II. Matrix Multiplication

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UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

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SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
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SQUARE-MATRIX-MULTIPLY(A, B) takes time $\Theta(n^3)$.

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

This definition suggests that $n \cdot n^2 = n^3$ arithmetic operations are necessary.

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$n = A.rows$
let C be a new $n \times n$ matrix
for $i = 1$ **to** n
    for $j = 1$ **to** n
        $c_{ij} = 0$
        for $k = 1$ **to** n
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C

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Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Each equation specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their products.

Divide & Conquer: First Approach (Pseudocode)

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
$$C_{11} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$$

Divide & Conquer: First Approach (Pseudocode)

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SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
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return C
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\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
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\end{aligned}
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Divide & Conquer: First Approach (Pseudocode)

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SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
1   n = A.rows
2   let C be a new n × n matrix
3   if n == 1
4       c11 = a11 · b11
5   else partition A, B, and C as in equations (4.9)
6       C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
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7       C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
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9       C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
              + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
10  return C
```

Line 5: Handle submatrices implicitly through index calculations instead of creating them.

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
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Let $T(n)$ be the runtime of this procedure.

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

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Solution: $T(n) =$

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Solution: $T(n) = \Theta(8^{\log_2 n})$

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

No improvement over the naive algorithm!

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
  $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
  $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
  $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
  $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
  $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ \boxed{8} \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(\boxed{8}^{\log_2 n}) = \Theta(n^{\boxed{3}})$

Goal: Reduce the number of multiplications

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.
3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.
3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Time for steps 1,2,4: $\Theta(n^2)$, hence $\underline{T(n) = 7 \cdot T(n/2) + \Theta(n^2)}$ $\boxed{\Rightarrow}$ $\boxed{T(n) = \Theta(n^{\log 7})}$.

Solving the Recursion

$$
\begin{aligned}
T(n) &= 7 \cdot T(n/2) + c \cdot n^2 \\
&= 7 \cdot \left(7 \cdot T(n/4) + c \cdot (n/2)^2\right) + c \cdot n^2 \\
&= 7^2 \cdot T(n/4) + 7c \cdot (n/2)^2 + c \cdot n^2 \\
&= 7^2 \cdot \left(7 \cdot T(n/8) + c \cdot (n/4)^2\right) + 7c \cdot (n/2)^2 + cn^2 \\
&= 7^3 \cdot T(n/8) + \underbrace{7^2 c \cdot (n/4)^2 + 7c \cdot (n/2)^2 + cn^2} \\
&= \ldots \\
&= 7^{\log_2 n} \cdot T(1) + \sum_{i=0}^{\log_2 n - 1} 7^i \cdot c \cdot \left(\frac{n}{2^i}\right)^2 \\
&= 7^{\log_2 n} \cdot \Theta(1) + \sum_{i=0}^{\log_2 n - 1} \underbrace{\left(\frac{7}{4}\right)}_{>1}^{i} \cdot c \cdot n^2 \\
&= 7^{\log_2 n} \cdot \Theta(1) + \Theta\left(\left(\frac{7}{4}\right)^{\log_2 n - 1} \cdot n^2\right) \\
&= 7^{\log_2 n} \cdot \Theta(1) + \Theta\left(7^{\log_2 n - 1}\right) = \Theta\left(2^{\log_2 7 \cdot \log_2 n}\right) \\
&= \Theta\left(n^{\log_2 7}\right)
\end{aligned}
$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}
$$

Claim

$$
\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}
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Proof:

Details of Strassen's Algorithm

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$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

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$$P_5 + P_4 - P_2 + P_6 =$$

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$$

Proof:

$$
\begin{aligned}
P_5 + P_4 - P_2 + P_6 = {} & A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22} + A_{22}B_{21} - A_{22}B_{11} \\
& - A_{11}B_{22} - A_{12}B_{22} + A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}
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Proof:

$$\begin{aligned}
P_5 + P_4 - P_2 + P_6 = {} & \underline{A_{11}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} \\
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\begin{aligned}
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$$

Other three blocks can be verified similarly.

Proof:

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\end{aligned}$$

$\alpha_{ij}, \beta_{ij} \in \{-1, 0, 1\}$

$$\boxed{P_i = (\alpha_{i1} \cdot A_{11} + \alpha_{i2} \cdot A_{12} + \alpha_{i3} \cdot A_{21} + \alpha_{i4} \cdot A_{22}) \cdot (\beta_{i1} \cdot B_{11} + \beta_{i2} \cdot B_{12} + \beta_{i3} \cdot B_{21} + \beta_{i4} \cdot B_{22})}$$

Claim

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Other three blocks can be verified similarly.

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\end{aligned}$$

Current State-of-the-Art

Conjecture: Does a quadratic-time algorithm exist?

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Asymptotic Complexities:

- $O(n^3)$, naive approach

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- $O(n^{2.796})$, Pan (1978)
- $O(n^{2.522})$, Schönhage (1981)
- $O(n^{2.517})$, Romani (1982)
- $O(n^{2.496})$, Coppersmith and Winograd (1982)
- $O(n^{2.479})$, Strassen (1986)
- $O(n^{2.376})$, Coppersmith and Winograd (1989)

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Conjecture: Does a quadratic-time algorithm exist?

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- $O(n^{2.374})$, Stothers (2010)
- $O(n^{2.3728642})$, V. Williams (2011)
- $O(n^{2.3728639})$, Le Gall (2014)
- ...

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Memory Models

Distributed Memory

- Each processor has its private memory
- Access to memory of another processor via messages

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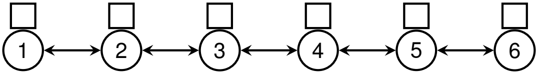
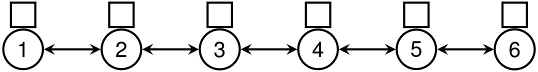


Memory Models

Distributed Memory

- Each processor has its private memory
- Access to memory of another processor via messages



Shared Memory

- Central location of memory
- Each processor has direct access

Memory Models

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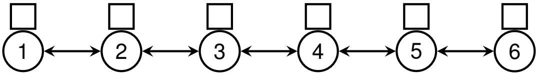

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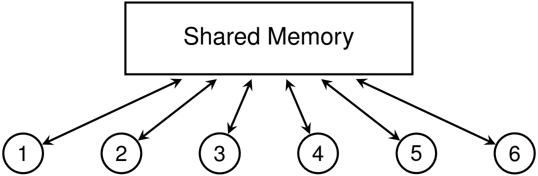

Dynamic Multithreading

- Programming shared-memory parallel computer difficult

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- Programming shared-memory parallel computer difficult
- Use concurrency platform which coordinates all resources

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Scheduling jobs, communication protocols, load balancing etc.

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Functionalities:

Dynamic Multithreading

- Programming shared-memory parallel computer difficult
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Functionalities:

- **`spawn`**

Dynamic Multithreading

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Functionalities:

- `spawn`
  - (optional) prefix to a procedure call statement
  - procedure is executed in a separate thread
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  - (optinal) prefix to the standard loop **`for`**
  - each iteration is called in its own thread

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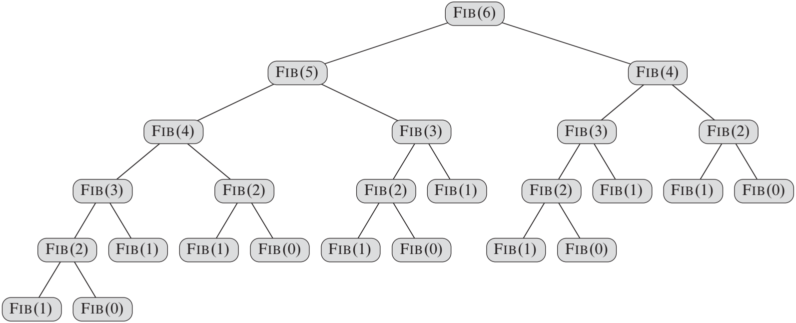
- **`sync`**
  - wait until all spawned threads are done
- **`parallel`**
  - (optinal) prefix to the standard loop **`for`**
  - each iteration is called in its own thread

Only logical parallelism, but not actual!
Need a scheduler to map threads to processors.

Computing Fibonacci Numbers Recursively (Fig. 27.1)

