III. Linear Programming

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UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Introduction

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

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- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters

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Example: Political Advertising

- Imagine you are a politician trying to win an election
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- Aim: at least half of the registered voters in each of the three regions should vote for you

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Example: Political Advertising

- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters
- Aim: at least half of the registered voters in each of the three regions should vote for you
- Possible Actions: Advertise on one of the primary issues which are (i) building more roads, (ii) gun control, (iii) farm subsidies and (iv) a gasoline tax dedicated to improve public transit.

Political Advertising Continued

policy	urban	suburban	rural
build roads	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

Political Advertising Continued

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The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

- Possible Solution:
  - $20,000 on advertising to building roads
  - $0 on advertising to gun control
  - $4,000 on advertising to farm subsidies
  - $9,000 on advertising to a gasoline tax

Political Advertising Continued

policy	urban	suburban	rural
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The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

$$\text{urban}: 20{,}000 \times (-2) + 0 \times (+8) + 4{,}000 \times 0 + 9{,}000 \times 10 = 50{,}000 \checkmark$$

- Possible Solution:
  - $20,000 on advertising to building roads
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- Total cost: $33,000

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- Total cost: $33,000

What is the best possible strategy?

Towards a Linear Program

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The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

- $x_1 =$ number of thousands of dollars spent on advertising on building roads
- $x_2 =$ number of thousands of dollars spent on advertising on gun control
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Constraints:

Towards a Linear Program

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq \underline{50}$

Towards a Linear Program

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$

Towards a Linear Program

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

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- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Objective: Minimize $x_1 + x_2 + x_3 + x_4$

The Linear Program

Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcl}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{9}{r}{\boxed{x_1, x_2, x_3, x_4 \qquad\qquad\qquad\qquad\qquad\qquad \geq \quad 0}}
\end{array}
$$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & \multicolumn{5}{l}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
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& 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
\end{array}$$

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Formal Definition of Linear Program

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$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
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 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{7}{c}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n. = \langle a, x \rangle$$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr} \text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & & & & & & & & & \\ & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\ & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\ & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\ & & & \multicolumn{5}{l}{x_1, x_2, x_3, x_4} & \geq & 0 \end{array}$$

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$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$

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Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcr}
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\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
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 & & & \multicolumn{3}{c}{x_1, x_2, x_3, x_4} & & & \geq & 0
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- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \overset{\geq}{\leq} b$

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Linear Program for the Advertising Problem

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- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \underset{\leq}{\geq} b$

Linear Constraints

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & \multicolumn{7}{l}{\underline{x_1 \quad + \quad x_2 \quad + \quad x_3 \quad + \quad x_4}} & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{3}{c}{x_1, x_2, x_3, x_4} & & & & & \geq & 0
\end{array}$$

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- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

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- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \substack{\geq \\ \leq} b$

Linear Constraints

- Linear-Progamming Problem: either minimize or maximize a linear function subject to a set of linear constraints

A Small(er) Example

$$
\begin{array}{llcccc}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}
$$

A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
\end{array}
$$

$$
\left.\begin{array}{rcrcr}
4x_1 & - & x_2 & \leq & 8 \\
2x_1 & + & x_2 & \leq & 10 \\
5x_1 & - & 2x_2 & \geq & -2 \\
x_1, x_2 & & & \geq & 0
\end{array}\right\}
$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

A Small(er) Example

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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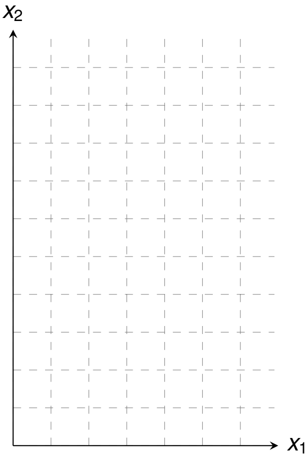

