A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $\boxed{O}(\log n)$.

Quite elaborate construction, and involves huges constants.

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Perfect Halver

A perfect halver is a comparator network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\log_2 n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

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Approximate Halver

$\varepsilon = \frac{1}{100}$

An (n, ϵ)-approximate halver, $\epsilon < 1$, is a comparator network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

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We will prove that such networks can be constructed in constant depth!

Expander Graphs

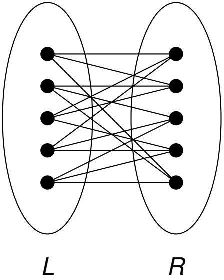
Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is the union of d matchings (perfect)
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| \geq \min\{\mu \cdot |S|, n/2 - |S|\}$$

$n = 10$



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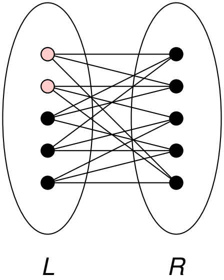

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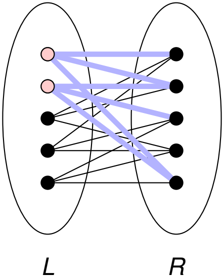

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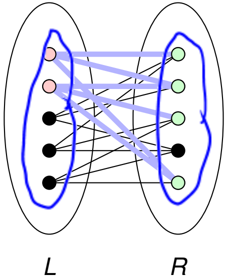

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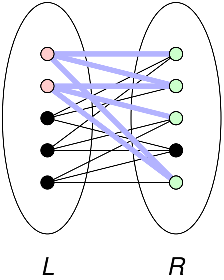
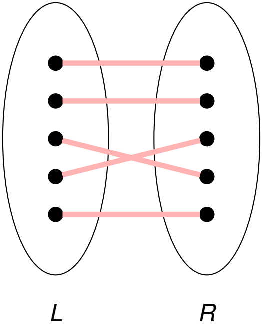


Expander Graphs:

- **probabilistic construction** "easy": take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers



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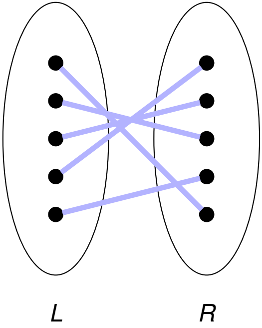

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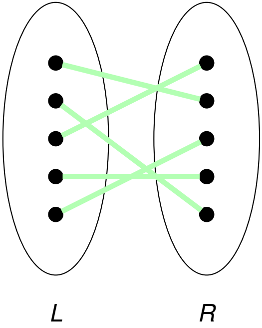

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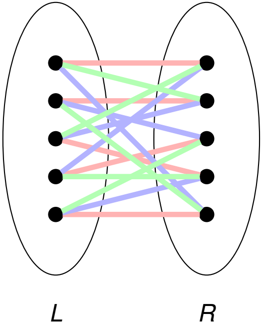

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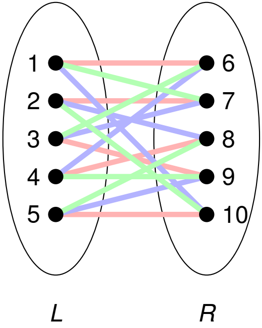

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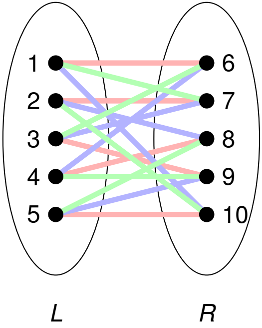

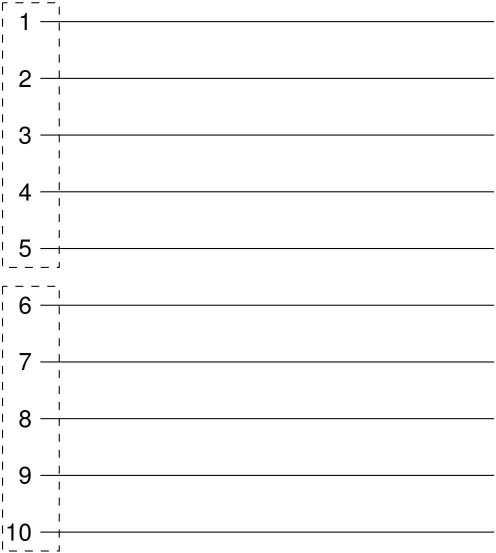

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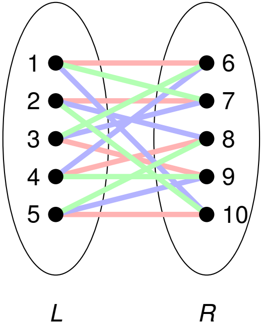

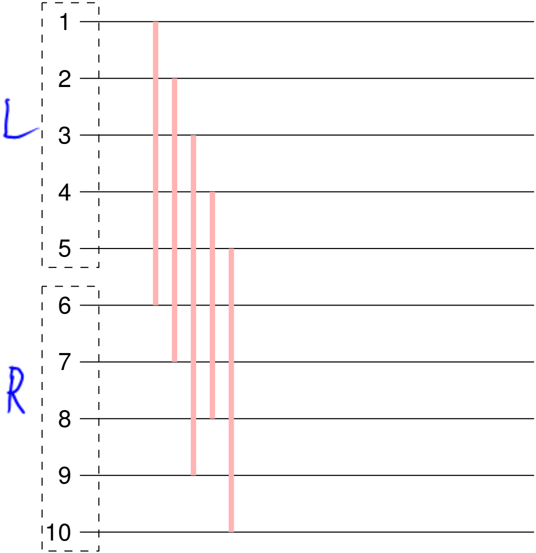

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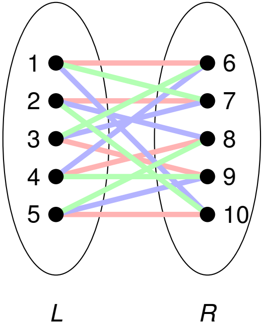