```
FIB(n)
  if n<=1 return n
  else x=FIB(n-1)
       y=FIB(n-2)
       return x+y
```

Computing Fibonacci Numbers Recursively (Fig. 27.1)




```
FIB(n)
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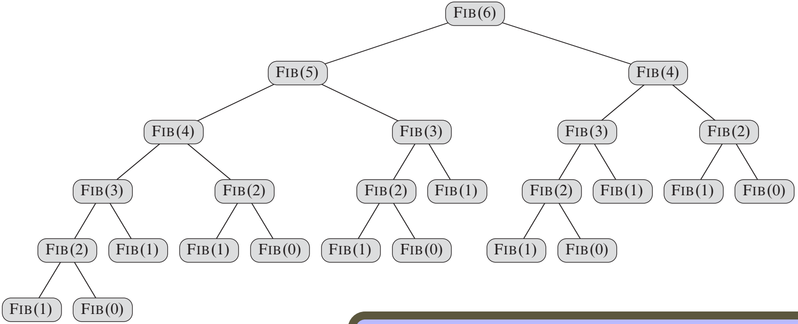
Computing Fibonacci Numbers Recursively (Fig. 27.1)



Very inefficient – exponential time!

```
FIB(n)
  if n<=1 return n
  else x=FIB(n-1)
       y=FIB(n-2)
       return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

```
P-FIB(n)
  if n<=1 return n
  else x=spawn P-FIB(n-1)
       y=P-FIB(n-2)
       sync
       return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

- Without **spawn** and **sync** same pseudocode as before
- **spawn** does not imply parallel execution (depends on scheduler)

```
P-FIB(n)
  if n<=1 return n
  else x=spawn P-FIB(n-1)
       y=P-FIB(n-2)
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       return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

Computation Dag $G = (V, E)$

```
P-FIB(n)
  if n<=1 return n
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       return x+y
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