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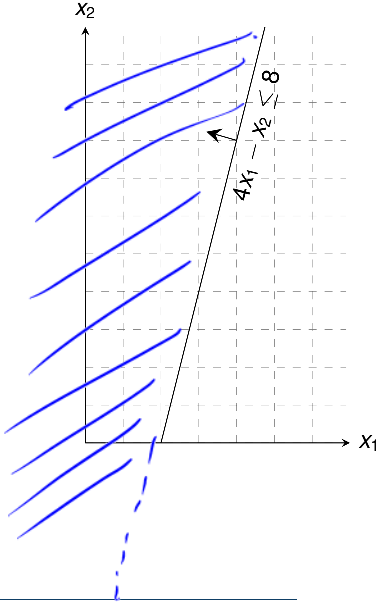

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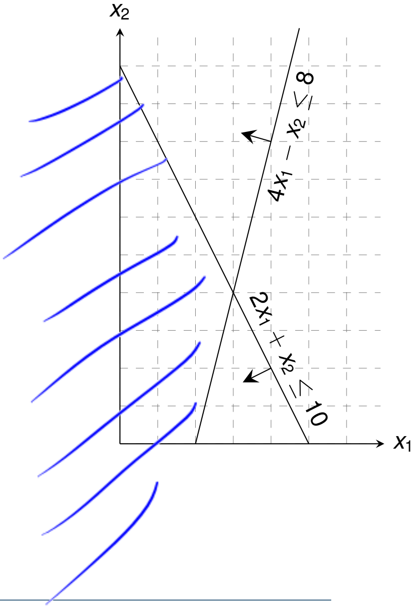

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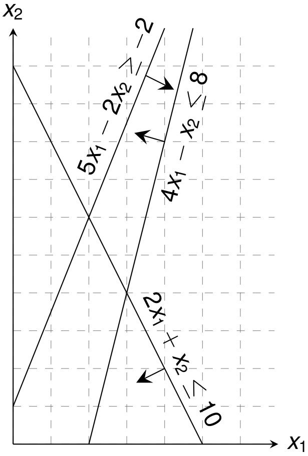

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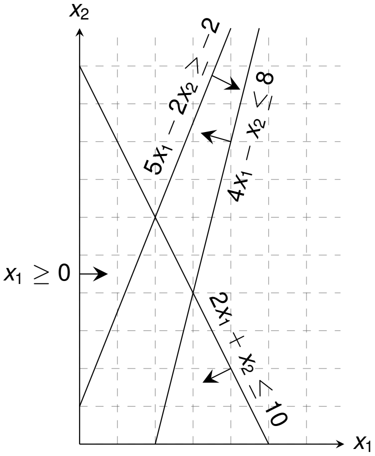

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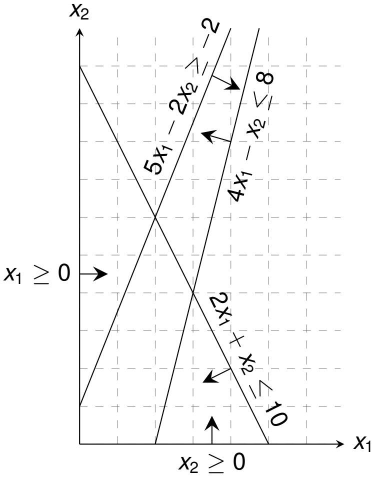

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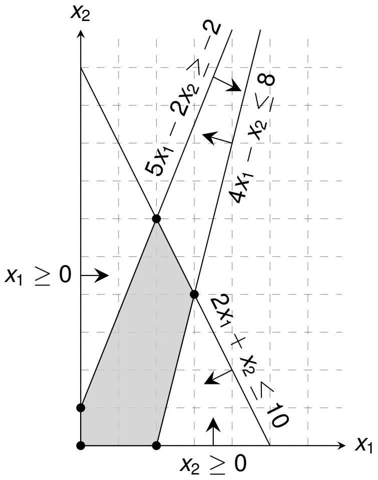