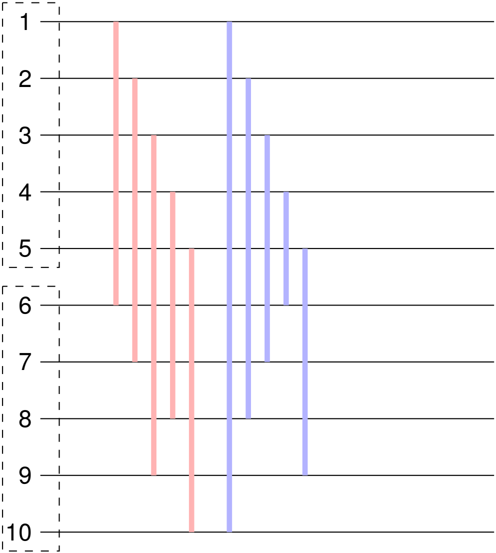

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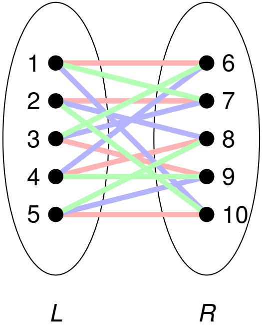
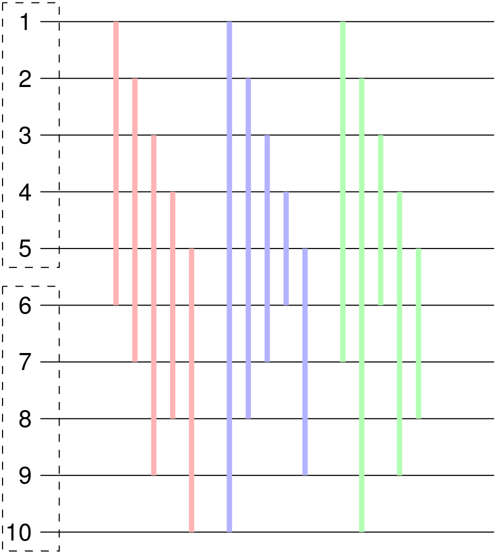
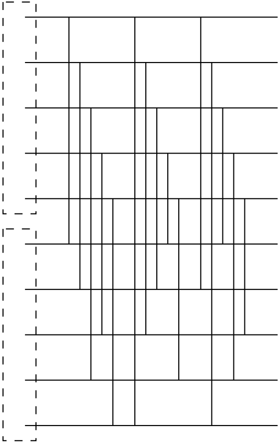




Existence of Approximate Halvers

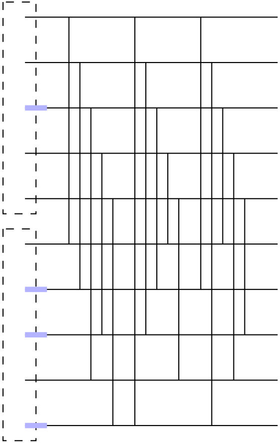
Proof:



Existence of Approximate Halvers

Proof:

- $X :=$ wires with the k smallest inputs

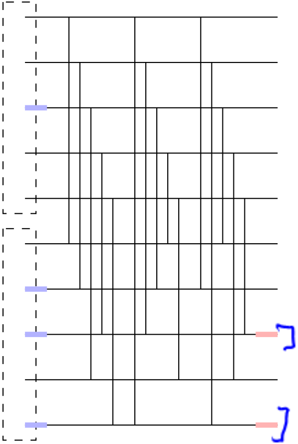


Existence of Approximate Halvers

Proof Strategy:
1. $|N(Y)| \gg |Y|$
2. $|N(Y)| \leq |X| = k$

Proof: keys

- $X :=$ wires with the k smallest inputs
- $Y :=$ wires in lower half with k smallest outputs



Existence of Approximate Halvers

Proof:

- $X :=$ wires with the k smallest inputs
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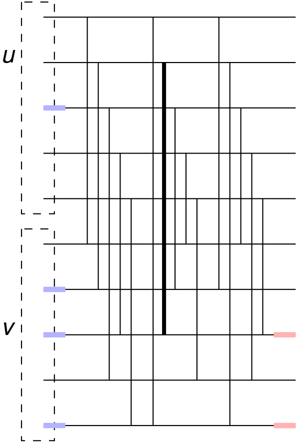

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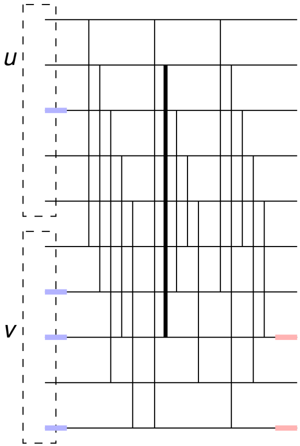

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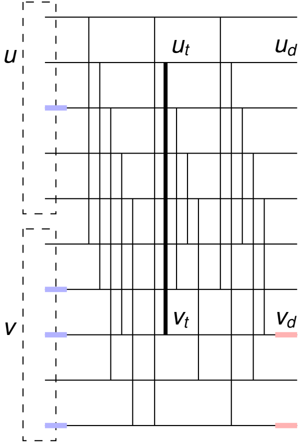

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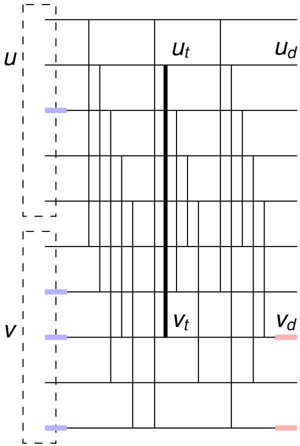

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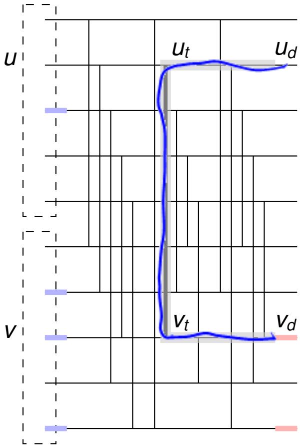

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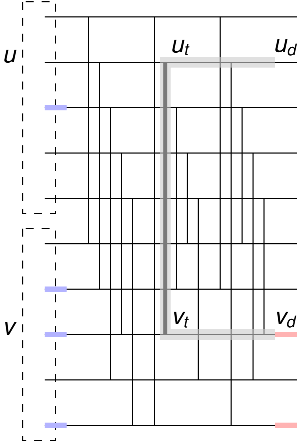
$$\boxed{|Y| + |N(Y)| \leq k.}$$



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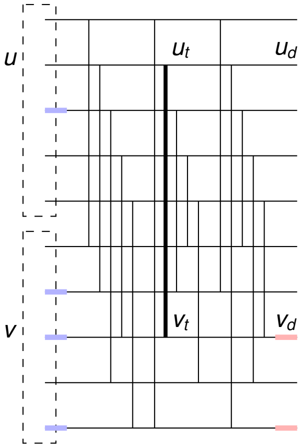

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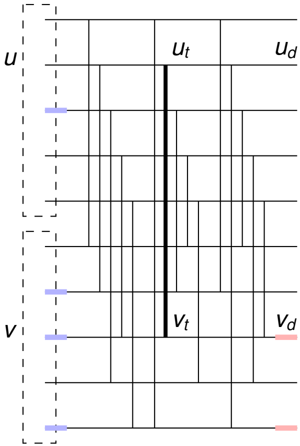