Computation Dag $G = (V, E)$

- V set of threads (instructions/strands without parallel control)

```
P-FIB(n)
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

Computation Dag $G = (V, E)$

- V set of threads (instructions/strands without parallel control)
- E set of dependencies

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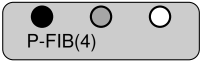


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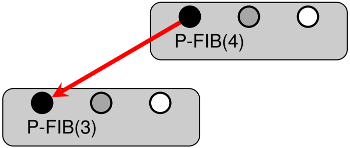


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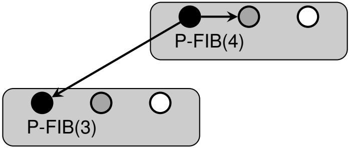


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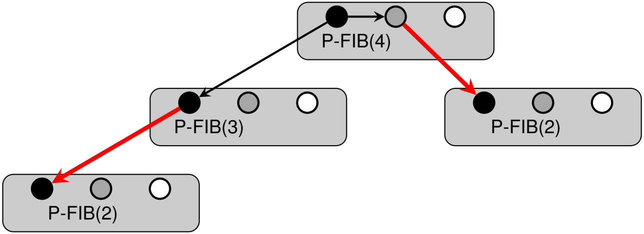


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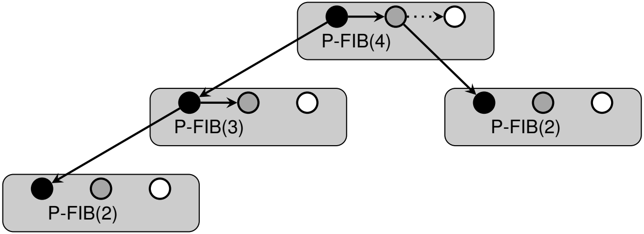


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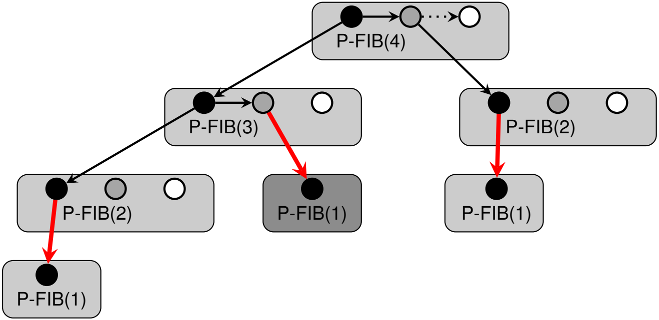


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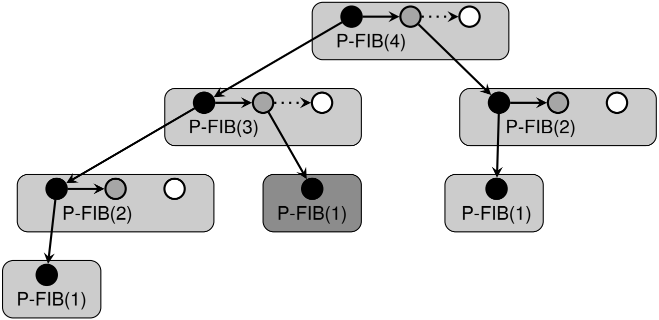


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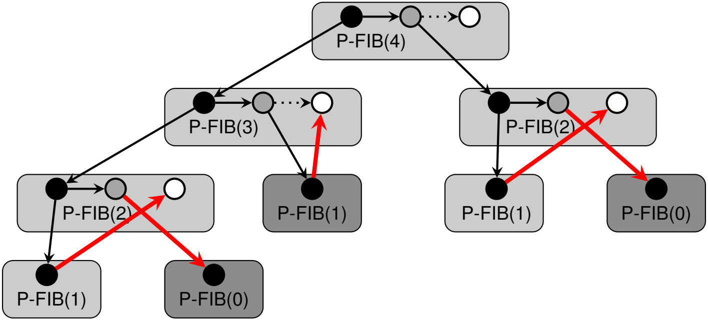


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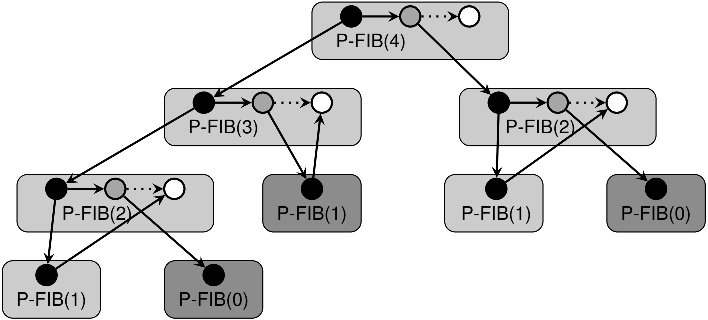


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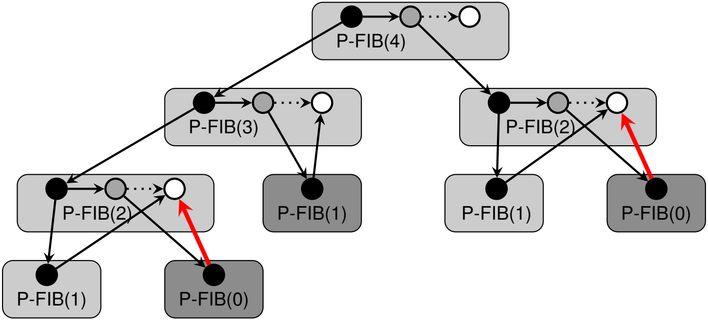


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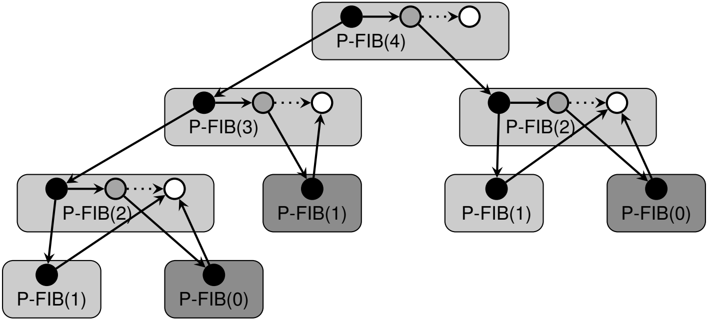


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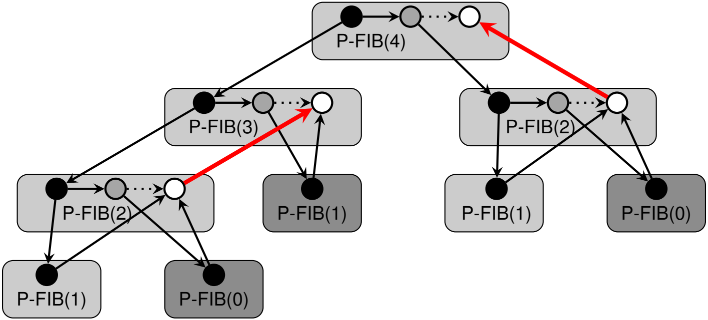


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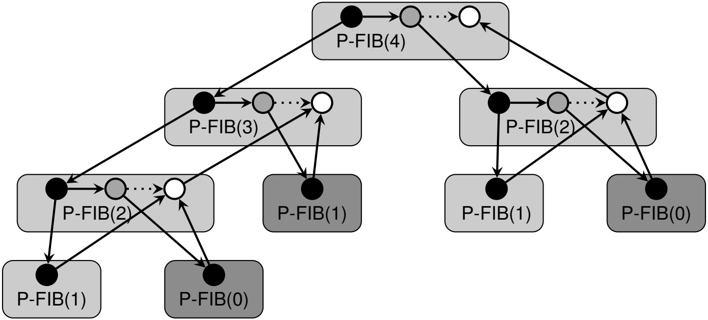


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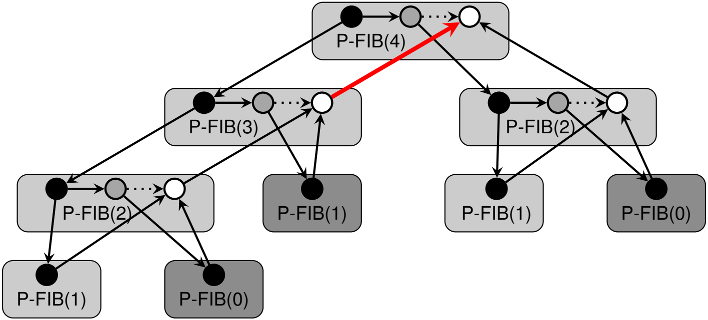


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0: P-FIB(n)
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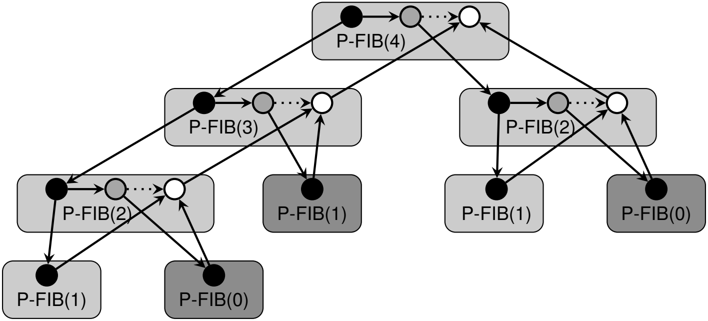


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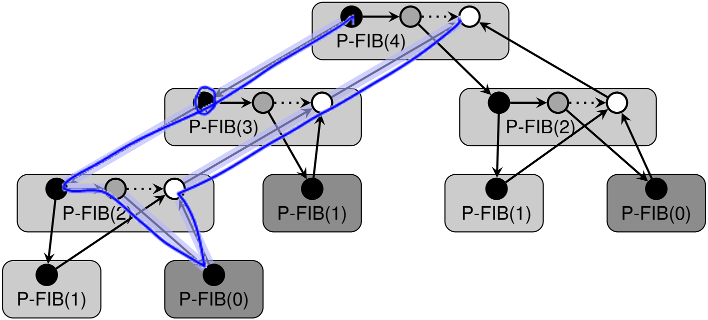


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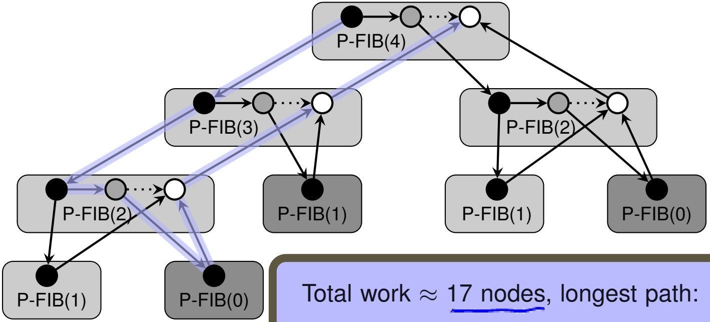

Total work ≈ 17 nodes, longest path: 8 nodes

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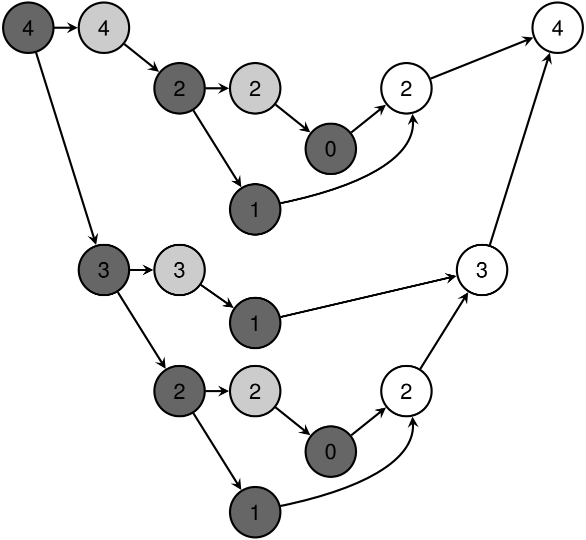

