Analysis of Greedy for Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ⊔ {u, v}
    remove from E' every edge incident on either u or v
return C
```

Line 5 in math form: $C = C\,\boxed{\cup \{u, v\}}$

A "vertex-based" Greedy-Alg. which adds one vertex at each iteration does not achieve approx. ratio of 2 (Exercise 18)

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

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Proof:

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Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V + E)$ (using adjacency lists to represent E')
- Let $A \subseteq E$ denote the set of edges picked in line 4
- Every optimal cover C^* must include at least one endpoint of edges in A, and edges in A do not share a common endpoint: $\boxed{|C^*| \geq |A|}$
- Every edge in A contributes 2 vertices to $|C|$: $\boxed{|C| = 2|A| \leq 2|C^*|.}$ $\square$

Analysis of Greedy for Vertex Cover

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```

We can bound the size of the returned solution without knowing the (size of an) optimal solution!

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof: Key idea: A is a maximal matching

- Running time is $O(V + E)$ (using adjacency lists to represent E')
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Solving Special Cases

Strategies to cope with NP-complete problems

1. If inputs are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

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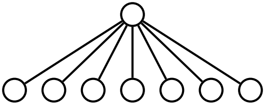

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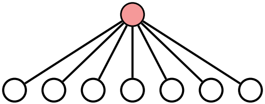

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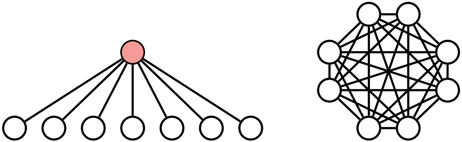

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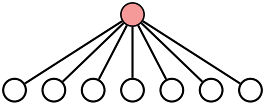

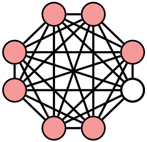

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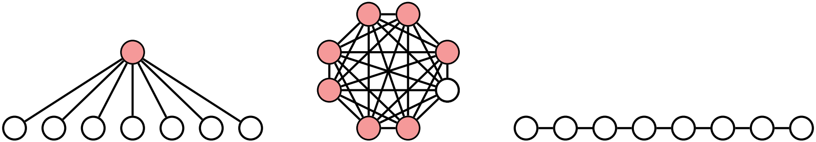

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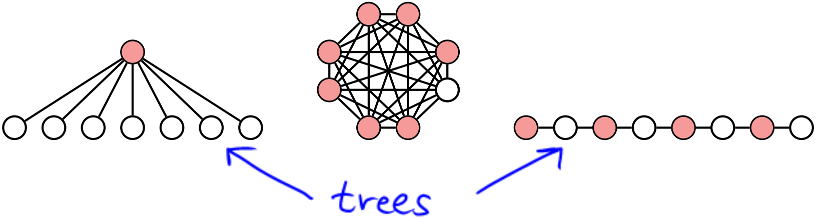
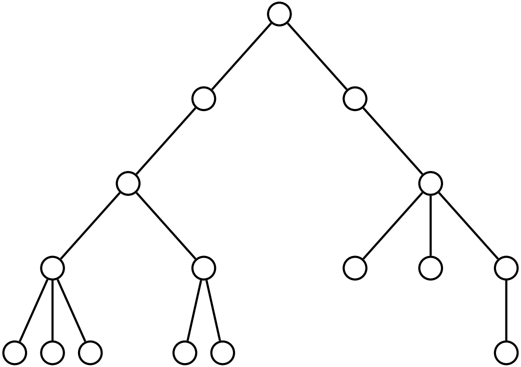


Vertex Cover on Trees



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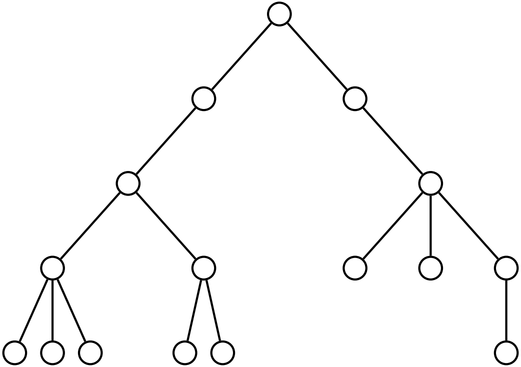

There exists an optimal vertex cover which does not include any leaves.

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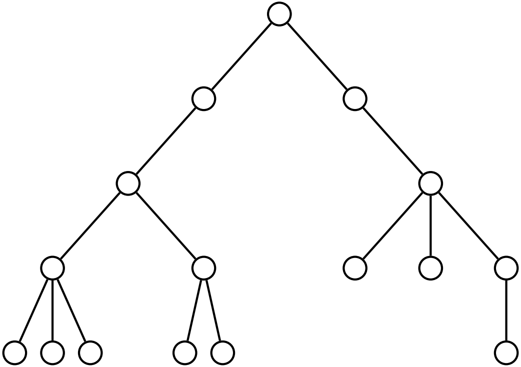

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Exchange-Argument: Replace any leaf in the cover by its parent.