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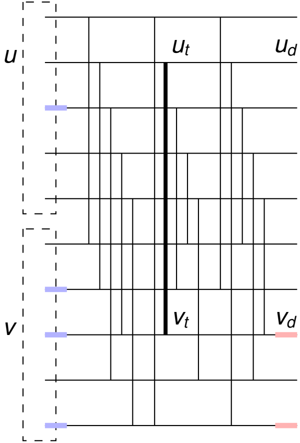

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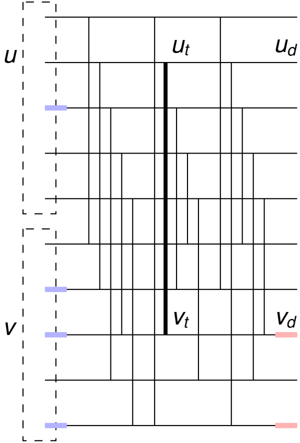
$$= \min\{(1 + \mu)|Y|, n/2\}.$$



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- Combining the two bounds above yields:

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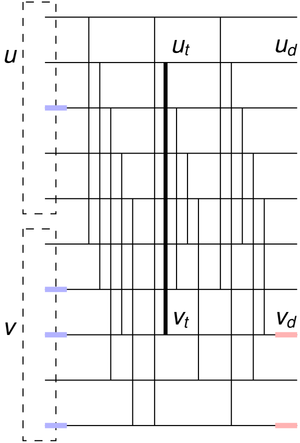

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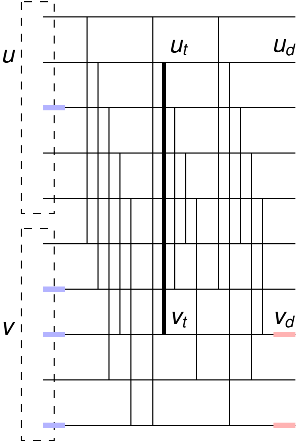
Here we used that $\boxed{k \leq n/2}$



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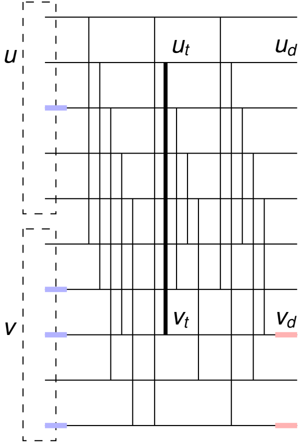
- The same argument shows that at most $\epsilon \cdot k$, $\epsilon := 1/(\mu+1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. $\square$



AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"

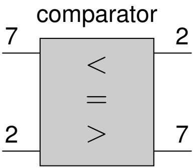
Richard J. Lipton (Georgia Tech)

*"The AKS sorting network is **galactic**: it needs that n be larger than 2^{78} or so to finally be smaller than Batcher's network for n items."*

Siblings of Sorting Network

Sorting Networks

- sorts any input of size *n*
- special case of Comparison Networks



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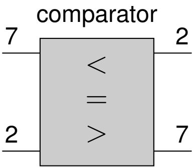
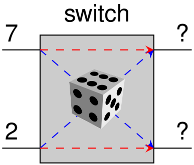


Switching (Shuffling) Networks

- creates a random permutation of *n* items
- special case of Permutation Networks



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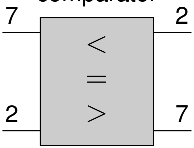
- special case of Comparison Networks

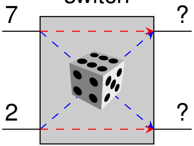
comparator



Switching (Shuffling) Networks

- creates a random permutation of n items
- special case of Permutation Networks

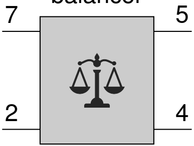
switch



Counting Networks

- balances any stream of tokens over n wires
- special case of Balancing Networks

balancer



Outline

Outline of this Course

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Values could represent addresses in memories or destinations on an interconnection network

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Balancing Networks

- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,. . .)

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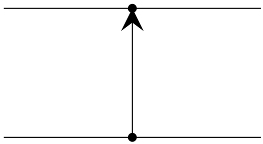

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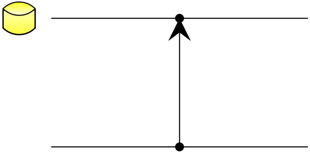

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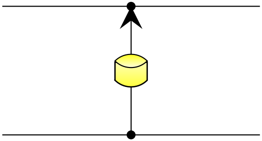

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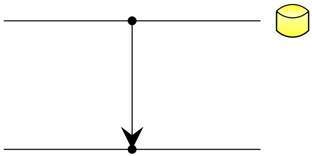

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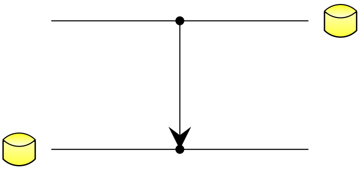

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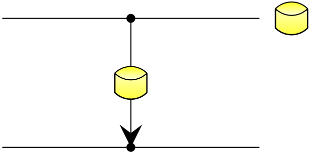

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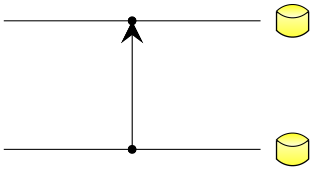

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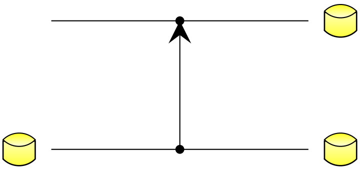

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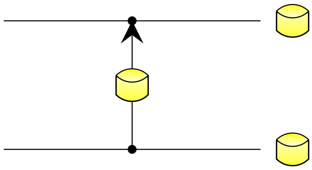

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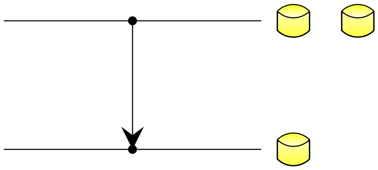

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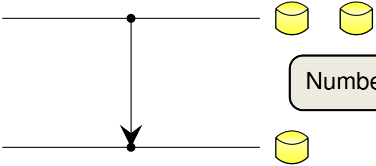

Number of tokens differs by at most one

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires

Bitonic Counting Network

Counting Network (Formal Definition)

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2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires
3. In a quiescent state: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$
4. A counting network is a balancing network with the **step-property**:

$$\boxed{0 \leq y_i - y_j \leq 1 \text{ for any } i < j.}$$

(4, 4, 4, 3, 3, 3, 3, 3)

Bitonic Counting Network

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$$0 \leq y_i - y_j \leq 1 \text{ for any } i < j.$$

Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
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$(4, 4, 3, 3, 3, 3, 3, 3)$

$(4, 4, 4, 3, 3, 3, 3, 3)$

Key Lemma

Consider a MERGER[n]. Then if the inputs $x_1, \dots, x_{n/2}$ and $x_{n/2+1}, \dots, x_n$ have the step property, then so does the output $y_1, \dots, y_n$.