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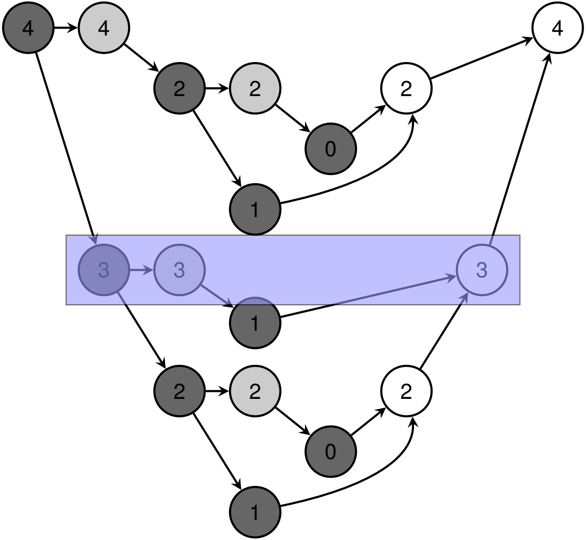

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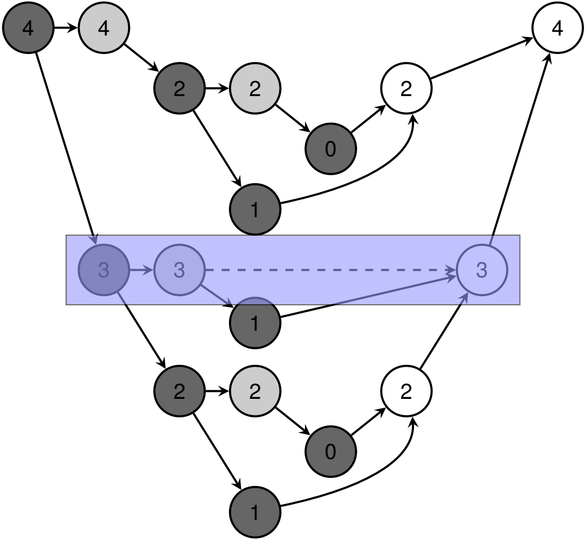

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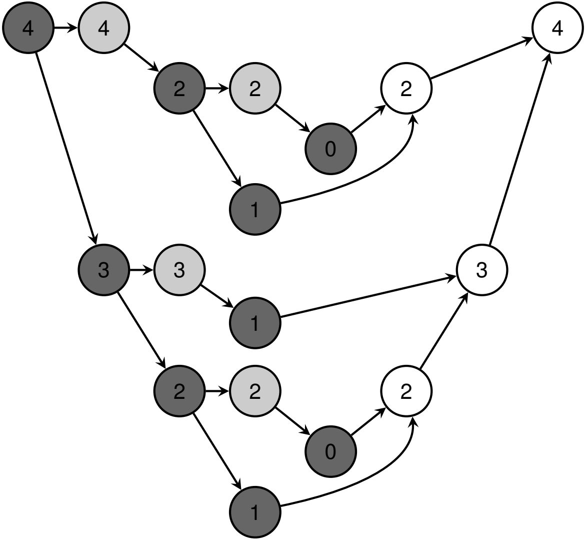

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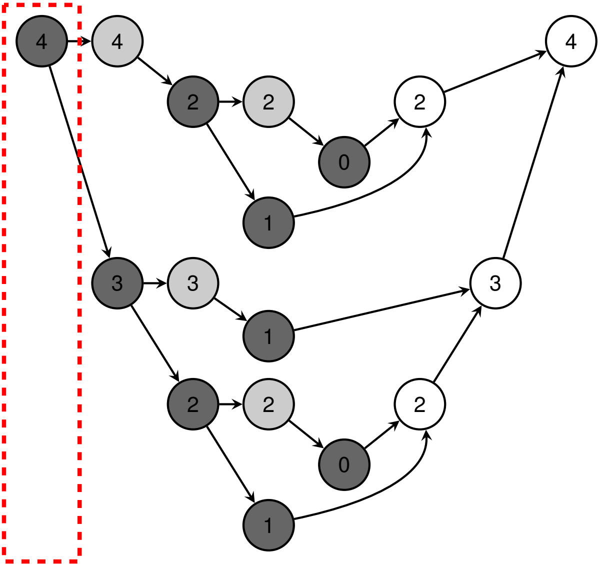

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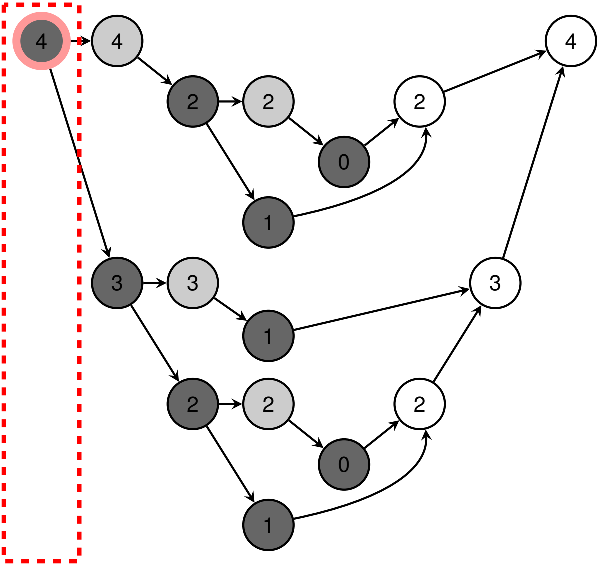

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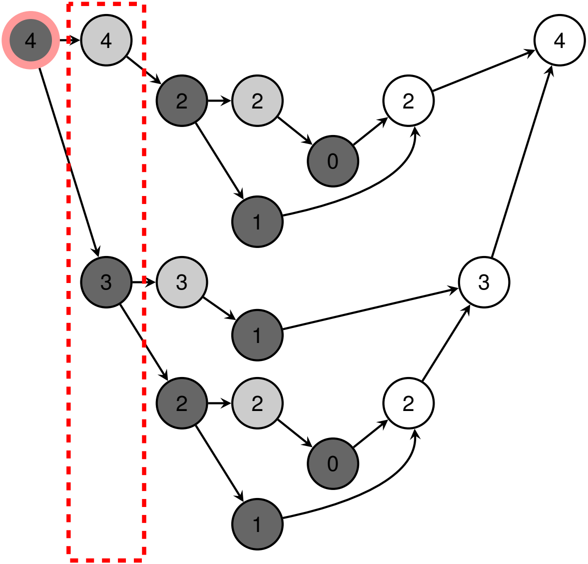

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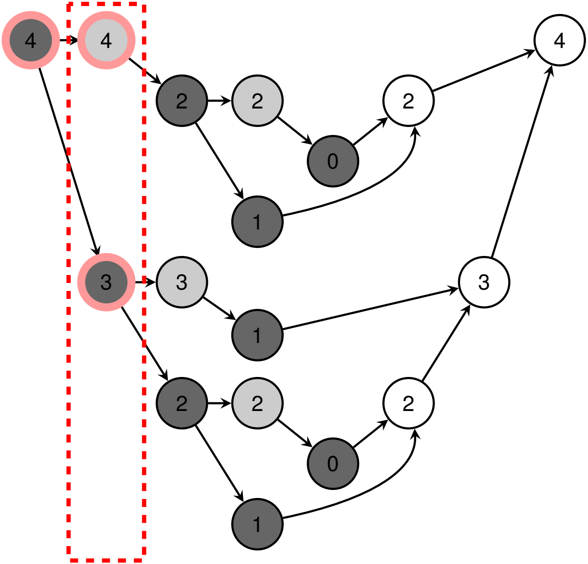

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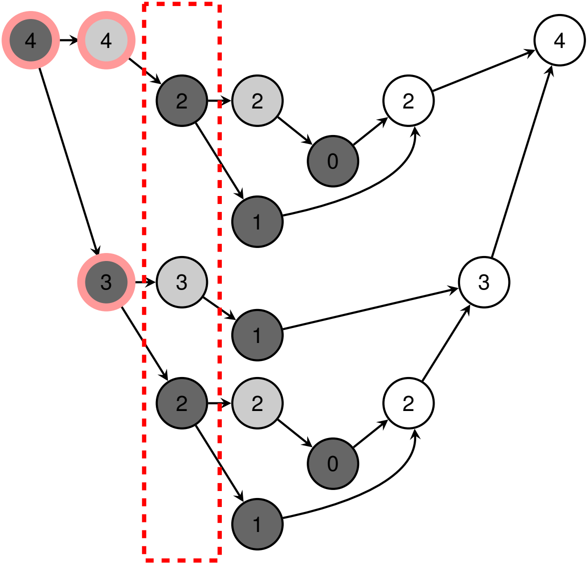

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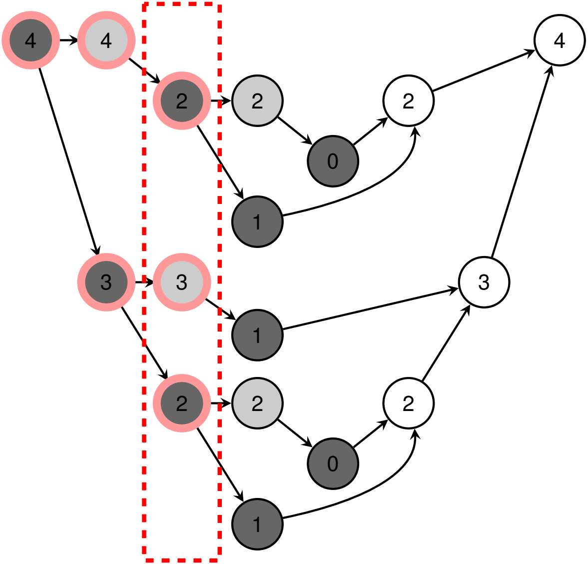

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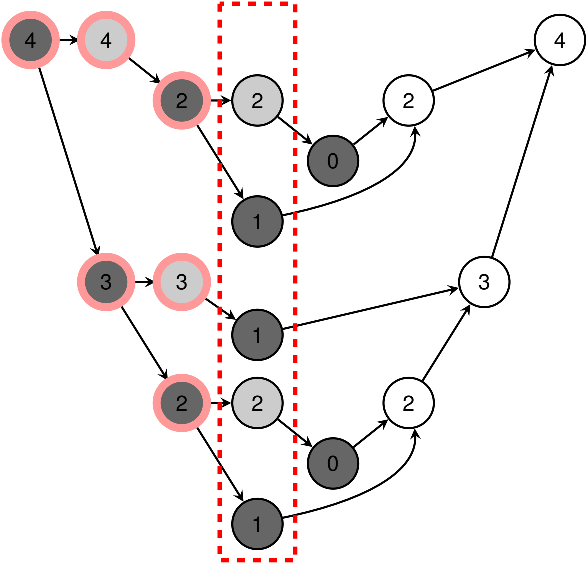

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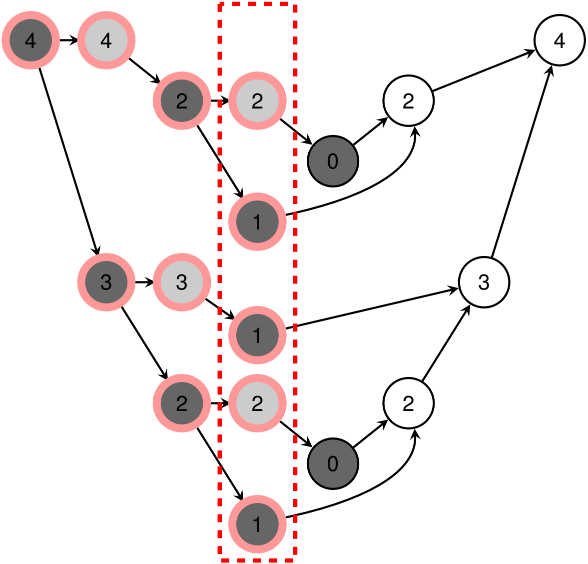

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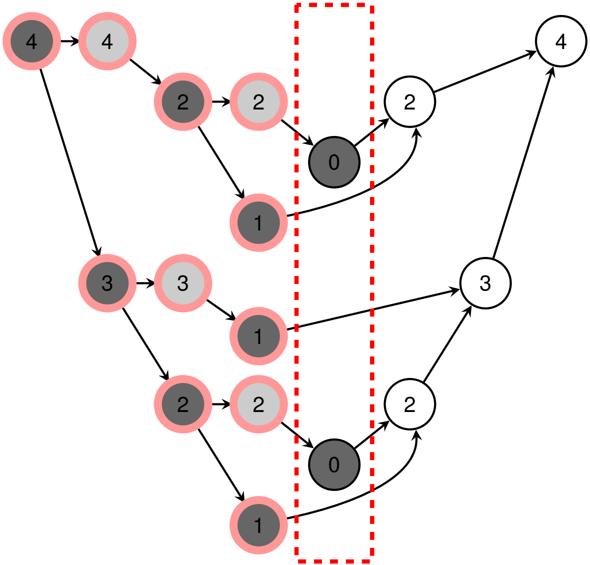

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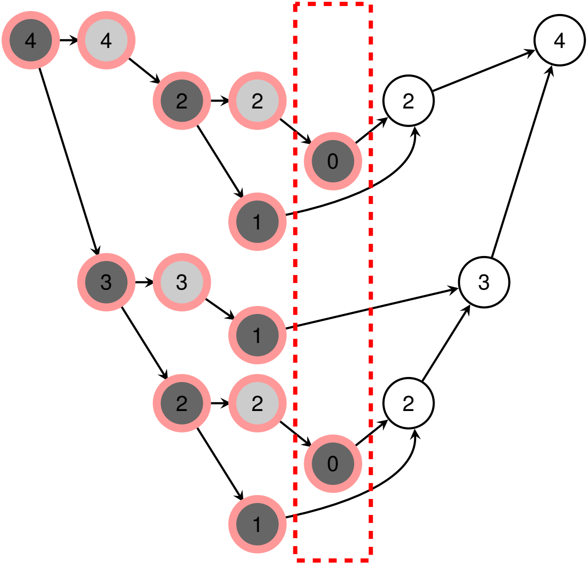

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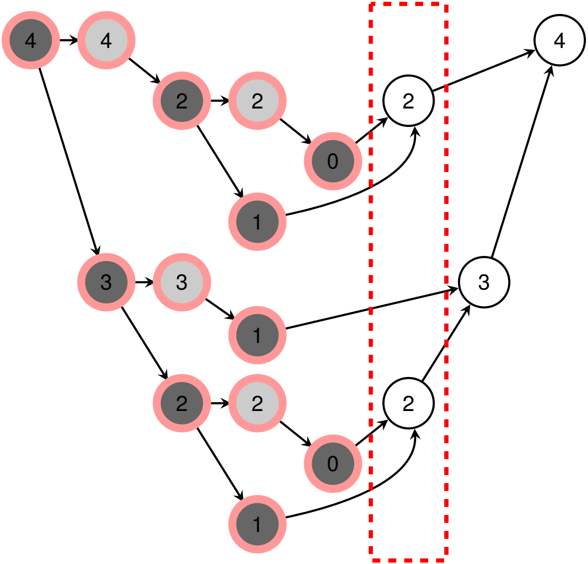

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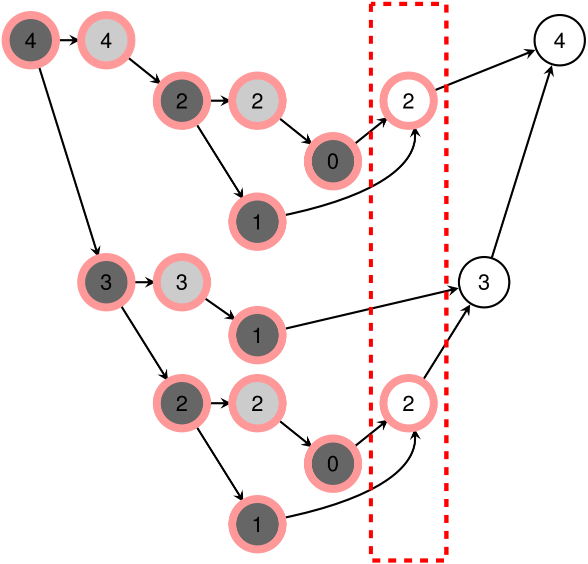

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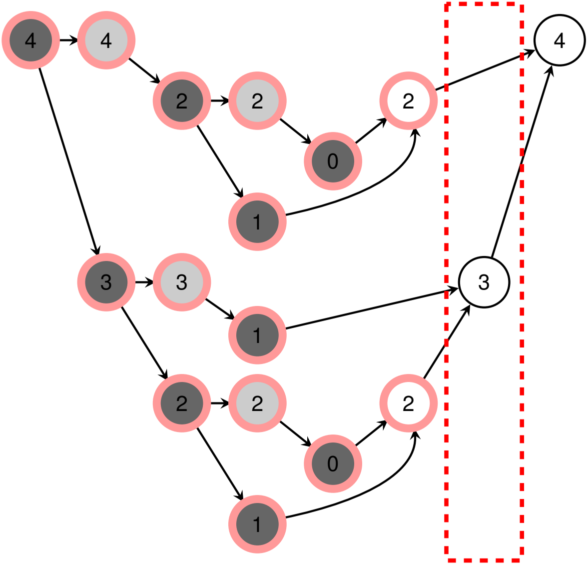

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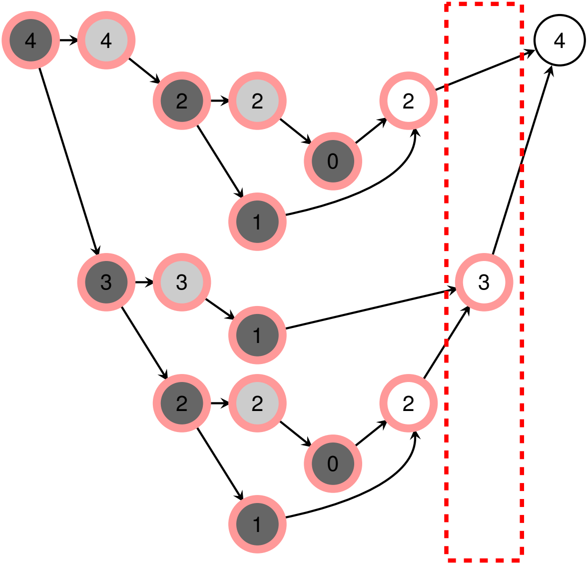

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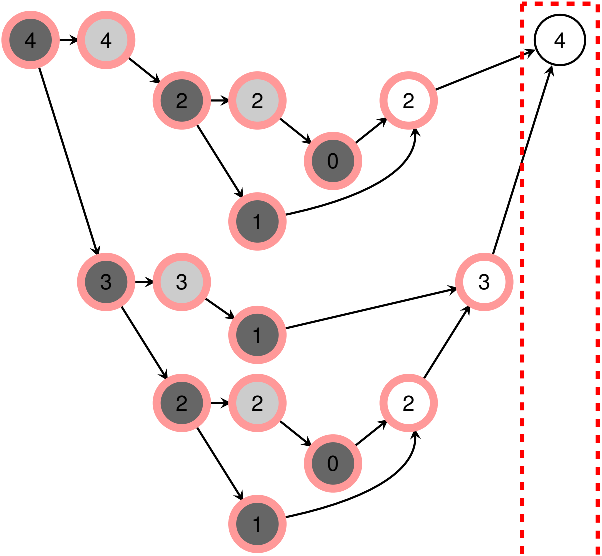

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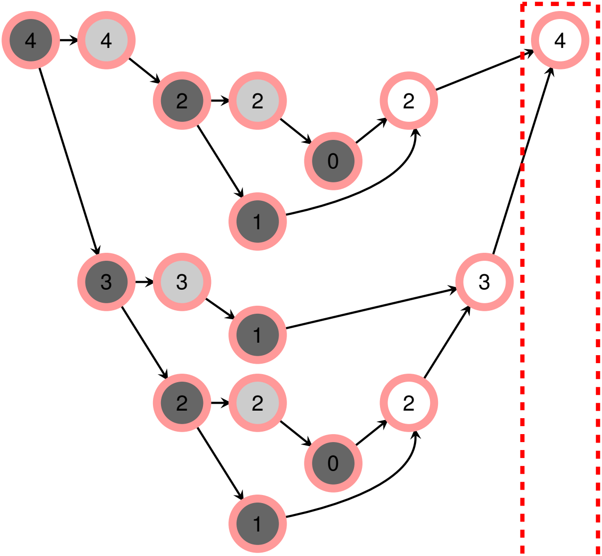

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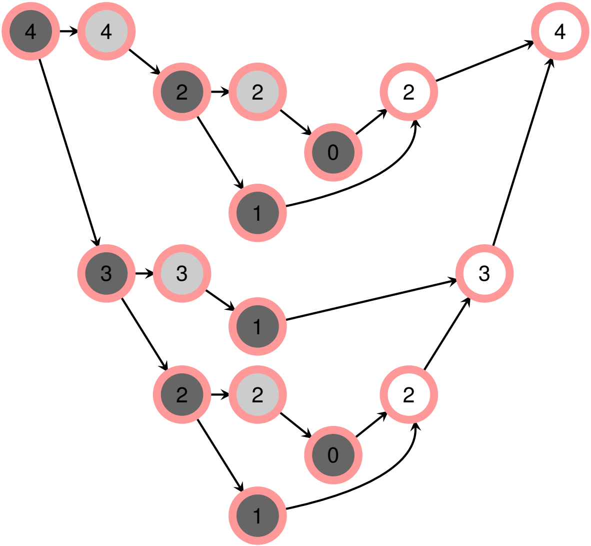

Performance Measures

Work

Total time to execute everything on single processor.

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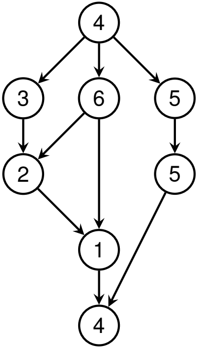