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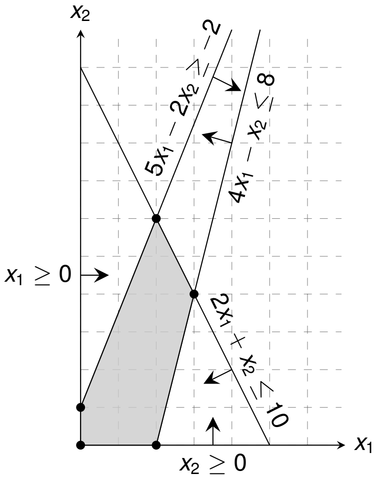
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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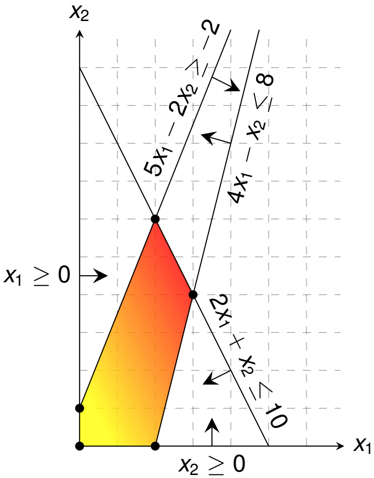

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\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
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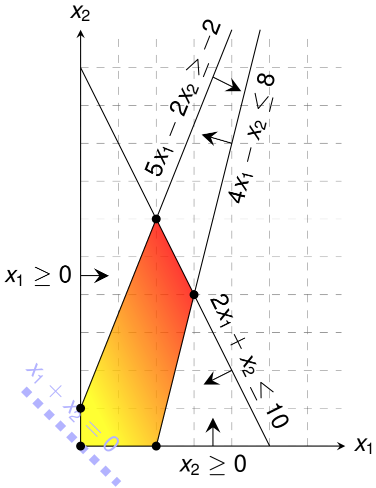
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1\,, x_2 & & & \geq & 0
\end{array}
$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
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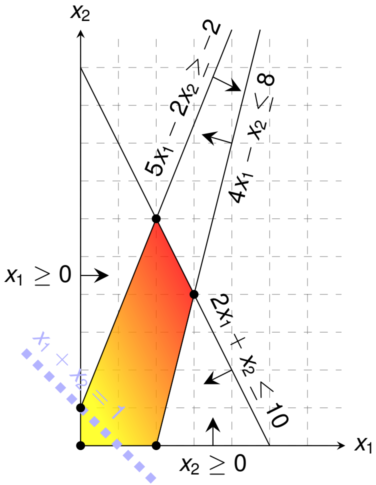

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\text{maximize} & x_1 & + & x_2 & & \\
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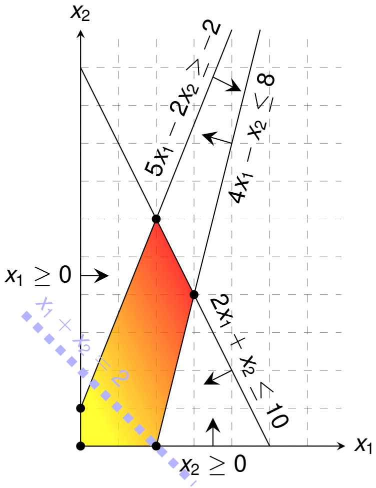

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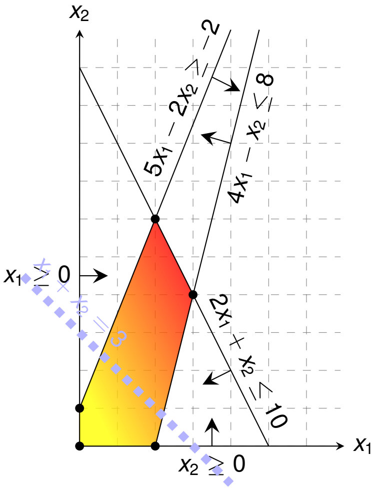

