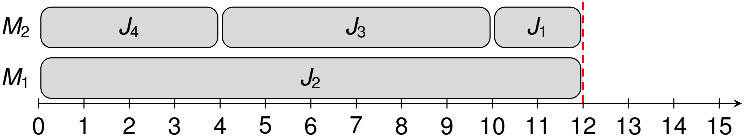

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How good is this most basic Greedy Approach?

List Scheduling Analysis (Observations)

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

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Proof:

b. ~~a.~~ The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

~~b.~~ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time $\boxed{\text{2-approximation}}$ algorithm.

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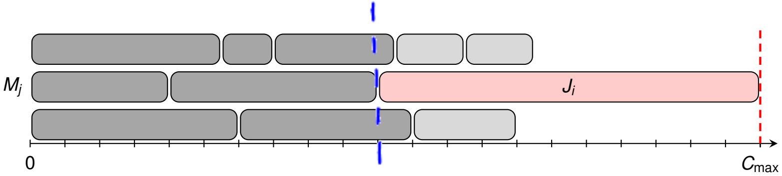

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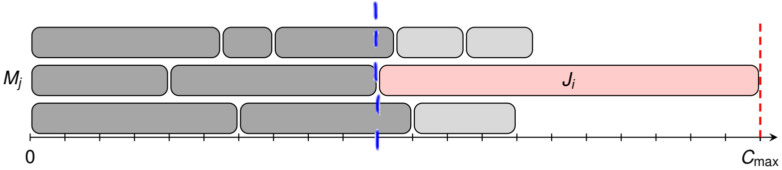

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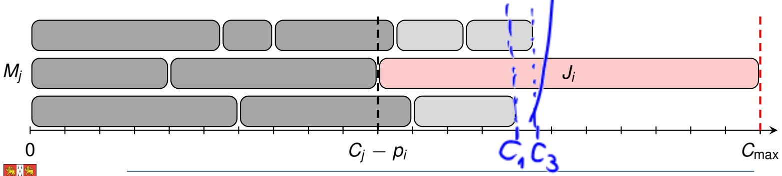
"Greedy"-Property



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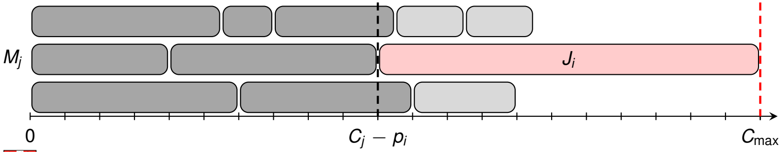

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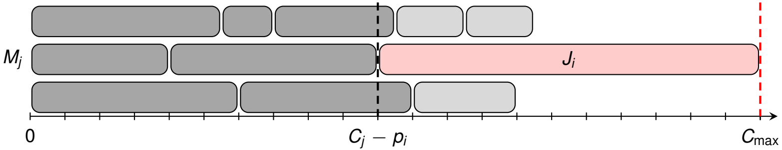

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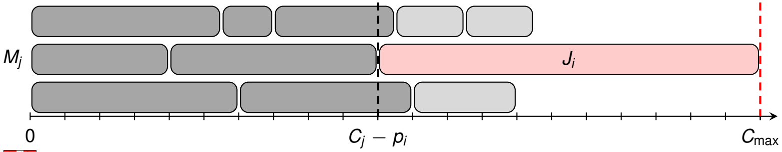

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$C_j =$

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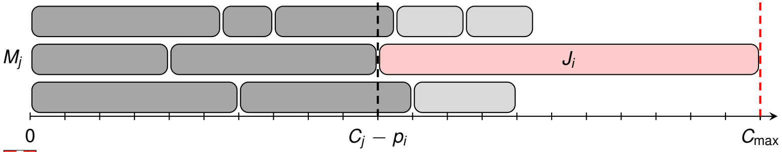

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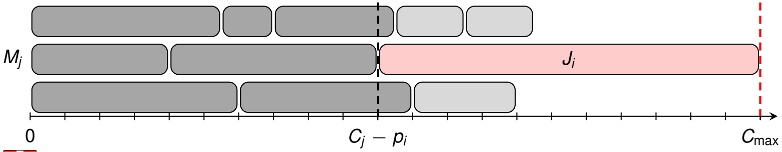

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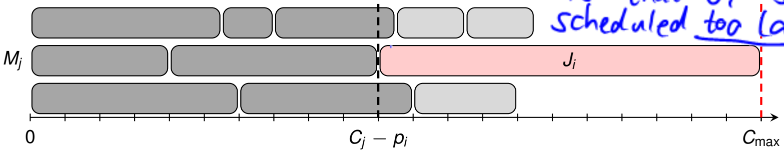
Idea: Problem is that J_i is scheduled too late!



Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

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give them
"higher priority"

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LEAST PROCESSING TIME$(J_1, J_2, \ldots, J_n, m)$

```
Sort jobs decreasingly in their processing times
for i = 1 to m
    C_i = 0
    S_i = ∅
end for
for j = 1 to n
    i = argmin_{1≤k≤m} C_k
    S_i = S_i ∪ {j}, C_i = C_i + p_j
end for
return S_1, ..., S_m
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Runtime:

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Runtime:

- $O(n \log n)$ for sorting

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Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting the minimum (use priority queue).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

exactly!!

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→ a bit easier to prove

Proof (of approximation ratio $3/2$).

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- Observation 1: If there are at most m jobs, then the solution is optimal.

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- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.

$\exists$ machine which has to process two jobs from $J_1, J_2, \ldots, J_{m+1}$

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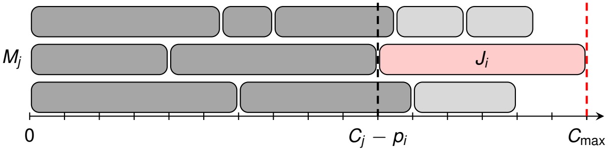

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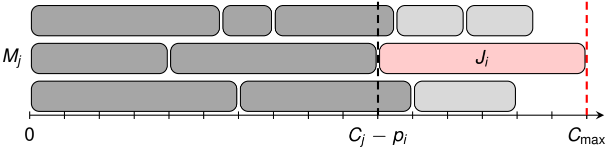

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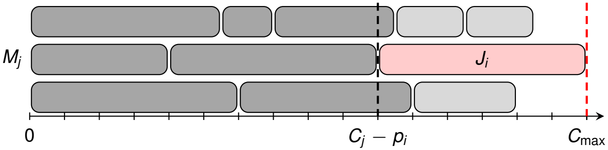
$$C_{max} = C_j = \boxed{(C_j - p_i)} + \boxed{p_i} \leq \boxed{C^*_{\max}} + \boxed{\frac{1}{2} C^*_{\max}}$$

$$\boxed{C_j - p_i \leq C_k \ \forall k}$$

This is for the case $i \geq m + 1$ (otherwise, an even stronger inequality holds)



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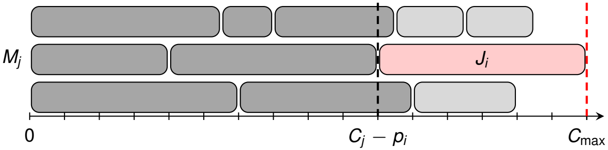

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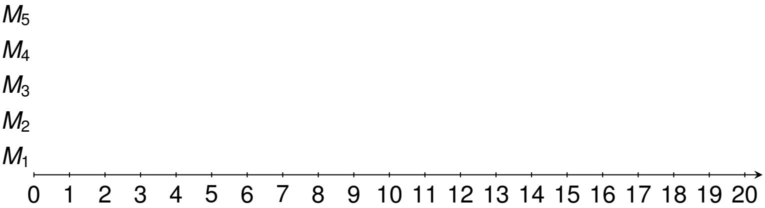

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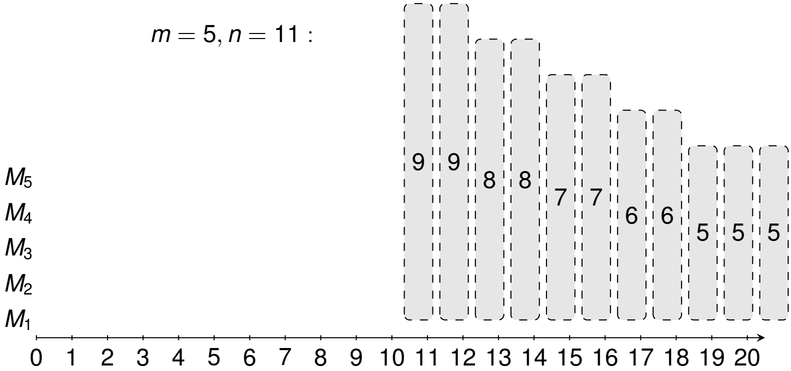