Proof (by induction on n) → power of 2

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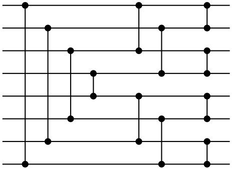

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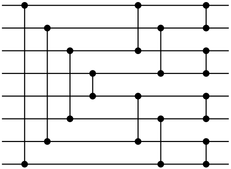

Proof (by induction on n)

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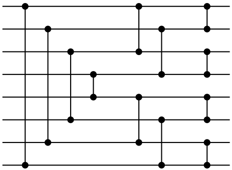

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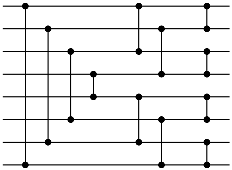

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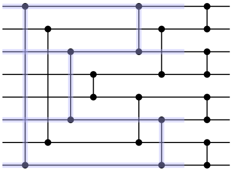

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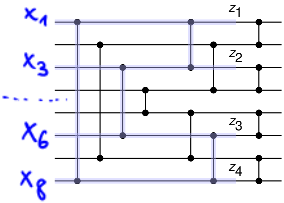

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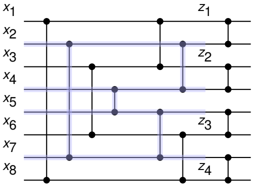

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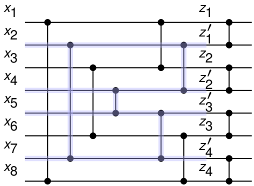

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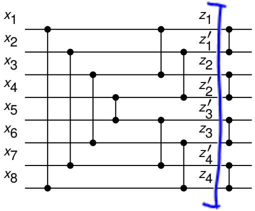

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- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property

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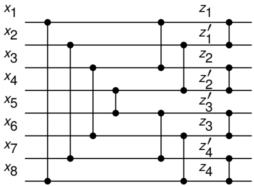


Proof (by induction on n)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$

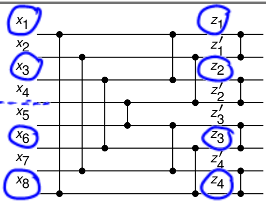
Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \lceil \frac{1}{2} \sum_{i=1}^{n} x_i \rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.

$Z = \lceil \frac{1}{2} \cdot (x_1 + x_2 + x_3 + x_4) \rceil + \lfloor \frac{1}{2} \cdot (x_5 + x_6 + x_7 + x_8) \rfloor$



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- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$
- F1 $\Rightarrow$ $Z = \lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \rceil + \lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rfloor$ and $Z' = \lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \rfloor + \lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rceil$

Correctness of the Bitonic Counting Network

Facts

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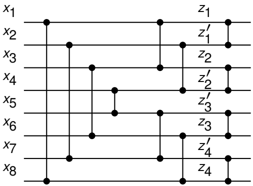

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Correctness of the Bitonic Counting Network

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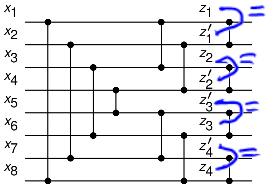

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- IH $\Rightarrow z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z_i'$
- F1 $\Rightarrow Z = \left\lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \right\rceil + \left\lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \right\rfloor$ and $Z' = \left\lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \right\rfloor + \left\lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \right\rceil$
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
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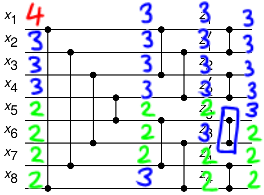

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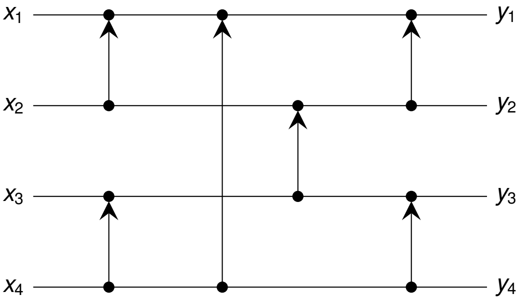
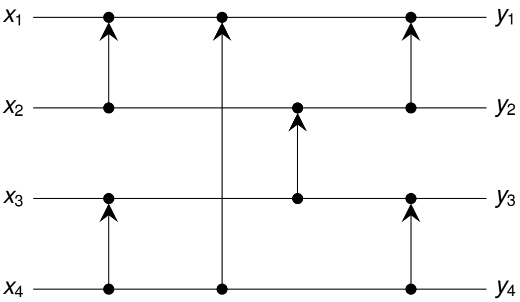
- F1 $\Rightarrow$ $Z = \left\lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \right\rceil + \left\lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \right\rfloor$ and $Z' = \left\lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \right\rfloor + \left\lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \right\rceil$
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$. Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action

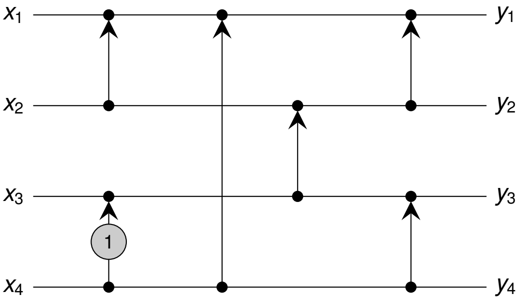


Bitonic Counting Network in Action

1



Bitonic Counting Network in Action



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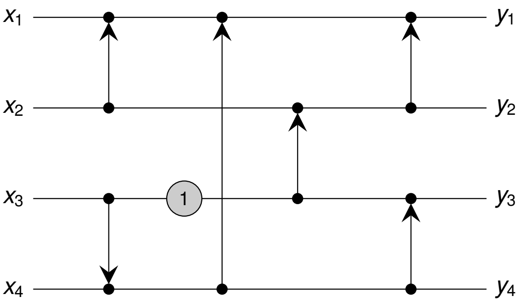

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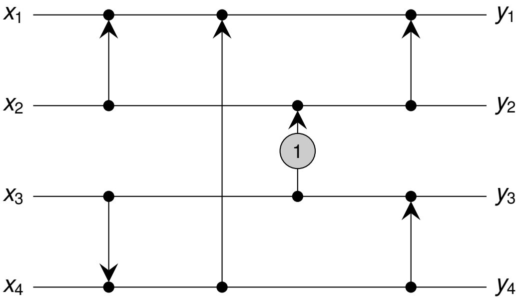

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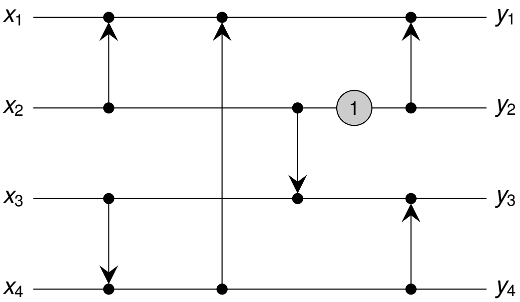