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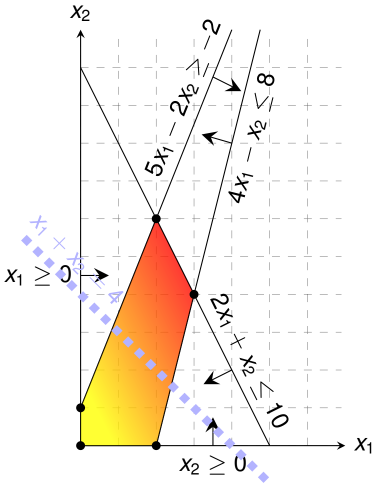

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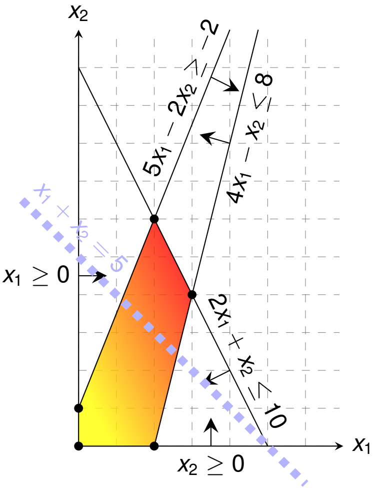

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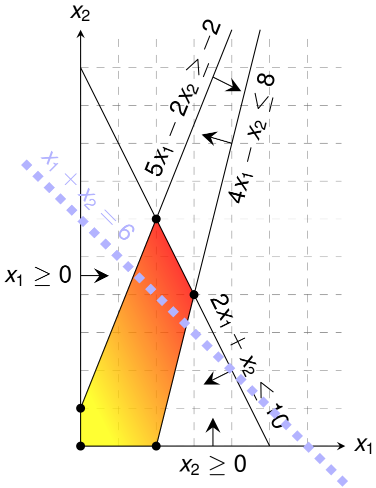

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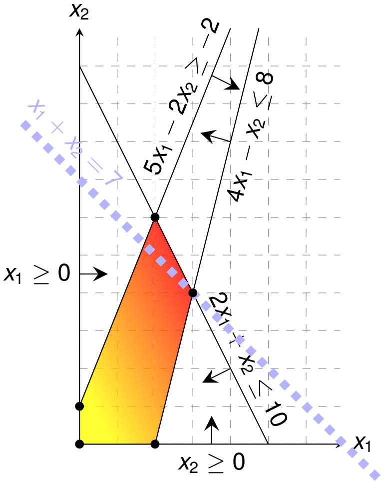

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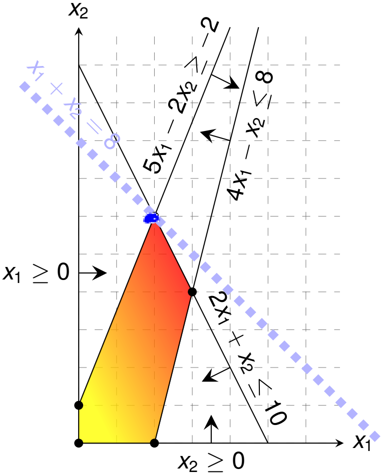

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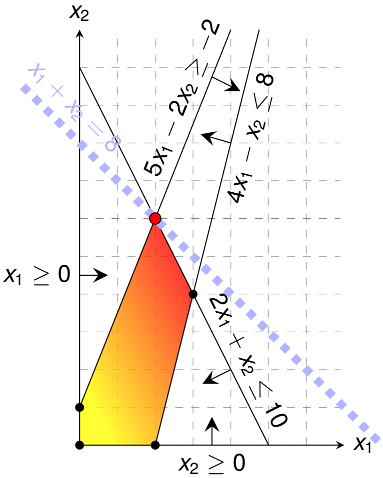

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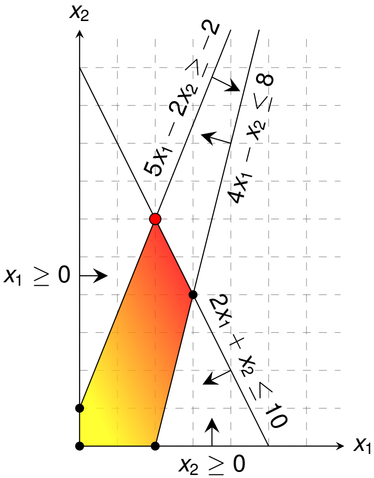