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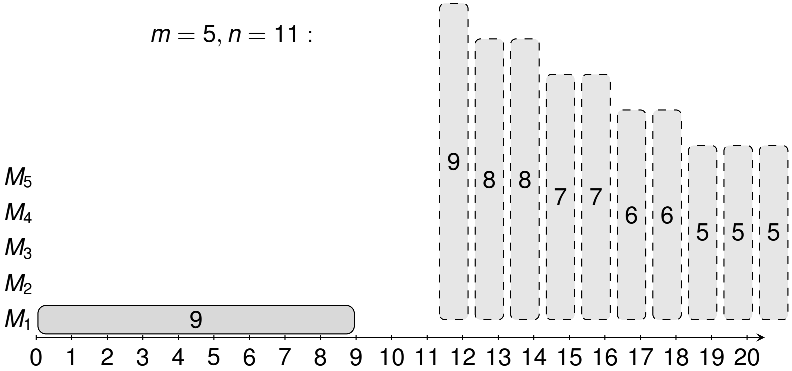

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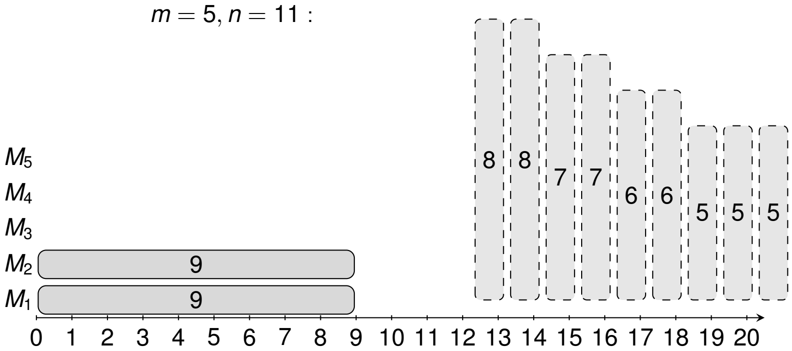

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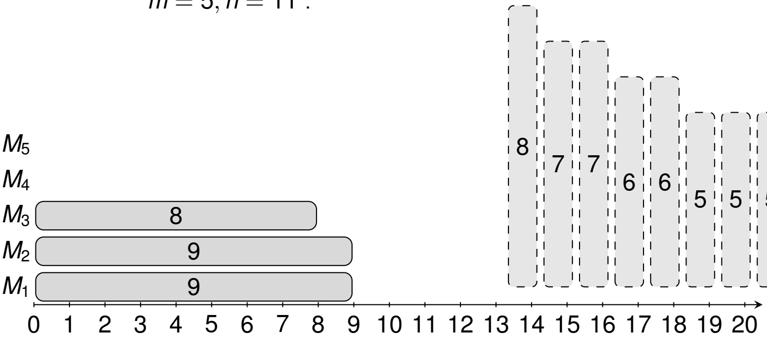

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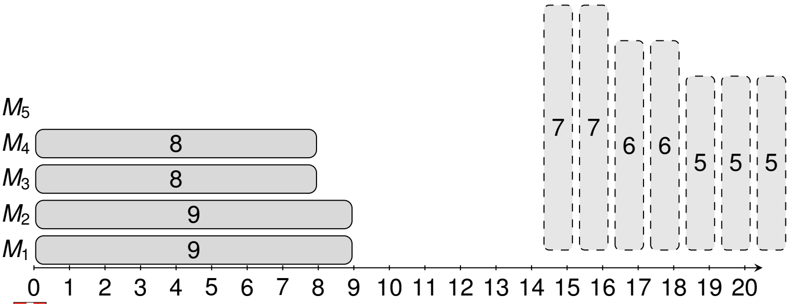

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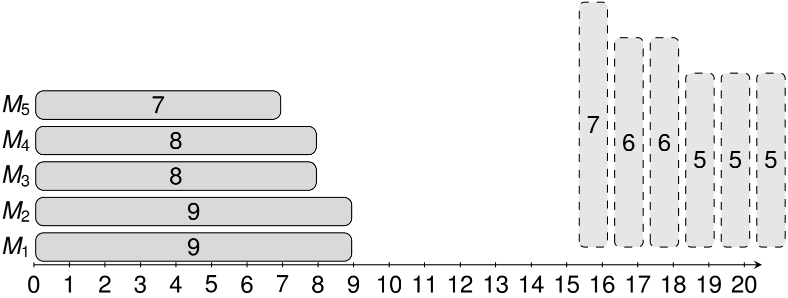

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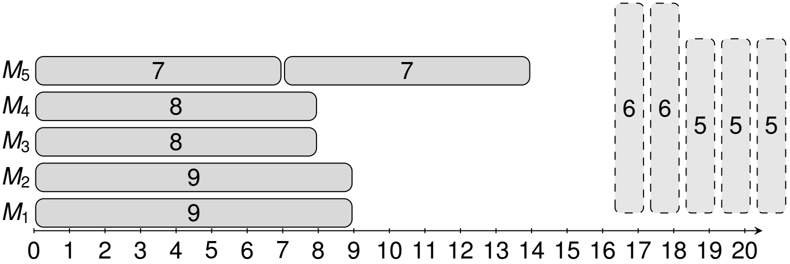

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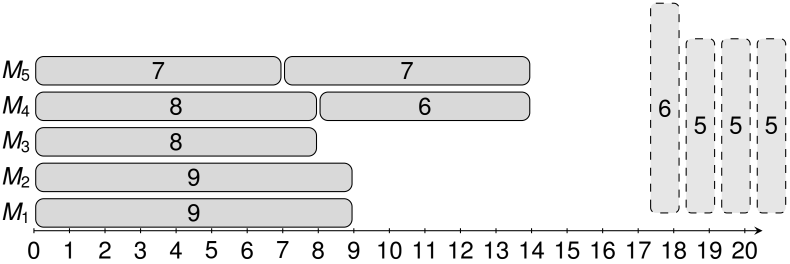

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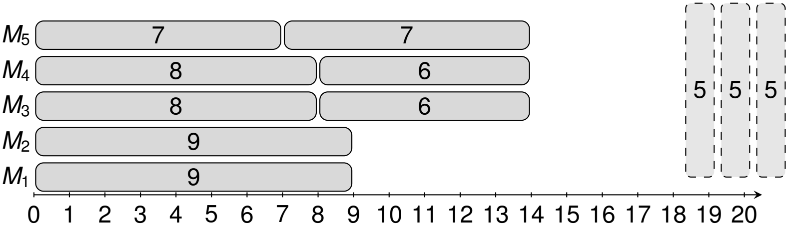

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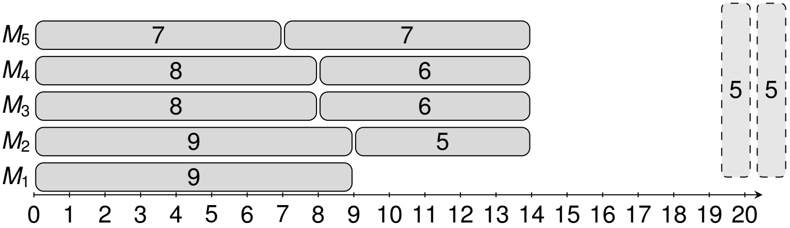

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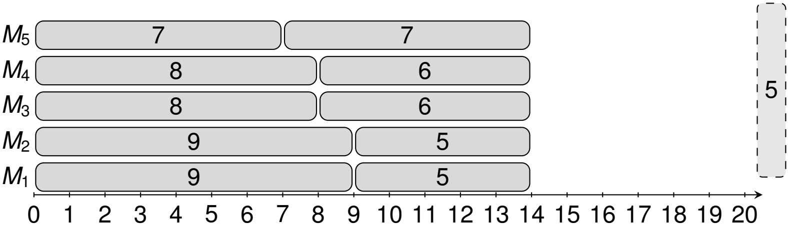

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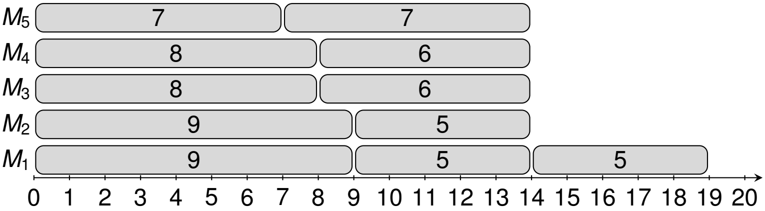