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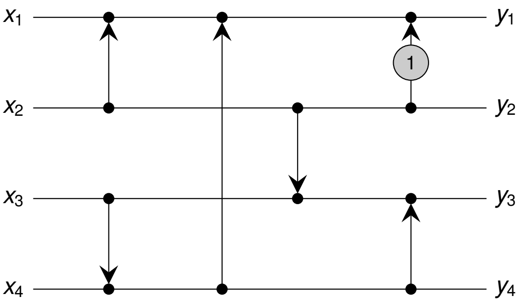

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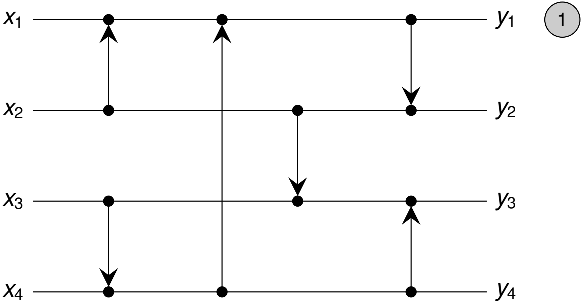

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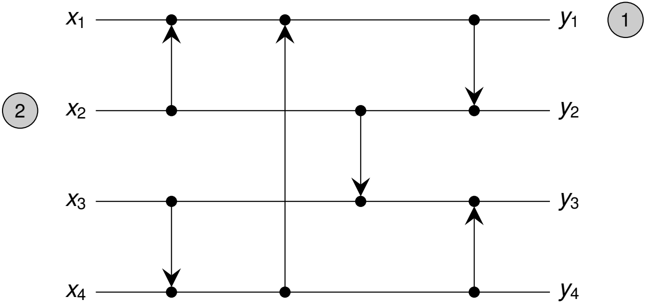

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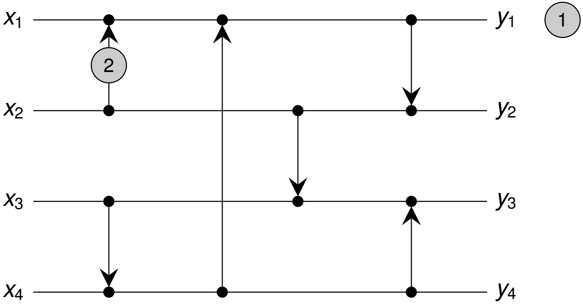

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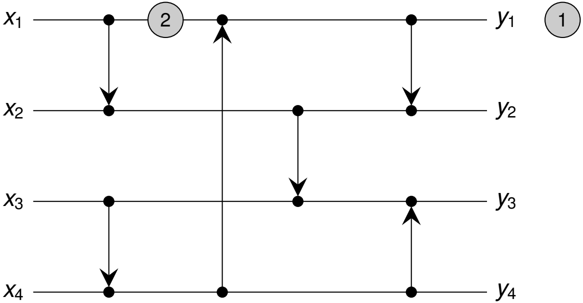

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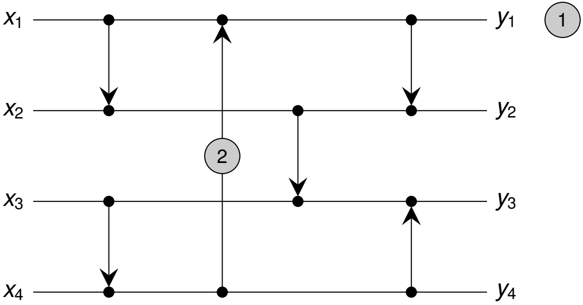

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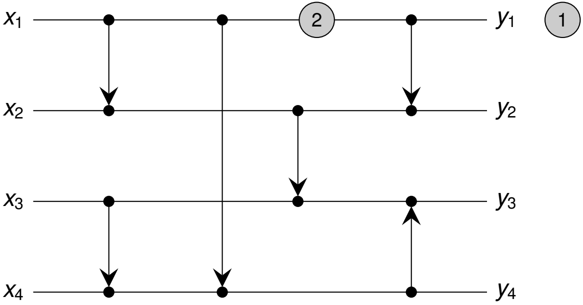

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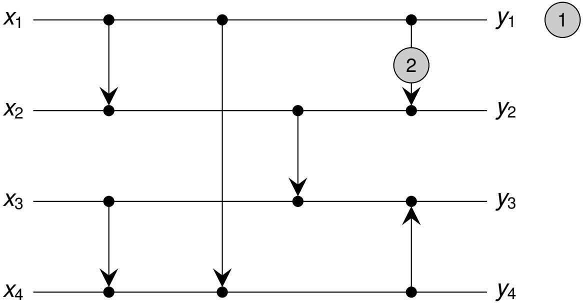

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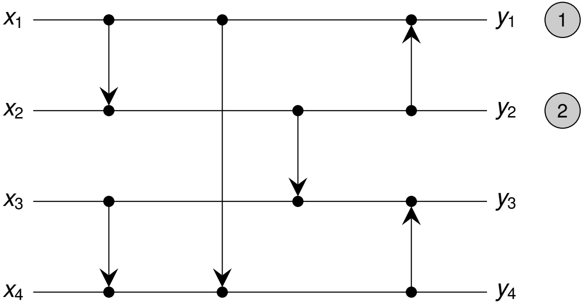

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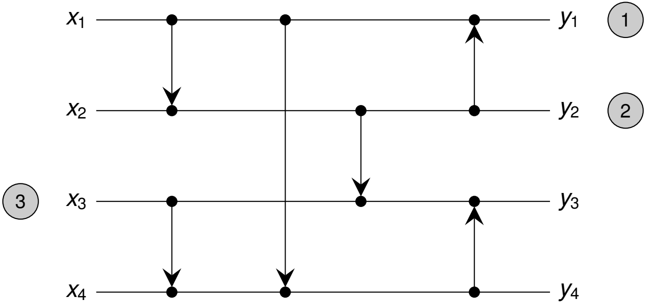

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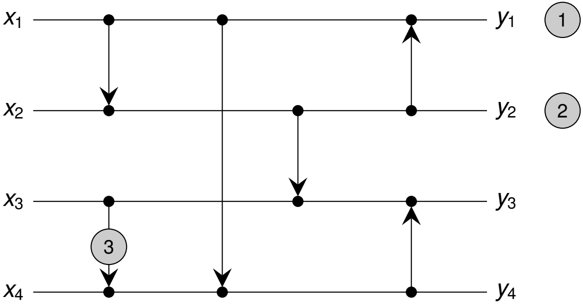