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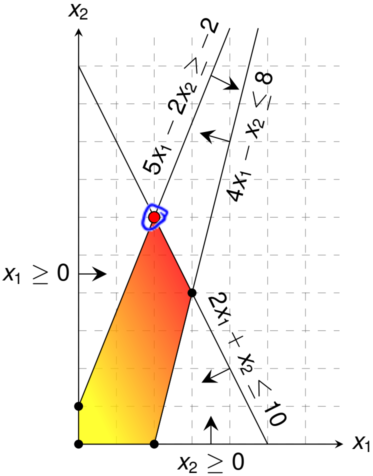

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Standard and Slack Forms

Standard Form

maximize $$\sum_{j=1}^{n} \underline{c_j} x_j$$

subject to

$$\sum_{j=1}^{n} \underline{a_{ij}} x_j \underline{\leq b_i} \qquad \text{for } i = 1, 2, \ldots, m$$

$$\underline{x_j \geq 0} \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximize $\sum_{j=1}^{n} c_j x_j$

Objective Function

subject to

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Standard and Slack Forms

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subject to

$n + m$ Constraints

$$\left\{\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i && \text{for } i = 1, 2, \ldots, m \\ x_j &\geq 0 && \text{for } j = 1, 2, \ldots, n \end{aligned}\right.$$

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$n + m$ Constraints

Non-Negativity Constraints

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$n + m$ Constraints

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

maximize $c^T x$

Inner product of two vectors

subject to

$$Ax \leq b$$

$$x \geq 0$$

Matrix-vector product

similar to
Gaussian Elimination
$(A \cdot x = b)$

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimization rather than maximization.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

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Equivalence: a correspondence (not necessarily a bijection) between solutions so that their objective values are identical.

When switching from maximization to minimization, sign of objective value changes.

Converting into Standard Form (1/5)

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$$\begin{array}{lrcrcr} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

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\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
\end{array}
$$

$$\boxed{x_1 \geq 0}$$

$\downarrow$ Replace x_2 by two non-negative variables x_2' and x_2''

$x_2 = x_2' - x_2''$

$x_2' \geq 0, x_2'' \geq 0$

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$$\begin{array}{llllll}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
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& x_1 & - & 2x_2 & \leq & 4 \\
& \multicolumn{5}{l}{\boxed{x_1 \qquad\qquad\qquad\qquad \geq \quad 0}}
\end{array}$$

Replace x_2 by two non-negative variables x_2' and x_2''

$$\begin{array}{llllll}
\text{maximize} & 2x_1 & - & \boxed{\boxed{3x_2' \quad + \quad 3x_2''}} & & \\
\text{subject to} & & & & & \\
& x_1 & + & \boxed{x_2' \quad - \quad x_2''} & = & 7 \\
& x_1 & - & \boxed{2x_2' \quad + \quad 2x_2''} & \leq & 4 \\
& \multicolumn{5}{l}{\boxed{\boxed{x_1, x_2', x_2''} \qquad\qquad\qquad\qquad \geq \quad 0}}
\end{array}$$

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$$\begin{array}{llcccccccc}
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\text{subject to} & & & & & & & \\
& \boxed{x_1 & + & x_2' & - & x_2'' & = & 7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& x_1, x_2', x_2'' & & & & & \geq & 0
\end{array}$$

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Reasons for a LP not being in standard form:

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$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & \boxed{x_1 + x_2' - x_2'' = 7} \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

⇣ Replace each equality by two inequalities.

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$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
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& x_1 - 2x_2' + 2x_2'' \leq 4 \\
& x_1, x_2', x_2'' \geq 0
\end{array}
$$

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$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
& \boxed{\begin{array}{l} x_1 + x_2' - x_2'' \leq 7 \\ x_1 + x_2' - x_2'' \geq 7 \end{array}} \\
& x_1 - 2x_2' + 2x_2'' \leq 4 \\
& x_1, x_2', x_2'' \geq 0
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$$
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\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

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$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
\hline
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
\hline
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\end{array}
$$

$\Big\downarrow$ Negate respective inequalities.