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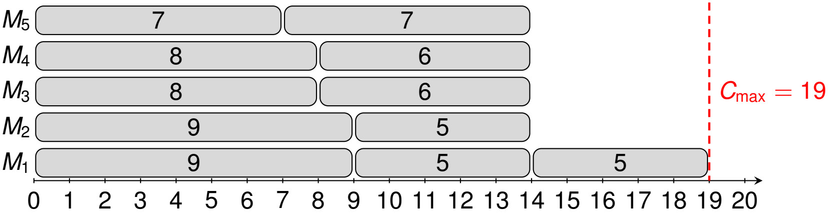

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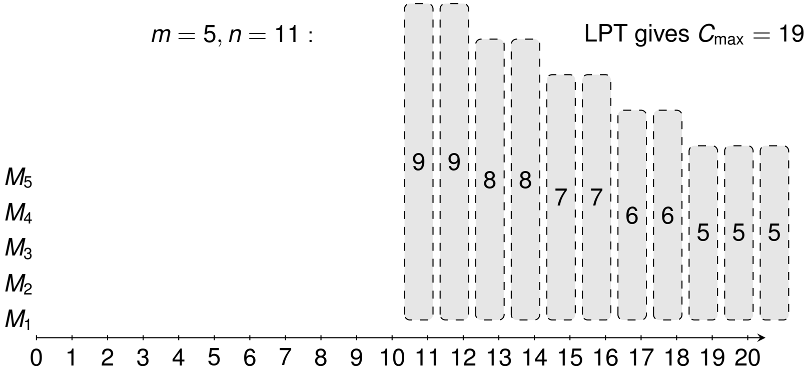

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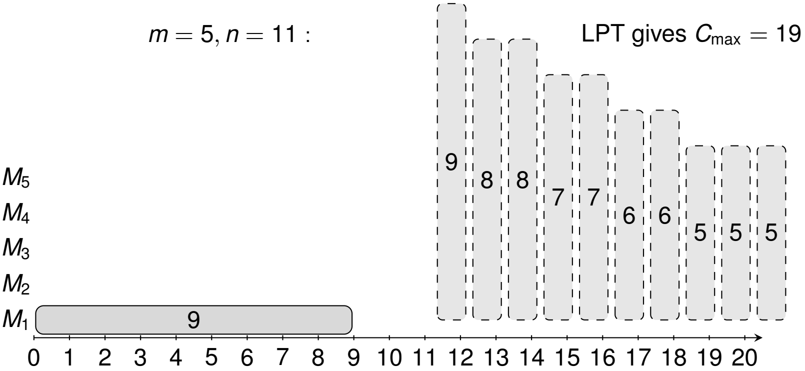

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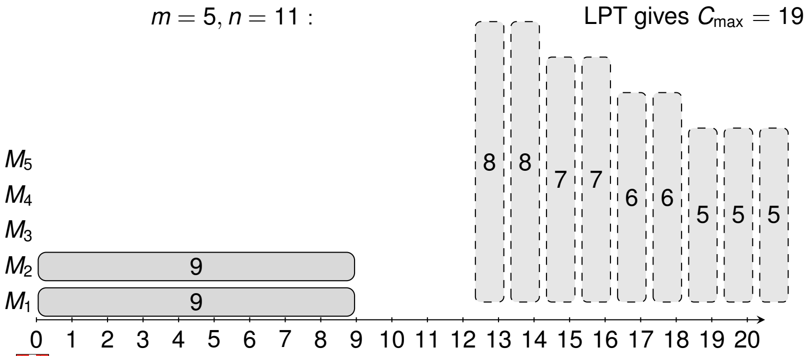

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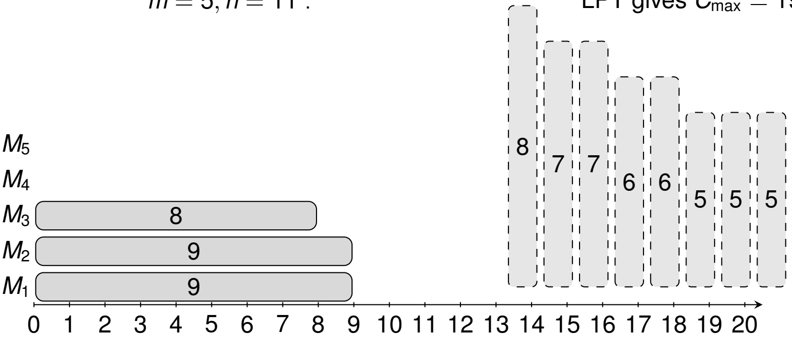

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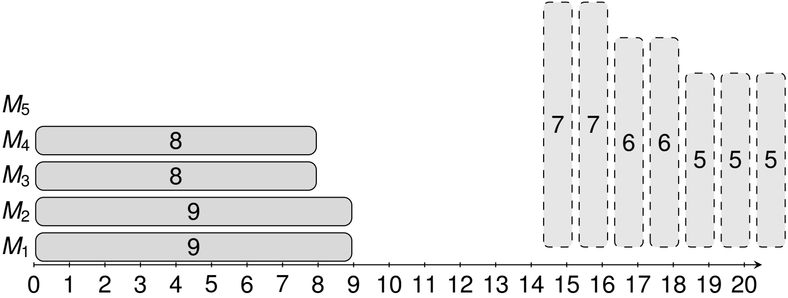

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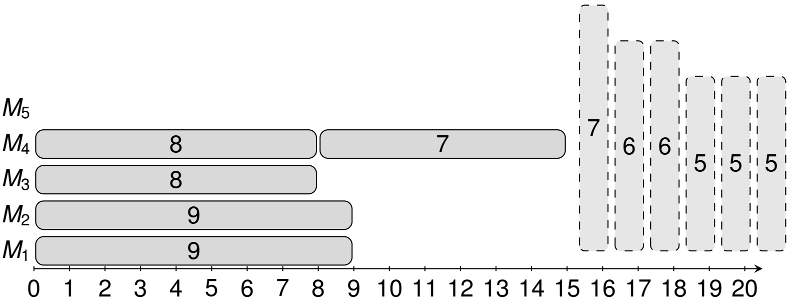

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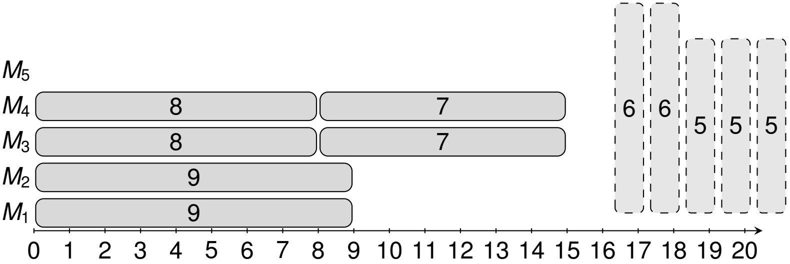

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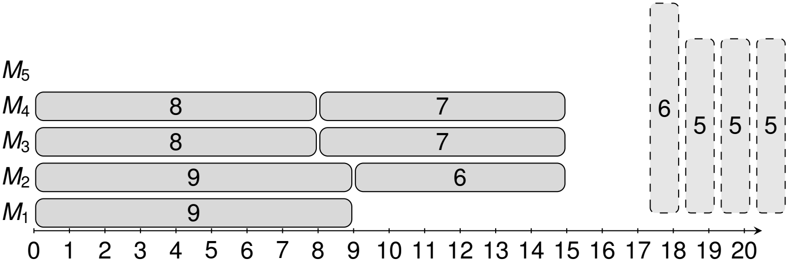

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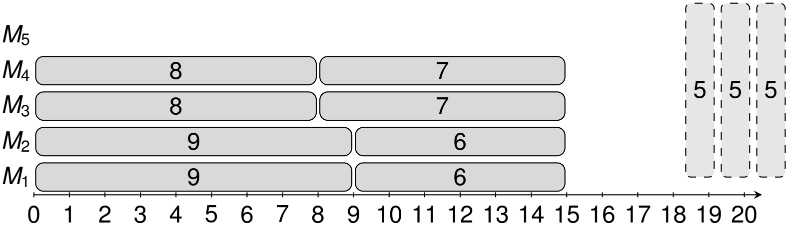

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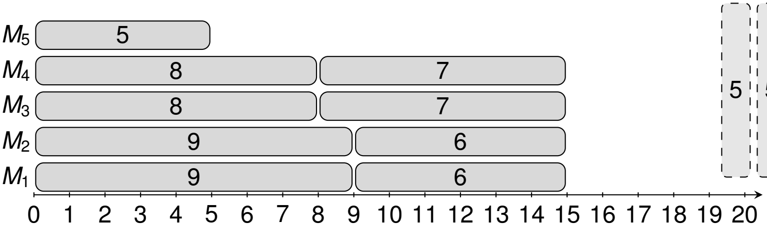