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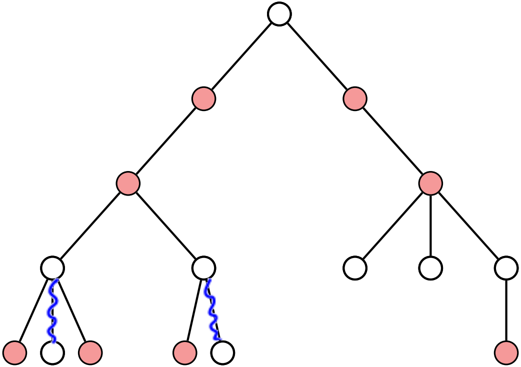

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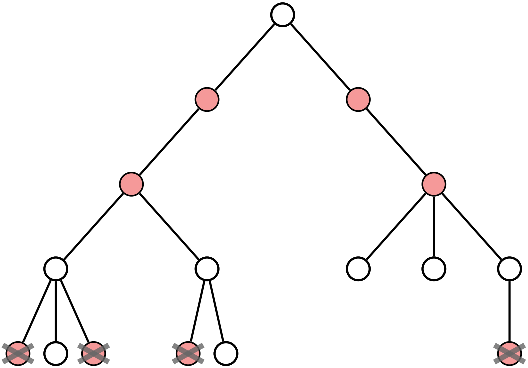

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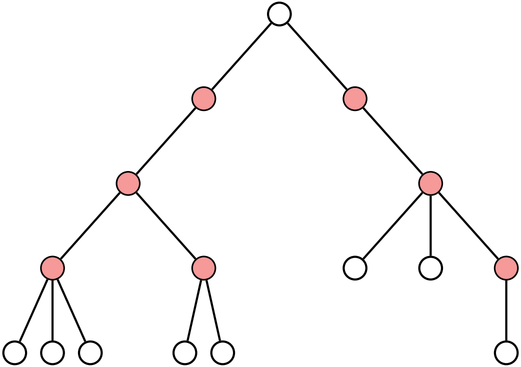

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Solving Vertex Cover on Trees

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VERTEX-COVER-TREES(G)

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C = ∅
while ∃ leaves in G
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Clear: Running time is $O(V)$, and the returned solution is a vertex cover.

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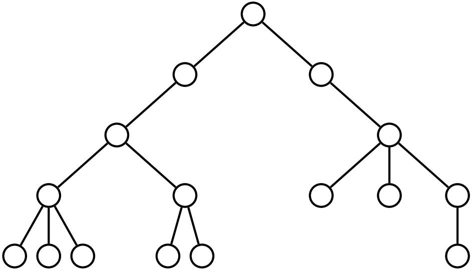
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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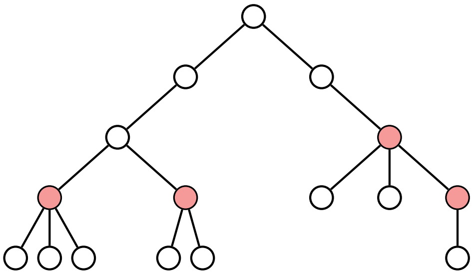

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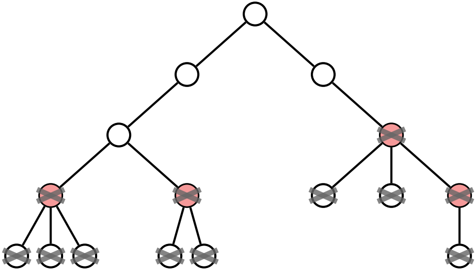

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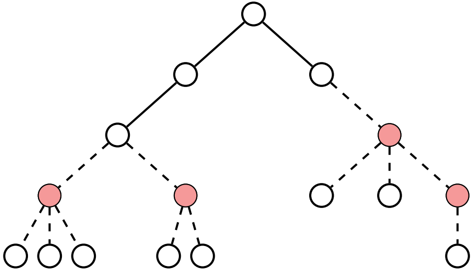

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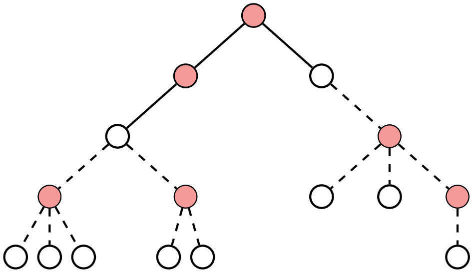

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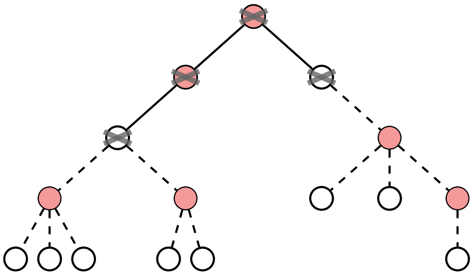

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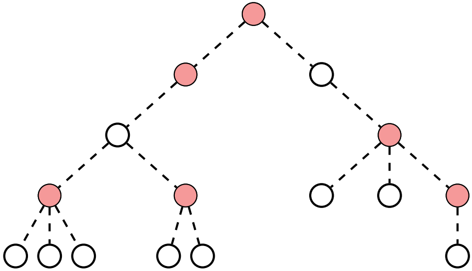

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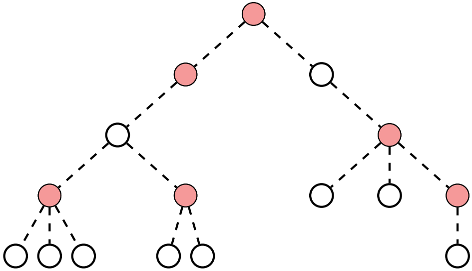

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

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Focus on instances of where the minimum vertex cover is small, that is, smaller than some given integer k.