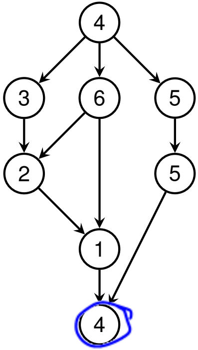


Performance Measures

Work

Total time to execute everything on single processor.

$\sum = 30$



Performance Measures

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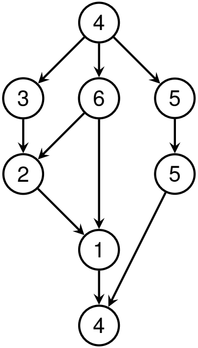
Total time to execute everything on single processor.

Span

Longest time to execute the threads along any path.



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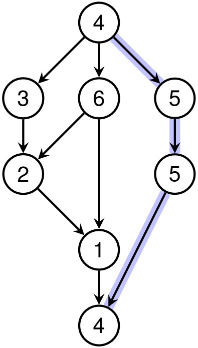
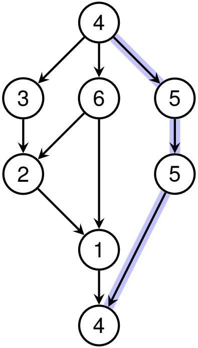
Longest time to execute the threads along any path.



Performance Measures

$\sum = 18$



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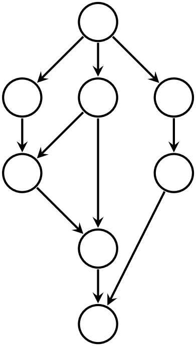

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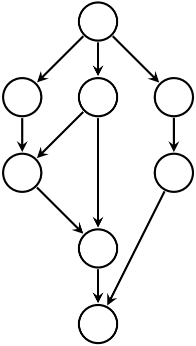
If each thread takes unit time, span is the length of the critical path.



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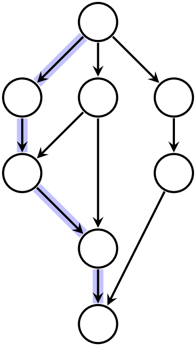

Performance Measures

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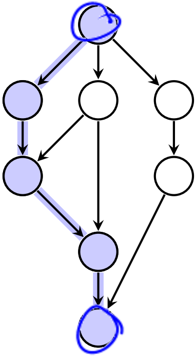
$\#\text{nodes} = 5$



Work Law and Span Law

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- $T_1 = \text{work}$, $T_\infty = \text{span}$

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- $P =$ number of (identical) processors
- $T_P =$ running time on P processors

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
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Running time actually also depends on scheduler etc.!

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
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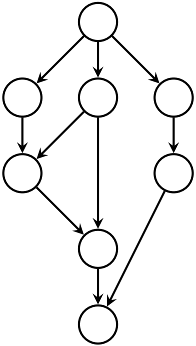
Work Law

$$T_P \geq \frac{T_1}{P}$$



Work Law and Span Law

- T_1 = work, T_∞ = span
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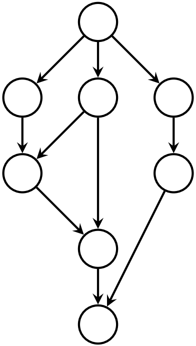
$T_1 = 8$, $P = 2$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

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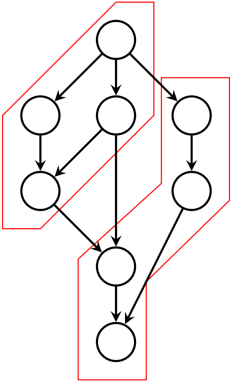
$T_1 = 8,\ P = 2$