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\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
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\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{lrcrcrcr} \text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & - & x_3 & {\color{red}\leq} & 7 \\ & -x_1 & - & x_2 & + & x_3 & {\color{red}\leq} & -7 \\ & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{5}{l}{x_1, x_2, x_3} & \geq & 0 \end{array}$$

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 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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For the simplex algorithm, it is more convenient to work with equality constraints.

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Introducing Slack Variables

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Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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$$\boxed{s \geq 0.}$$

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s measures the slack between the two sides of the inequality.

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$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the ith inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& \underline{x_1} & \underline{+} & \underline{x_2} & \underline{-} & \underline{x_3} & \underline{\leq} & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

$\Big\downarrow$ Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
& \boxed{x_4} & = & 7 & - & x_1 & - & x_2 & + & x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcl}
x_1 & + & x_2 & - & x_3 & \le & 7 \\
-x_1 & - & x_2 & + & x_3 & \le & -7 \\
x_1 & - & 2x_2 & + & 2x_3 & \le & 4 \\
\multicolumn{5}{c}{x_1, x_2, x_3} & \ge & 0
\end{array}
\end{array}
$$

Introduce slack variables

$$
\begin{array}{ll}
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{4}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \ge & 0 & & &
\end{array}
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{l}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

$\downarrow$ Use variable z to denote objective function and omit the nonnegativity constraints.

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{rrrrrrrrr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

$(x_1, x_2, x_3, x_4, x_5, x_6)$
$= (0, 0, 0, 7, -7, 4)$

↑

not feasible!

This is called slack form.

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z &=& 0 &+& 2x_1 &-& 3x_2 &+& 3x_3 \end{array}$$

$$\begin{array}{cccc} \begin{array}{r} x_4\\ x_5\\ x_6 \end{array} & \begin{array}{c} =\\ =\\ = \end{array} & \boxed{\begin{array}{r} 7\\ -7\\ 4 \end{array}} & \begin{array}{crcrcr} - & x_1 & - & x_2 & + & x_3\\ + & x_1 & + & x_2 & - & x_3\\ - & x_1 & + & 2x_2 & - & 2x_3 \end{array} \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple $(\underline{N}, \underline{B}, \underline{A}, \underline{b}, \underline{c}, \underline{v})$ so that

$$z = \underline{v} + \sum_{j \in N} \underline{c_j} x_j$$

$$x_i = \underline{b_i} - \sum_{j \in N} \underline{a_{ij}} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Variables on the right hand side are indexed by the entries of N.

Slack Form (Example)

$$\begin{array}{rcrcrcrcr}
z & = & \boxed{28} & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & \boxed{8} & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & \boxed{4} & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & \boxed{18} & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

$(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 4, 0, 18, 0, 0)$
is a feasible solution with objective value 28

Slack Form (Example)

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}$$

Slack Form Notation

Slack Form (Example)

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

Slack Form (Example)

$$
\begin{array}{rcr}
z & = & 28
\end{array}
\begin{array}{rrrrrr}
- & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3}
\end{array}
$$

$$
\begin{array}{rcr}
x_1 & = & 8 \\[6pt]
x_2 & = & 4 \\[6pt]
x_4 & = & 18
\end{array}
\boxed{\begin{array}{rrrrrr}
+ & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\[6pt]
- & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\[6pt]
- & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \underline{\{3, 5, 6\}}$
- $$A = \begin{pmatrix} \underline{a_{13}} & \underline{a_{15}} & \underline{a_{16}} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Slack Form (Example)

$$
\begin{array}{rclcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

Slack Form (Example)

$$\begin{array}{rcrcccccc} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$

Slack Form (Example)

$$\begin{array}{rclclclcl}
z & = & \boxed{28} & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$

The Structure of Optimal Solutions

Definition

A point *x* is a vertex if it cannot be represented as a $\boxed{\text{strict}}$ convex combination of two other points in the feasible set.

$$x = \lambda \cdot y + (1-\lambda) \cdot z$$

$$\lambda \in (0,1)$$

The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

The set of feasible solutions is a convex set.