(or equal)

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Focus on instances of where the minimum vertex cover is small, that is, smaller than some given integer k.

Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $(u, v) \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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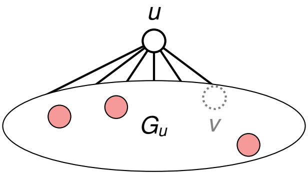

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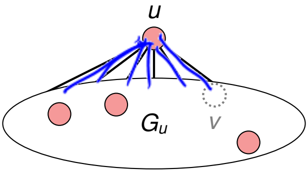
Adding u yields a vertex cover of G which is of size k



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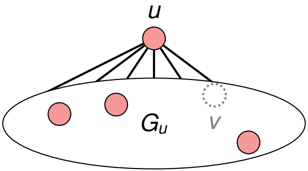

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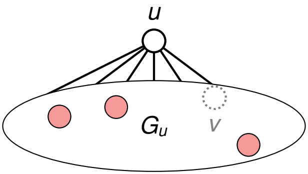
Removing u from C yields a vertex cover of G_u which is of size $k - 1$. $\square$



A More Efficient Search Algorithm

VERTEX-COVER-SEARCH(G, k)

If $E = \emptyset$ **return** $\{\perp\}$

If $k = 0$ and $E \neq \emptyset$ **return** $\emptyset$

Pick an arbitrary edge $(u, v) \in E$

$S_1 =$ VERTEX-COVER-SEARCH$(G_u, k-1)$

$S_2 =$ VERTEX-COVER-SEARCH$(G_v, k-1)$

if $S_1 \neq \emptyset$ **return** $S_1 \cup \{u\}$

if $S_2 \neq \emptyset$ **return** $S_2 \cup \{v\}$

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Correctness follows by the Substructure Lemma and induction.

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Running time:

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Running time:

- Depth k, branching factor 2

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Running time:

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Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

A More Efficient Search Algorithm

$\textsc{Vertex-Cover-Search}(G, k)$

```
If E = ∅ return {⊥}
If k = 0 and E ≠ ∅ return ∅
Pick an arbitrary edge (u, v) ∈ E
S_1 = VERTEX-COVER-SEARCH(G_u, k − 1)
S_2 = VERTEX-COVER-SEARCH(G_v, k − 1)
if S_1 ≠ ∅ return S_1 ∪ {u}
if S_2 ≠ ∅ return S_2 ∪ {v}
return ∅
```

Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Set-Covering Problem

Set Cover Problem

- Given: set X of size n and family of subsets $\mathcal{F}$
- Goal: Find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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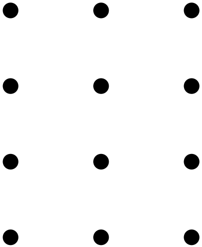

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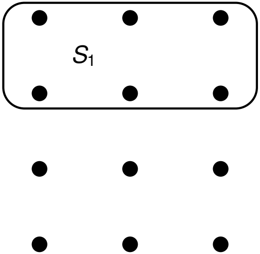