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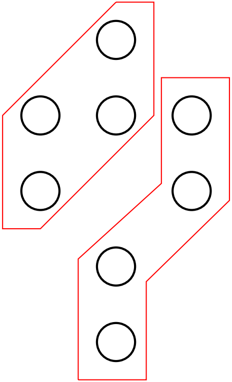
Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_1 = 8,\ P = 2$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

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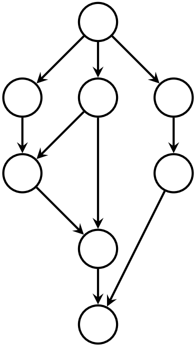

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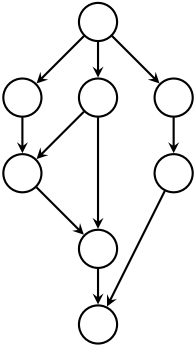

Work Law and Span Law

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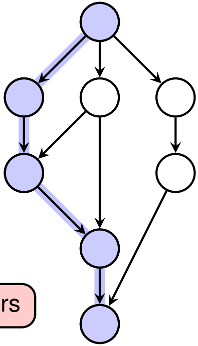
$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

Time on P processors can't be shorter than time on ∞ processors

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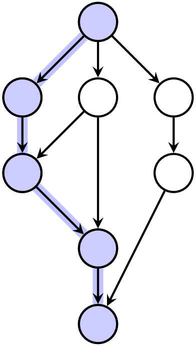

Work Law

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$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$

Work Law and Span Law

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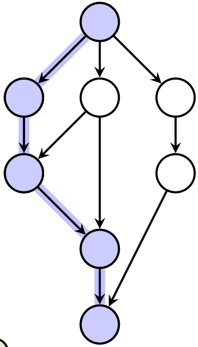

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$

Maximum Speed-Up bounded by P!

Work Law and Span Law

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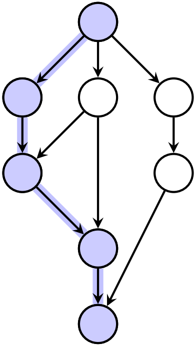

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Work Law and Span Law

- T_1 = work, T_∞ = span
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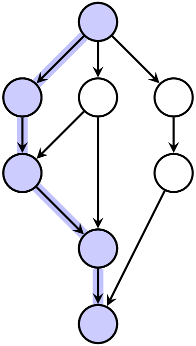

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Maximum Speed-Up for ∞ processors!

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $n \times n$ matrix $A = (a_{ij})$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

$$\boxed{y_i = \sum_{j=1}^{n} a_{ij} x_j} \quad \text{for } i = 1, 2, \ldots, n.$$

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MAT-VEC(A, x)
n = A.rows
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parallel for i = 1 to n
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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

How can a compiler implement the **parallel for**-loop?

Implementing `parallel for` based on Divide-and-Conquer