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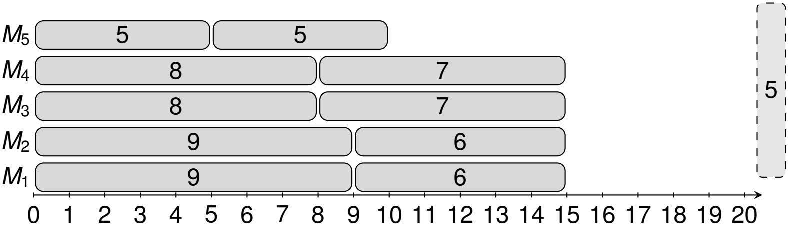

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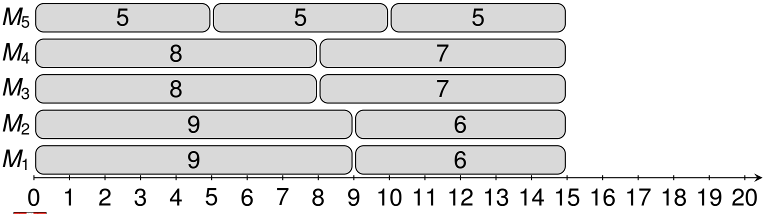

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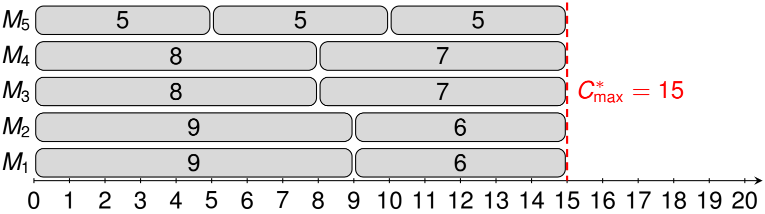

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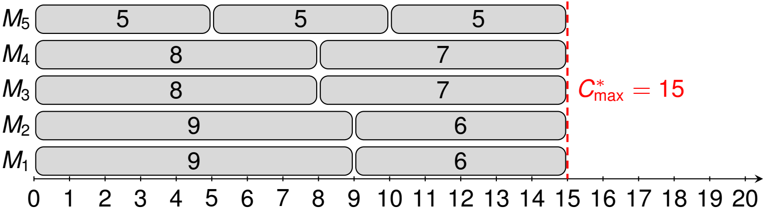

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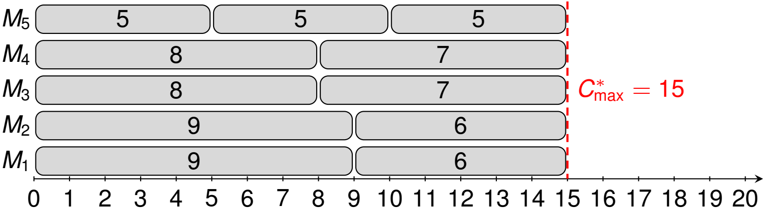

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Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load

Implementation of Subroutine

SUBROUTINE($J_1, J_2, \ldots, J_n, m, T$)

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$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k$$

the “well-known” formula

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Use Dynamic Programming to schedule J_{large} with makespan $(1 + \epsilon) \cdot T$.

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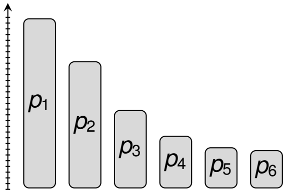