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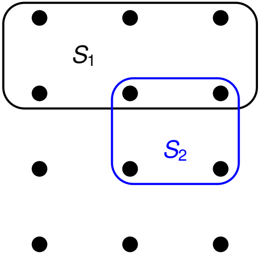

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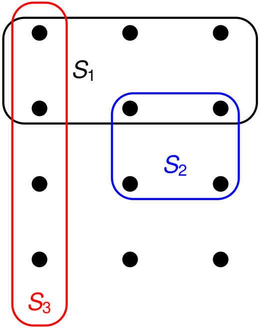

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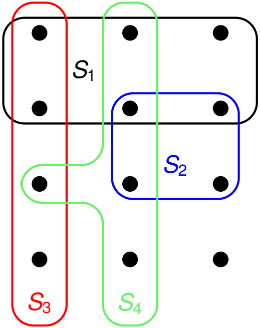

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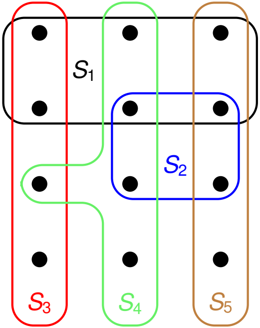

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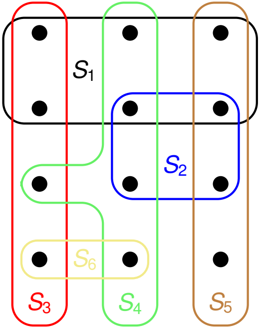

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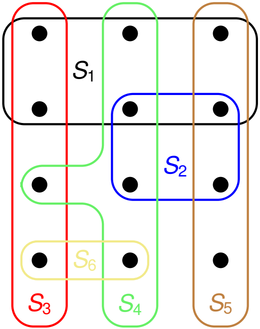

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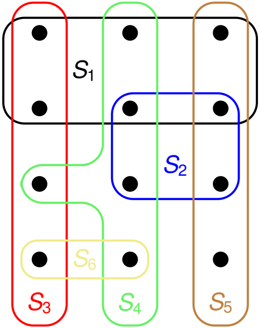

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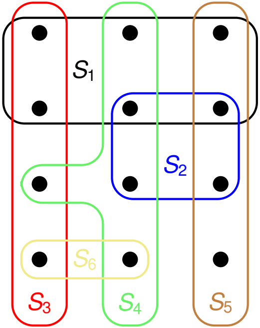

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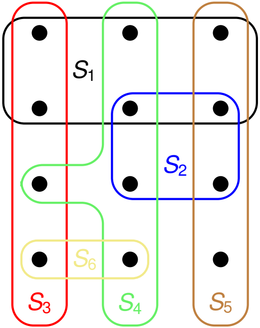

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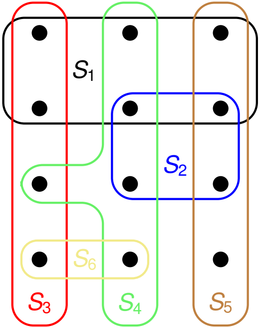

Remarks:

- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems

Greedy

Strategy: Pick the set S that covers the largest number of uncovered elements.

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GREEDY-SET-COVER$(X, \mathcal{F})$

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U = X
𝒞 = ∅
while U ≠ ∅
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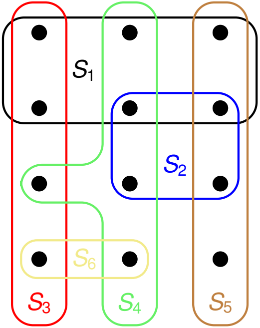

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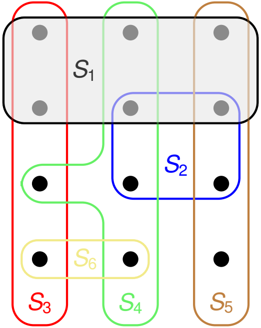

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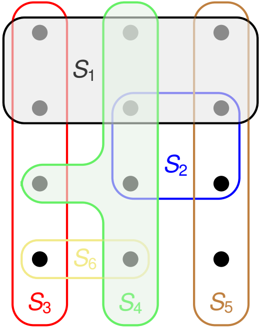

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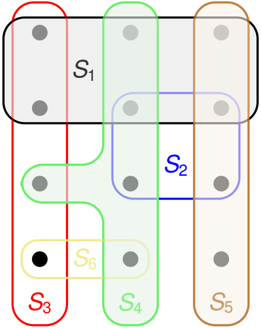

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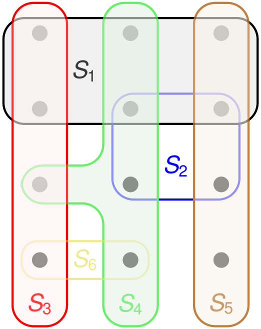

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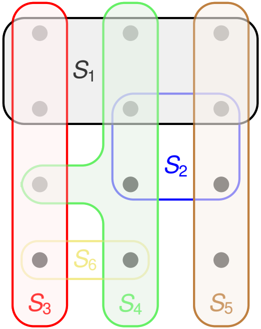

Greedy chooses S_1, S_4, S_5 and S_3 (or S_6), which is a cover of size 4.

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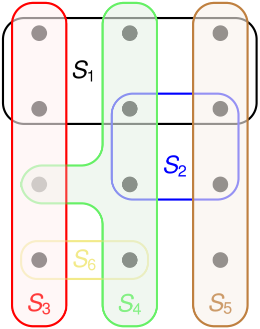

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Optimal cover is $\mathcal{C} = \{S_3, S_4, S_5\}$

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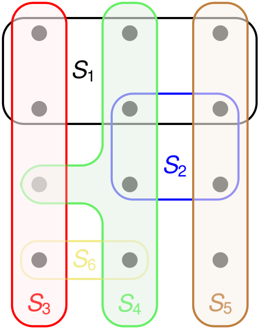

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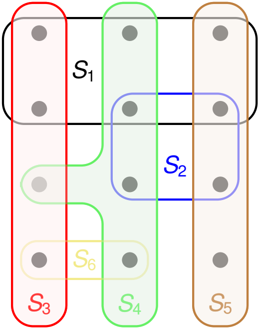

How good is the approximation ratio?