```
MAT-VEC-MAIN-LOOP(A, x, y, n, i, i')
if i == i'
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    spawn MAT-VEC-MAIN-LOOP(A, x, y, n, i, mid)
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```
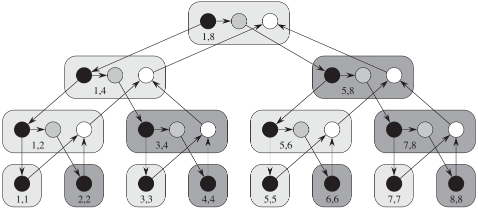

1

Implementing `parallel for` based on Divide-and-Conquer



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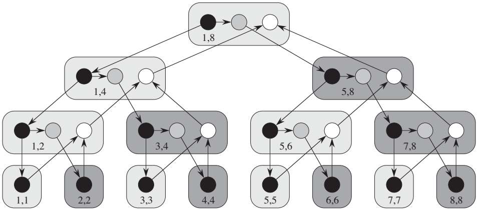

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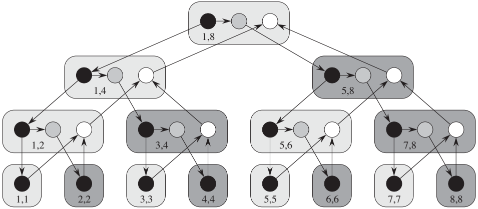

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$$T_1(n) =$$

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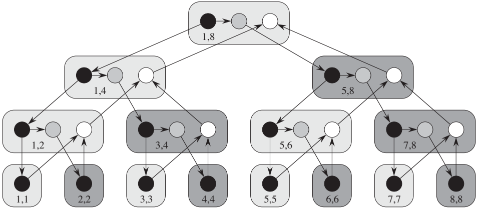

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Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