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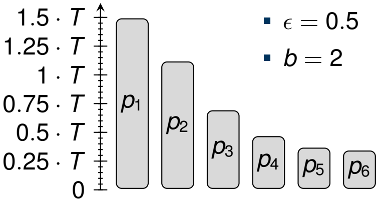

- $\epsilon = 0.5$
- $b = 2$

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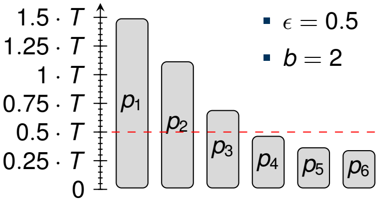

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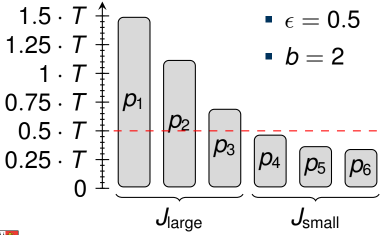

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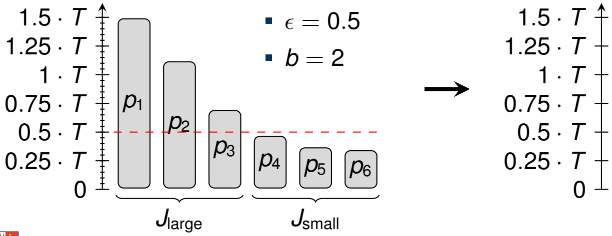

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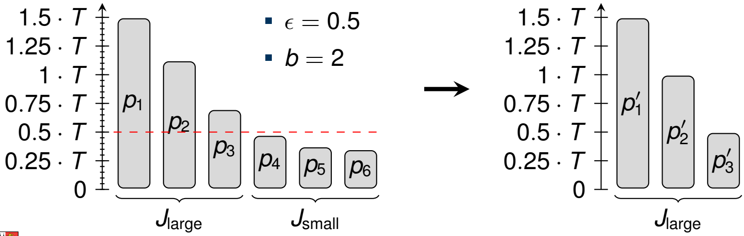

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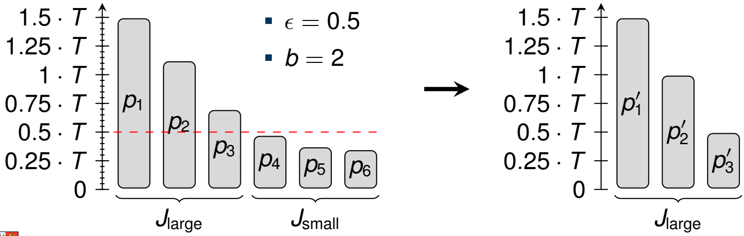
$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$ Can assume there are no jobs with $p_j \geq T$!



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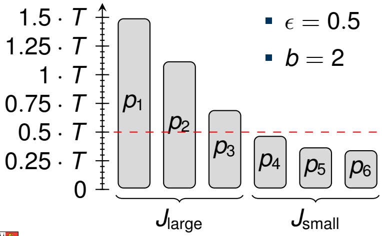

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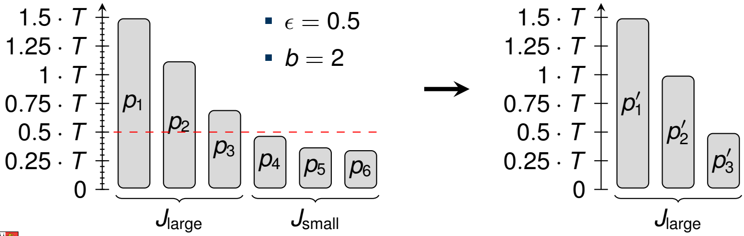
Assignments to one machine with makespan $\leq T$.



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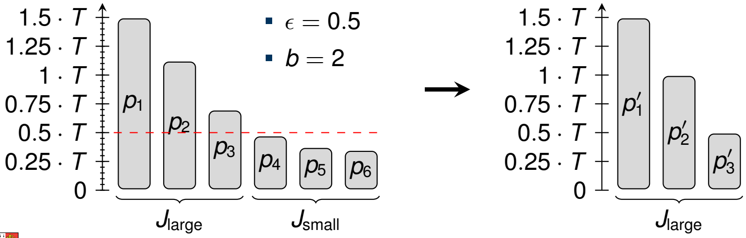
- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:



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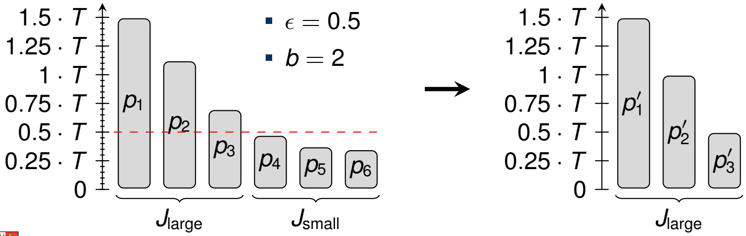

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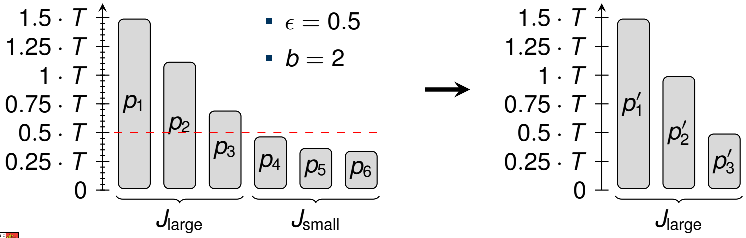

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Assign some jobs to one machine, and then use as few machines as possible for the rest.

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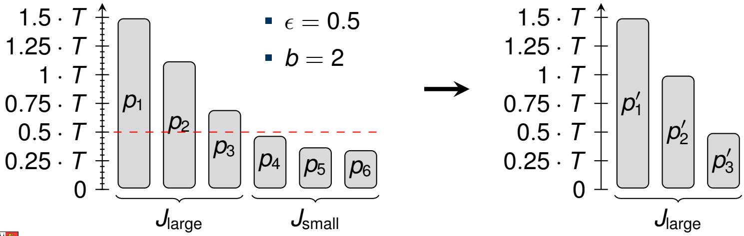

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- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

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- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.
- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

$$C_{\max} \leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i')$$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_i b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

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Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal,

and: Par. Mach. Sched. is strongly NP-hard!