Approximation Ratio of Greedy

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = \underbrace{H(\max\{|S| : |S| \in \mathcal{F}\})}_{\text{in general, not a constant}}$$

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If an element x is covered for the first time by set S_i in iteration i, then

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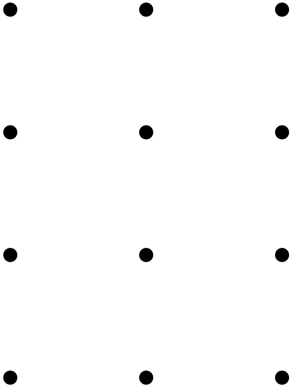
(in the mathematical analysis, S_i is the set chosen in iteration i - not to be confused with $S_1, S_2, \ldots, S_6$ in the example)

Illustration of Costs



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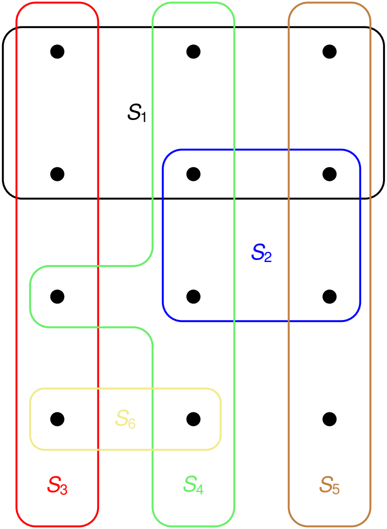

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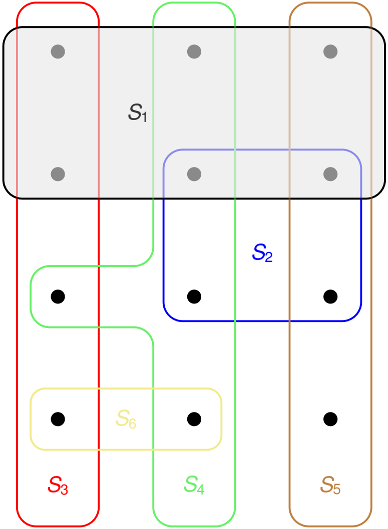

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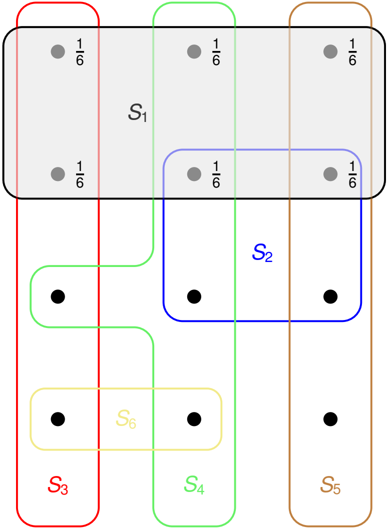

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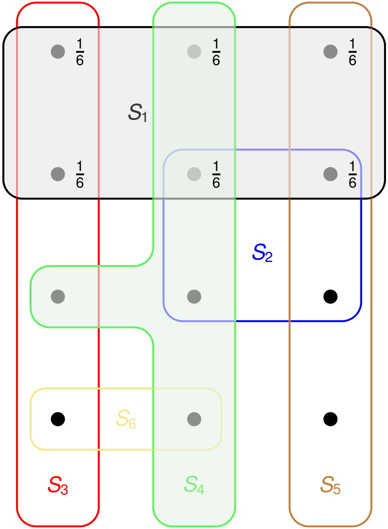

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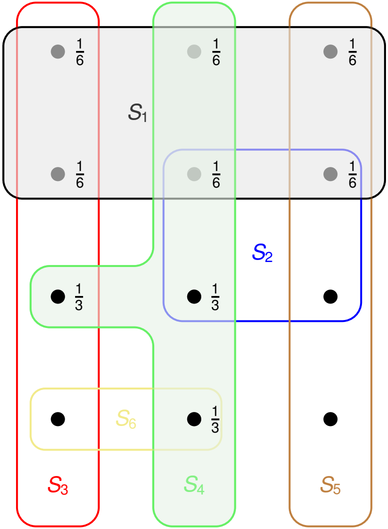

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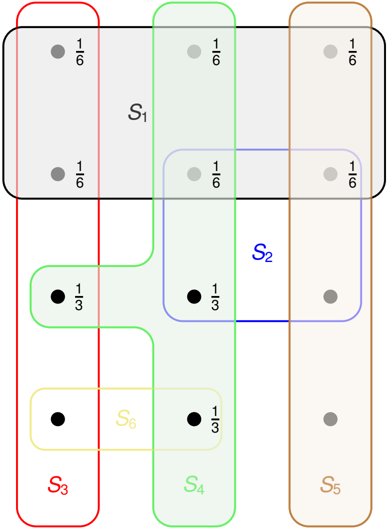

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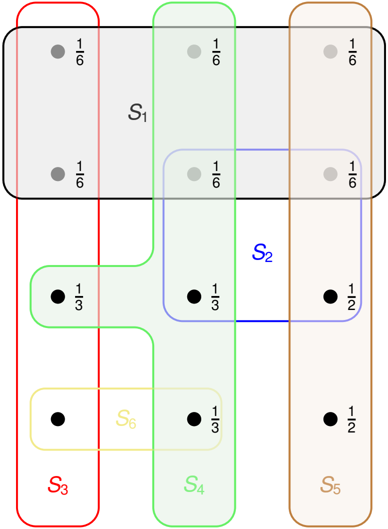

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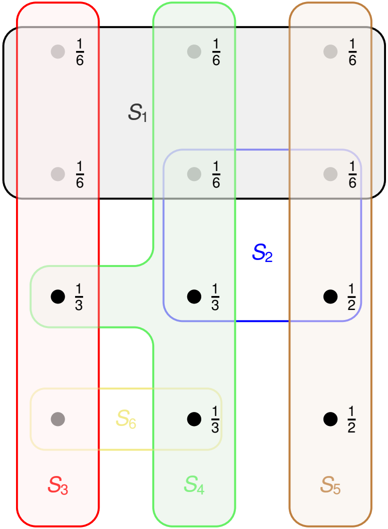