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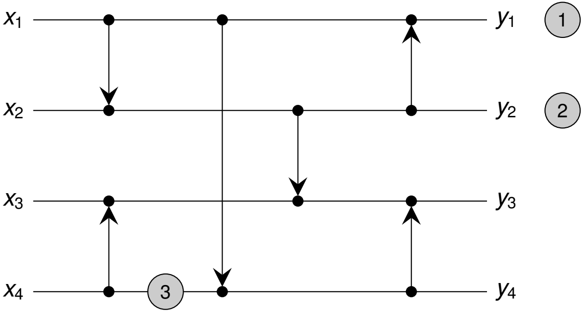

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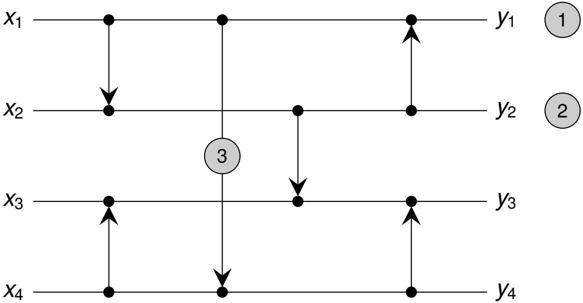

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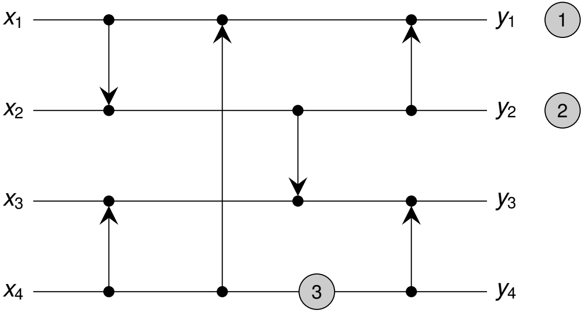

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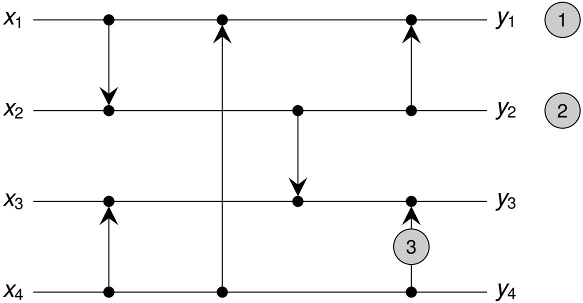

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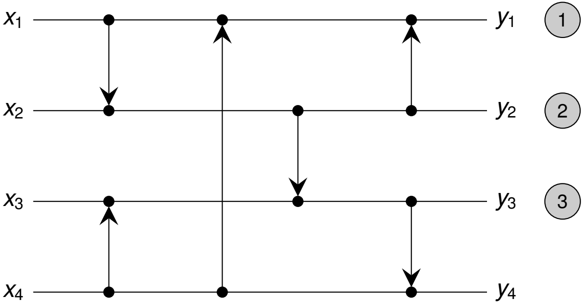

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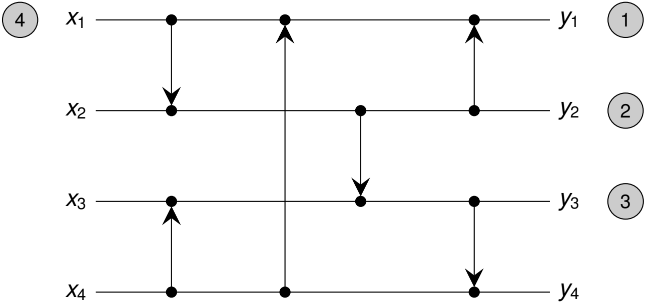

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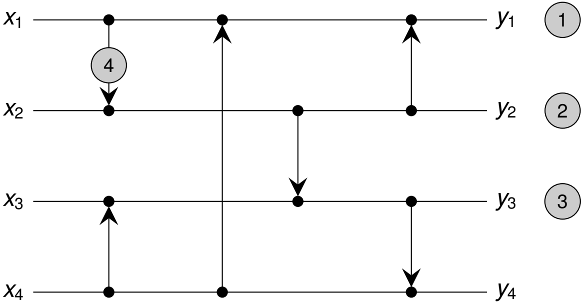

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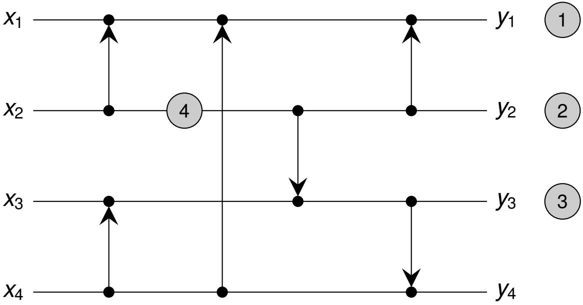

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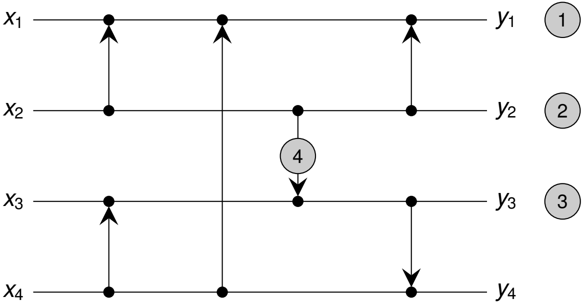

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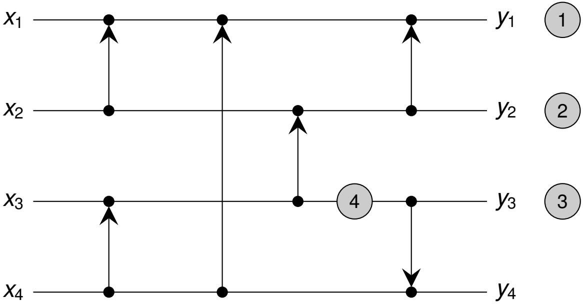

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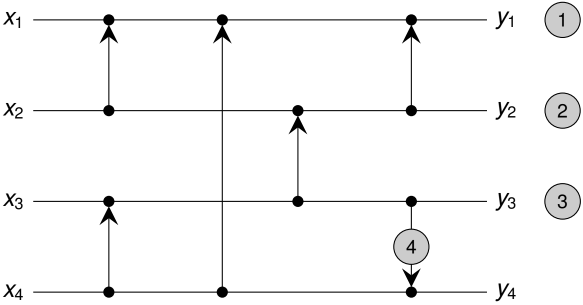

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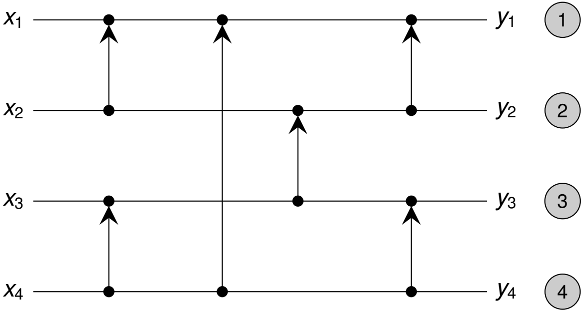