= finite intersection of halfspaces

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A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

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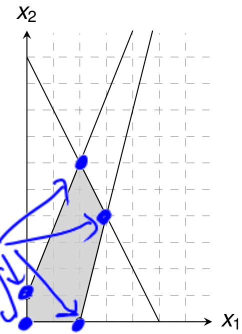
Theorem

If there exists an optimal solution, ~~it~~ occurs at a vertex of the polygon.

one of them

suffices to check a finite number of points



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A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

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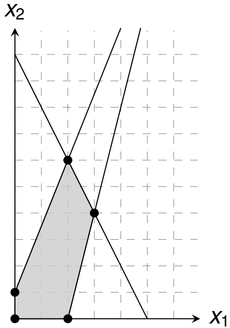
Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof: (non-examinable)

- Let x be an optimal solution which is not a vertex



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A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

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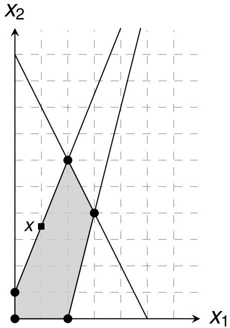
Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof:

- Let x be an optimal solution which is not a vertex



The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

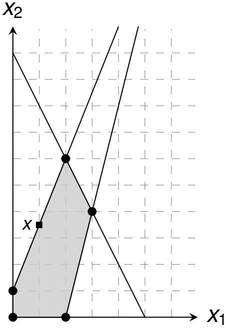
The set of feasible solutions is a convex set.

Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof:

- Let x be an optimal solution which is not a vertex $\Rightarrow \exists$ vector d so that $x - d$ and $x + d$ are feasible



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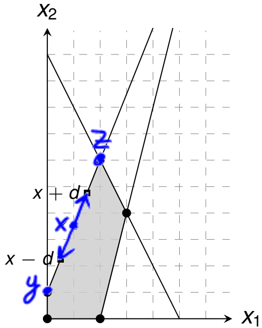

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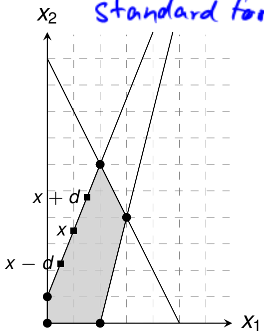
  slack form
- Since $\boxed{A(x+d) = b}$ and $\boxed{Ax = b}$ $\Rightarrow$ $\boxed{Ad = 0}$

this illustration is in standard form



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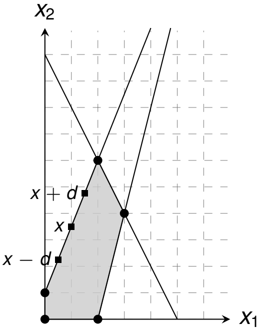

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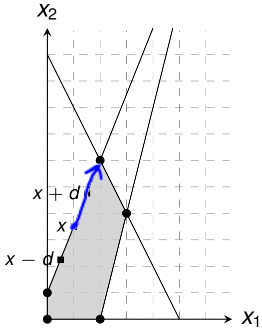

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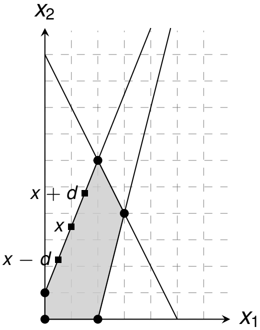
- Consider $x + \lambda d$ as a function of $\lambda \geq 0$
- Case 1: There exists $\underline{j}$ with $\underline{d_j < 0}$



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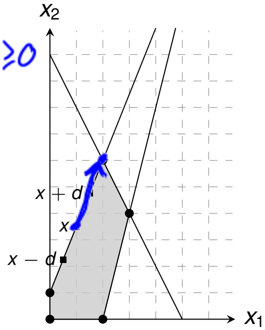
$x_j + d_j \geq 0$