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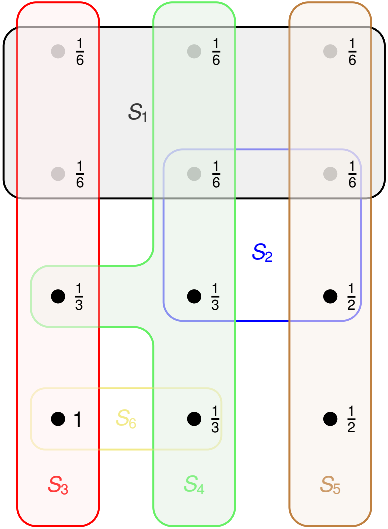

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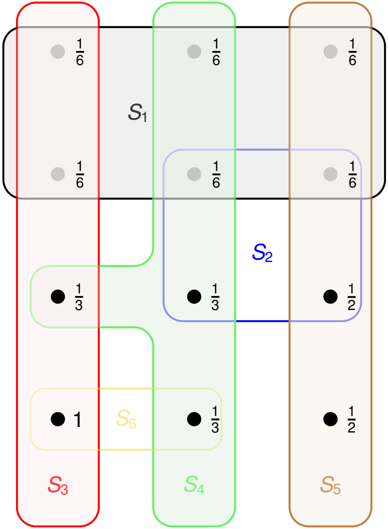

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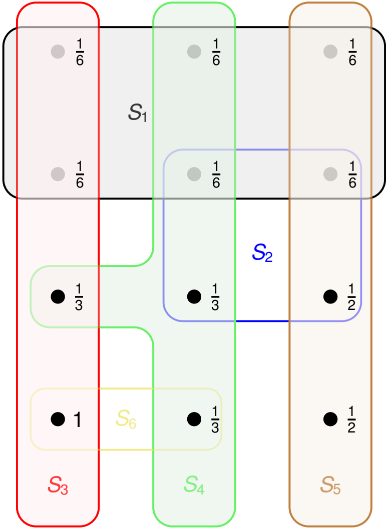

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Proof of Theorem 35.4 (1/2)

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If x is covered for the first time by a set S_i, then $c_x := \frac{1}{\left|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})\right|}$.

Proof.

- Each step of the algorithm assigns one unit of cost, so

$$|\mathcal{C}| = \sum_{x \in X} c_x \tag{1}$$

- Each element $x \in X$ is in at least one set in the optimal cover $\mathcal{C}^*$, so

$$\sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \geq \sum_{x \in X} c_x \tag{2}$$

- Combining 1 and 2 gives

$$|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x$$

Key Inequality: $\sum_{x \in S} c_x \leq H(|S|)$.

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Remaining uncovered elements in S

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Sets chosen by the algorithm

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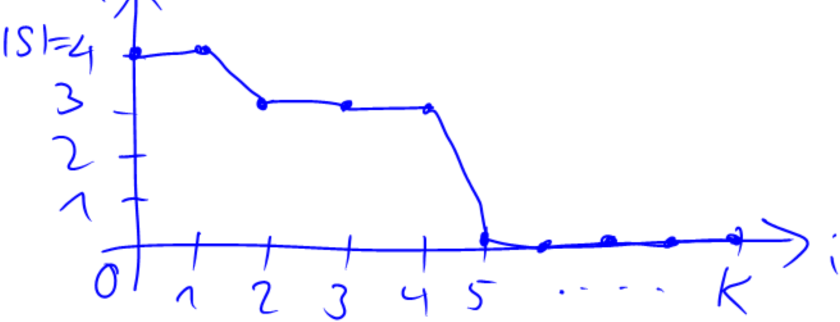

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Each factor is at most one.