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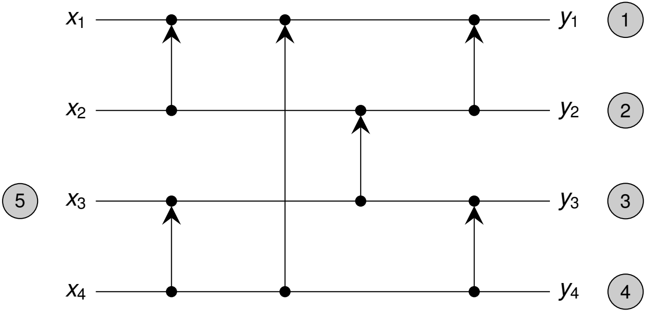

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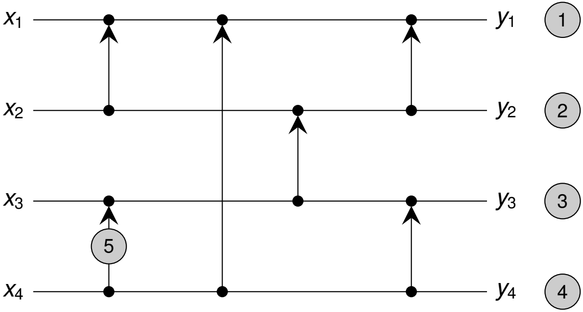

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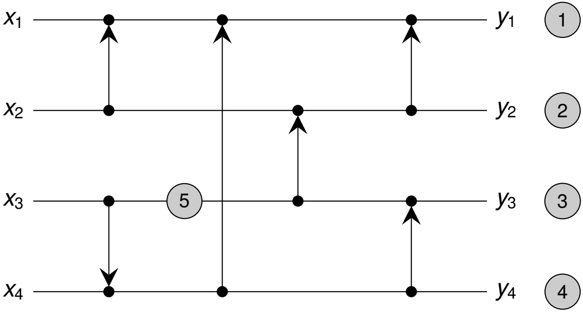

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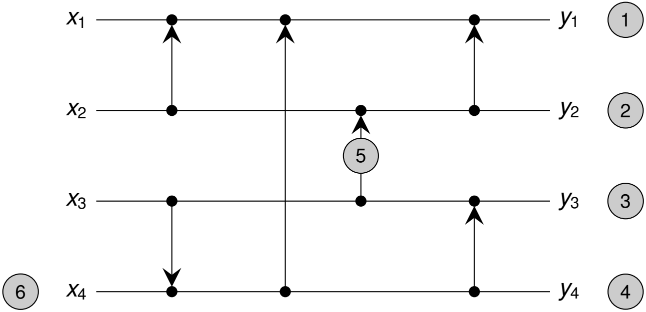

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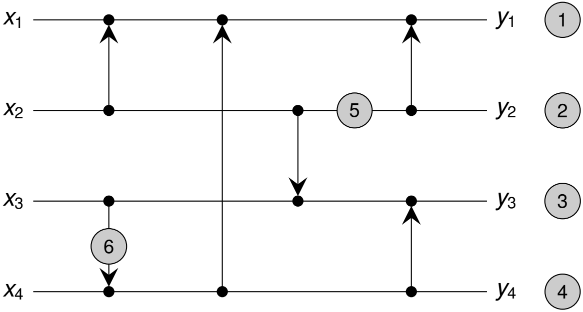

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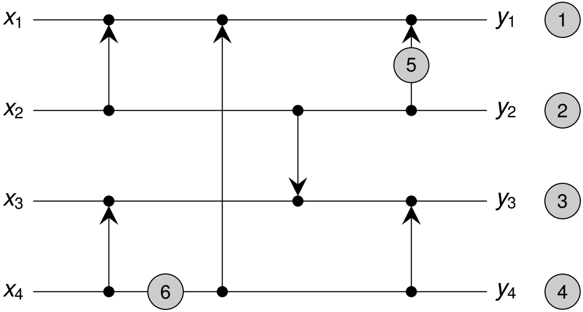

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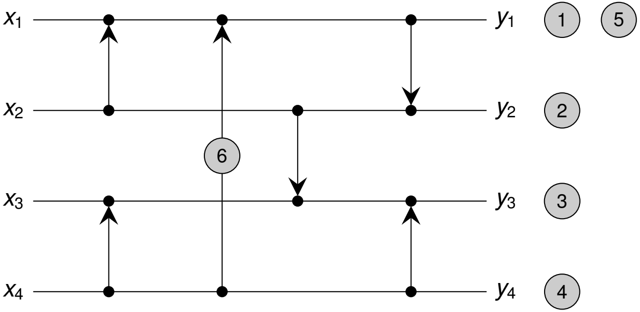

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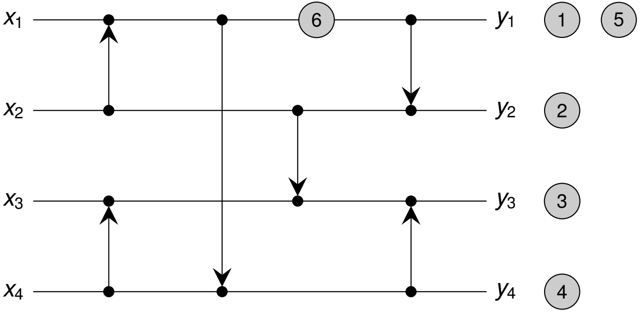

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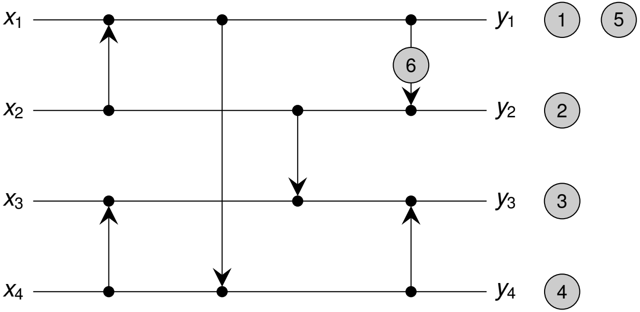

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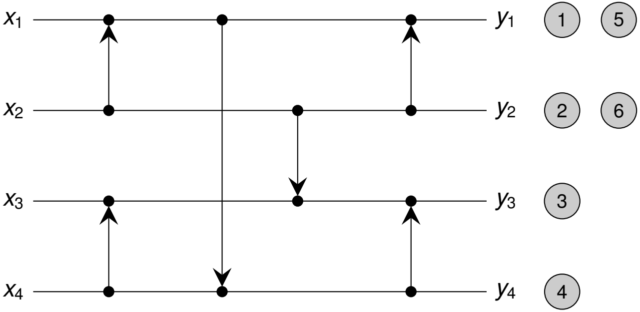
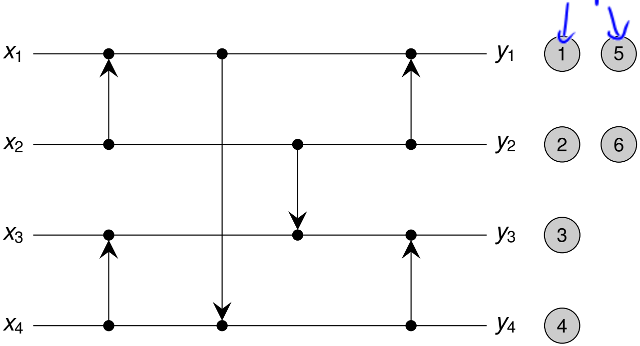