- Since $A(x + d) = b$ and $Ax = b \Rightarrow Ad = 0$
- W.l.o.g. assume $c^T d \geq 0$ (otherwise replace d by $-d$)
- Consider $x + \lambda d$ as a function of $\lambda \geq 0$

$\boxed{x_j > 0}$

- Case 1: There exists j with $d_j < 0$
  - Increase $\underline{\lambda}$ from 0 to λ' until a new entry of $\boxed{x + \lambda d}$ becomes zero



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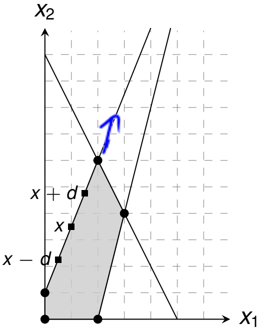
- Consider $x + \lambda d$ as a function of $\lambda \geq 0$

- Case 1: There exists j with $d_j < 0$
  - Increase λ from 0 to λ' until a new entry of $x + \lambda d$ becomes zero
  - $\boxed{x + \lambda' d \text{ feasible}}$ since $\boxed{A(x + \lambda' d) = Ax = b}$ and $\boxed{x + \lambda' d \geq 0}$



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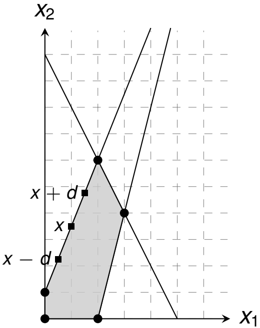
- Consider $x + \lambda d$ as a function of $\lambda \geq 0$

- Case 1 There exists j with $d_j < 0$
  - Increase λ from 0 to λ' until a new entry of $x + \lambda d$ becomes zero !
  - $x + \lambda' d$ feasible, since $A(x + \lambda' d) = Ax = b$ and $x + \lambda' d \geq 0$
  - $c^T(x + \lambda' d) = c^T x + c^T \lambda' d \geq c^T x$



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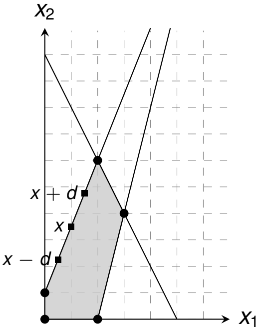
- Case 2: For all j, $d_j \geq 0$



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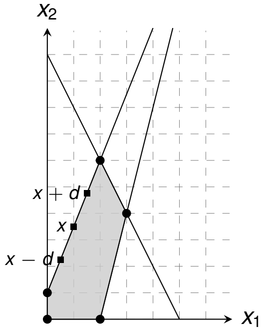
- Consider $x + \lambda d$ as a function of $\lambda \geq 0$

- Case 2: For all j, $d_j \geq 0$
  - $x + \lambda d$ is feasible for all $\lambda \geq 0$: $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$



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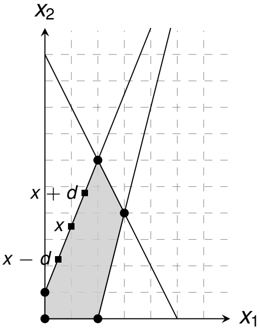
- Case 2: For all j, $d_j \geq 0$
  - $x + \lambda d$ is feasible for all $\lambda \geq 0$: $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
  - If $\lambda \to \infty$, then $c^T(x + \lambda d) \to \infty$



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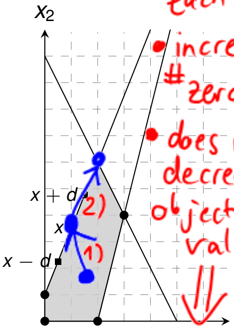
  - If $\lambda \to \infty$, then $c^T(x + \lambda d) \to \infty$

$\Rightarrow$ This contradicts the assumption that there exists an optimal solution.



each step:
- increases # zero's
- does not decrease objective value

⇓

has to hit vertex (eventually)