$$\sum_{x \in S} c_x = \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|}$$

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u_i's are integers

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&= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) = H(u_0) - H(u_k)
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0

$\shortparallel$

$H(0)$

$\shortparallel$

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Set-Covering Problem (Summary)

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| : |S| \in \mathcal{F}\}) \leq \ln(n) + 1.$$

"Toy" Application:

Vertex Cover for Graphs with maximum degree 3

$G = (V, E)$

$\mathcal{F} = \{S_1, S_2, \ldots, S_{|V|}\}$ $\quad X = E$

Apply GREEDY-SET-COVER $\Rightarrow \rho(n) = H(3) = 1 + \frac{1}{2} + \frac{1}{3} < 2$

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polynomial-time

Lower Bound

Unless P=NP, there is no $c \cdot \ln(n)$ approximation algorithm for set cover for some constant $0 < c < 1$.

Set-Covering Problem (Summary)

The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : S \to \mathbb{Z}^+$

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Example where Greedy is a $(1/2) \cdot \log_2 n$ factor off

Instance

- Given any integer $k \geq 3$

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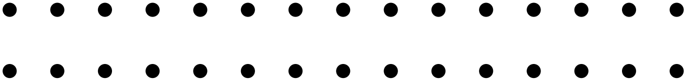

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- Given any integer $k \geq 3$
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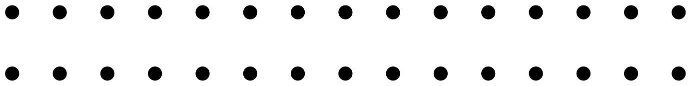

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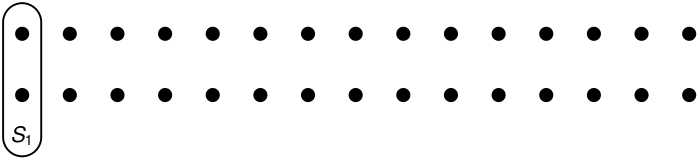

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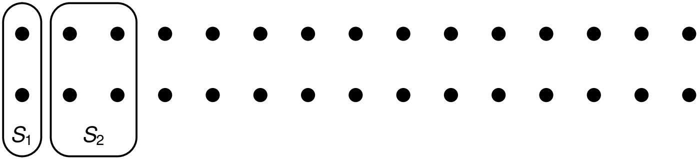

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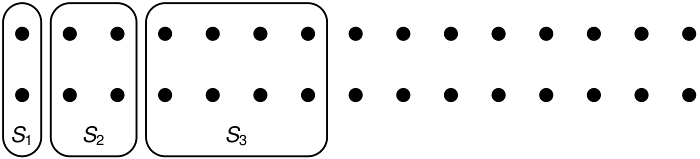