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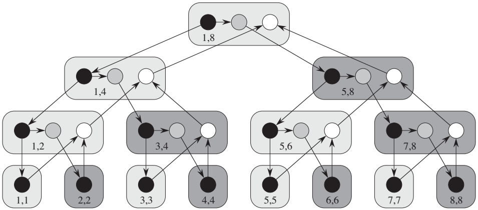

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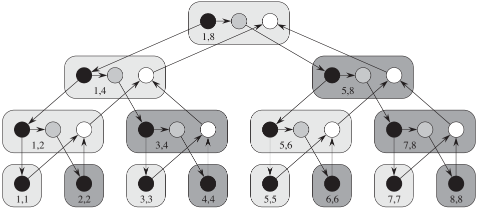


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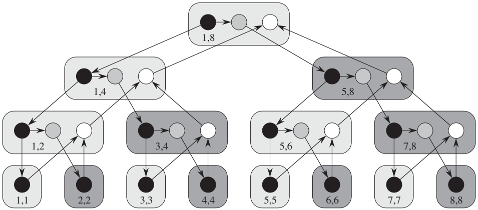

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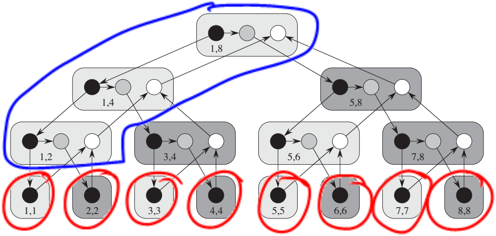
Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Implementing `parallel for` based on Divide-and-Conquer



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$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \boxed{\Theta(\log n)} + \boxed{\max_{1 \le i \le n} \text{iter}(n)}$$

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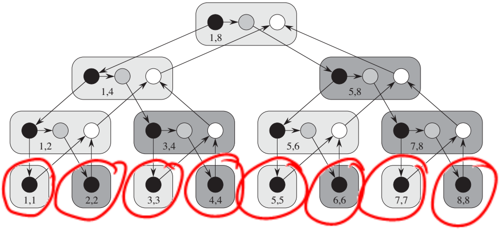
Implementing `parallel for` based on Divide-and-Conquer



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$$T_\infty(n) = \Theta(\log n) + \boxed{\max_{1 \leq i \leq n} \mathrm{iter}(n)}$$
$$= \boxed{\Theta(n)}.$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Naive Algorithm in Parallel

```
P-SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
parallel for i = 1 to n
    parallel for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

Naive Algorithm in Parallel

P-SQUARE-MATRIX-MULTIPLY(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
parallel for $i = 1$ **to** n
    parallel for $j = 1$ **to** n
        $c_{ij} = 0$
        for $k = 1$ **to** n
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C

P-SQUARE-MATRIX-MULTIPLY(A, B) has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(n)$.

The first two nested for-loops parallelise perfectly.

The Simple Divide&Conquer Approach in Parallel

P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)

$n = A.rows$
if $n == 1$
    $c_{11} = a_{11} b_{11}$
else let T be a new $n \times n$ matrix
    partition A, B, C, and $\boxed{T}$ into $n/2 \times n/2$ submatrices
        $A_{11}, A_{12}, A_{21}, A_{22}$; $B_{11}, B_{12}, B_{21}, B_{22}$; $C_{11}, C_{12}, C_{21}, C_{22}$;
        and $T_{11}, T_{12}, T_{21}, T_{22}$; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_{11}, A_{11}, B_{11})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_{12}, A_{11}, B_{12})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_{21}, A_{21}, B_{11})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_{22}, A_{21}, B_{12})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_{11}, A_{12}, B_{21})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_{12}, A_{12}, B_{22})
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_{21}, A_{22}, B_{21})
    P-MATRIX-MULTIPLY-RECURSIVE(T_{22}, A_{22}, B_{22})
    sync
    parallel for $i = 1$ **to** n
        parallel for $j = 1$ **to** n
            $c_{ij} = c_{ij} + t_{ij}$

spawn P-M..
+ spawn P-M

} Divide-Conquer

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
    c_11 = a_11 b_11
else let T be a new n × n matrix
    partition A, B, C, and T into n/2 × n/2 submatrices
        A_11, A_12, A_21, A_22; B_11, B_12, B_21, B_22; C_11, C_12, C_21, C_22;
        and T_11, T_12, T_21, T_22; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_11, A_11, B_11)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C_12, A_11, B_12)
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    spawn P-MATRIX-MULTIPLY-RECURSIVE(T_21, A_22, B_21)
    P-MATRIX-MULTIPLY-RECURSIVE(T_22, A_22, B_22)
    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            c_ij = c_ij + t_ij
```

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

The Simple Divide&Conquer Approach in Parallel

P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)

```
n = A.rows
if n == 1
    c11 = a11 b11
else let T be a new n × n matrix
    partition A, B, C, and T into n/2 × n/2 submatrices
        A11, A12, A21, A22; B11, B12, B21, B22; C11, C12, C21, C22;
        and T11, T12, T21, T22; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C11, A11, B11)
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    P-MATRIX-MULTIPLY-RECURSIVE(T22, A22, B22)
    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            c_ij = c_ij + t_ij
```

} $T_\infty(1) = \Theta(1)$

} $\Theta(1)$

8 multiplications in parallel

} $\Theta(\log n)$

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

$$T_\infty(n) = \boxed{T_\infty(n/2)} + \boxed{\Theta(\log n)}$$

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
    c11 = a11 b11
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    partition A, B, C, and T into n/2 × n/2 submatrices
        A11, A12, A21, A22; B11, B12, B21, B22; C11, C12, C21, C22;
        and T11, T12, T21, T22; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C11, A11, B11)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C12, A11, B12)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C21, A21, B11)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(C22, A21, B12)
    spawn P-MATRIX-MULTIPLY-RECURSIVE(T11, A12, B21)
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    spawn P-MATRIX-MULTIPLY-RECURSIVE(T21, A22, B21)
    P-MATRIX-MULTIPLY-RECURSIVE(T22, A22, B22)
    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            cij = cij + tij
```

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(\log^2 n)$.

$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

Strassen's Algorithm in Parallel

Strassen's Algorithm (parallelised)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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   Can create all 10 matrices with $\Theta(n^2)$ work and $\Theta(\log n)$ span using doubly nested **`parallel for`** loops.

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$$T_1(n) = \Theta(n^{\log 7})$$

Strassen's Algorithm in Parallel

	$T_1(n)$	$T_\infty(n)$
Naive	$\Theta(n^3)$	$\Theta(n)$
Simple DC	$\Theta(n^3)$	$\Theta(\log^2 n)$
Strassen	$\Theta(n^{2.81})$	$\Theta(\log^2 n)$

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Matrix Multiplication and Matrix Inversion

Speedups for Matrix Inversion by an equivalence with Matrix Multiplication.

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Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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- Define a $3n \times 3n$ matrix D by:

$$D = \begin{pmatrix} I_n & A & 0 \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix}$$

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$\Rightarrow$ We can compute AB in $O(I(n))$ time. $\square$

The Other Direction

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

Theorem 28.2 (Inversion is no harder than Multiplication)

Suppose we can multiply two $n \times n$ real matrices in time $M(n)$ and $M(n)$ satisfies the two regularity conditions $M(n+k) = O(M(n))$ for any $0 \leq k \leq n$ and $M(n/2) \leq c \cdot M(n)$ for some constant $c < 1/2$. Then we can compute the inverse of any real nonsingular $n \times n$ matrix in time $O(M(n))$.

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If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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Proof of this directon much harder (CLRS) – relies on properties of SPD matrices.

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Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

Allows us to use Strassen's Algorithm to invert a matrix!

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