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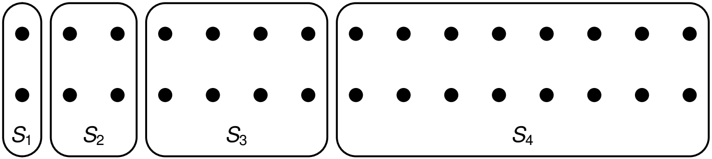

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Instance

- Given any integer $k \geq 3$
- There are $n = 2^{k+1} - 2$ elements overall
- Sets $S_1, S_2, \ldots, S_k$ are pairwise disjoint and each set contains $2, 4, \ldots, 2^k$ elements
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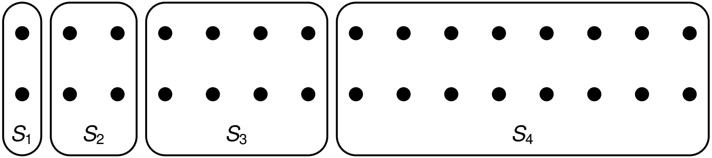
$k = 4$:



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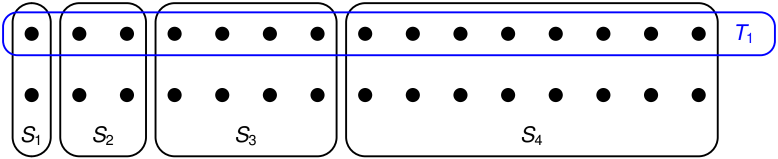

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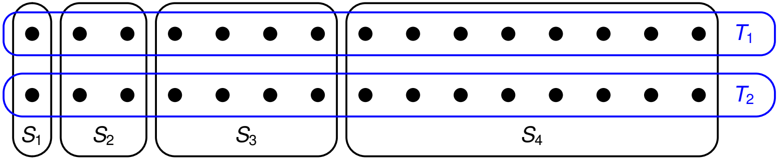
$k = 4$: $(n = 32 - 2 = 30)$



$= 15$

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$\| \; 16$

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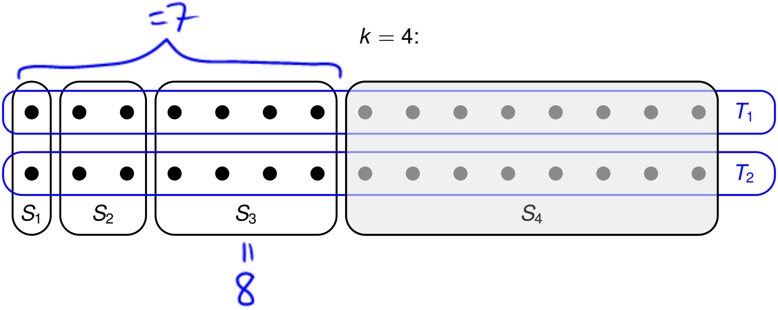

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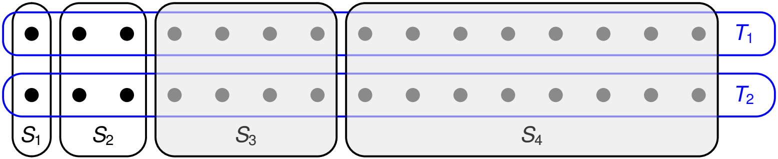

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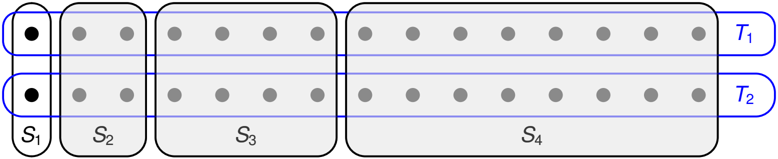

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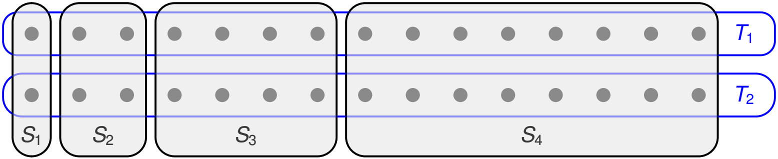

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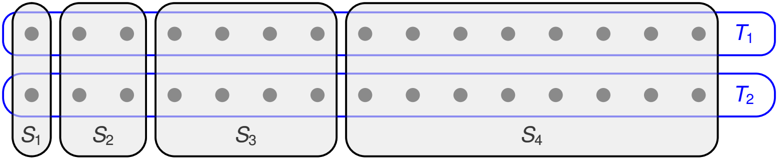

Solution of Greedy consists of k sets.

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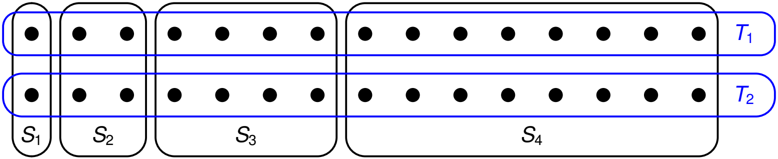

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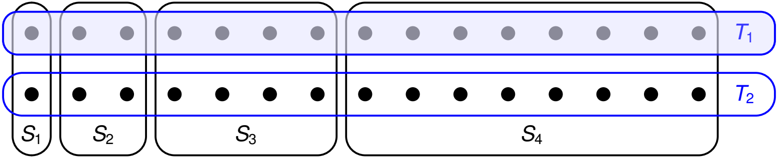

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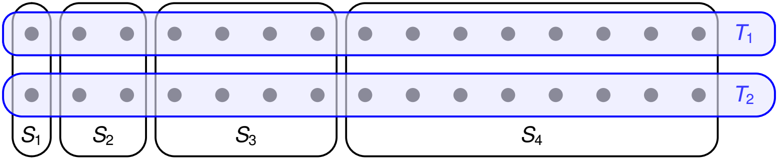

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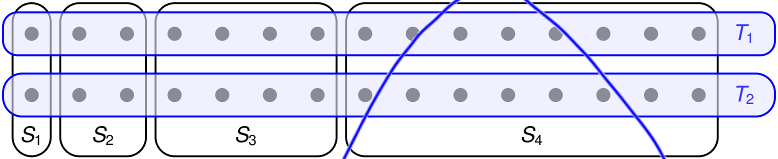
$k = 4$: $\rho(n) \geq \frac{k}{2} = \frac{\log_2(\frac{n}{2}) - 1}{2} = \frac{\log_2(n) - 2}{2}$



Solution of Greedy consists of k sets.

Optimum consists of 2 sets.