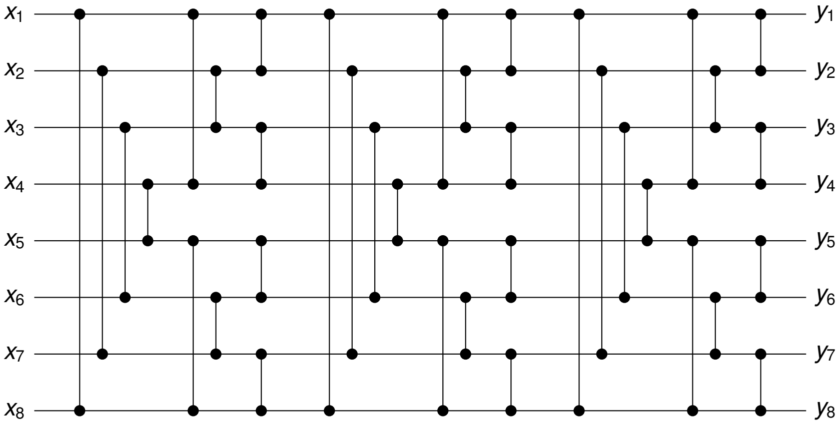
Bitonic Counting Network in Action

values are assigned at the output only

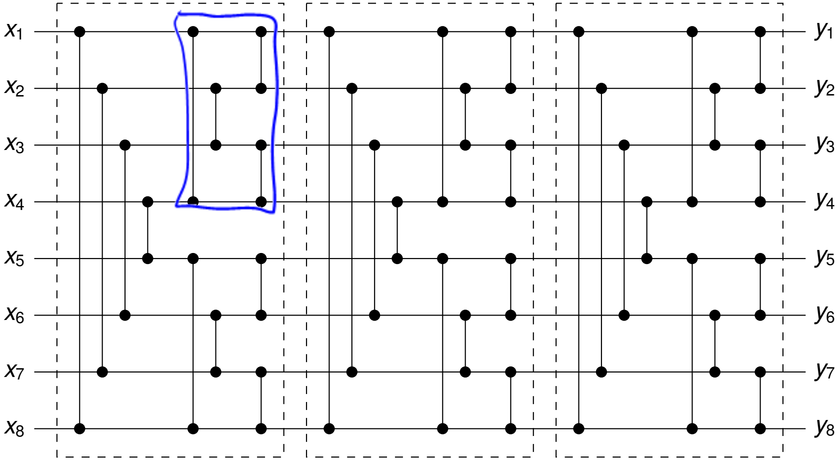


Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i+n, i+2\cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



Consists of $\log n$ BLOCK[n] networks each of which has depth $\log n$

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

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The converse is not true!

Counting vs. Sorting

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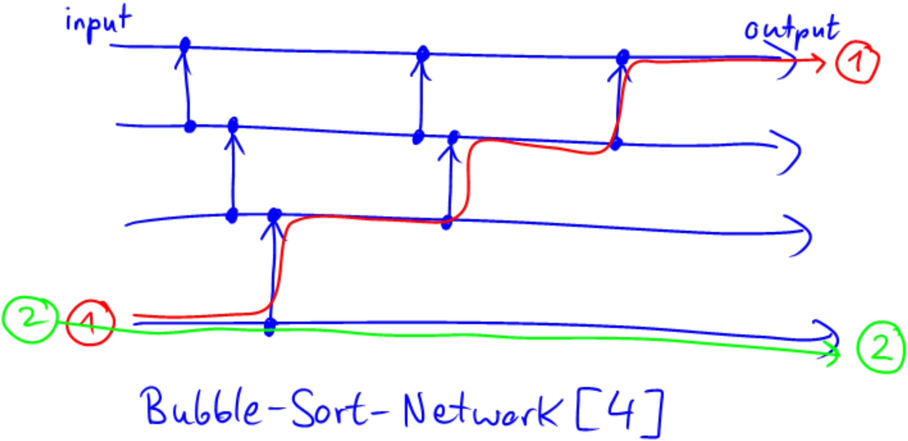

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Proof.

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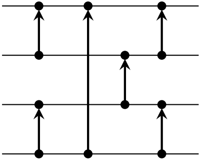
- Let C be a counting network, and S be the corresponding sorting network

C



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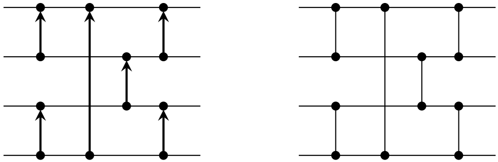

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- Let C be a counting network, and S be the corresponding sorting network
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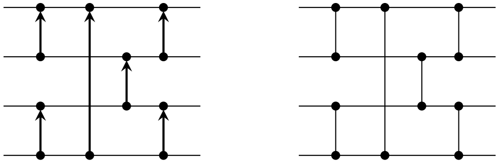

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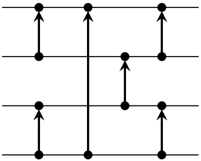

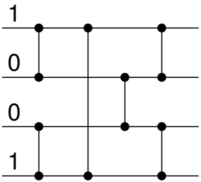

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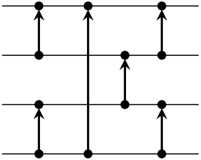

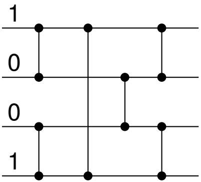

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- C is a counting network $\Rightarrow$ all ones will be routed to the lower wires

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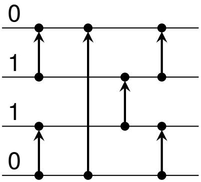

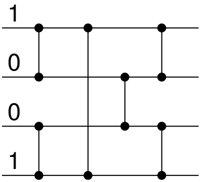

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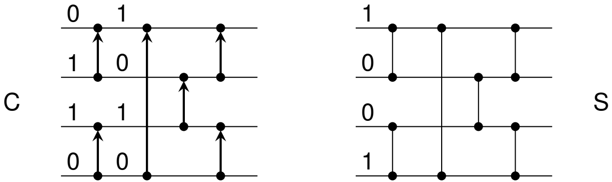

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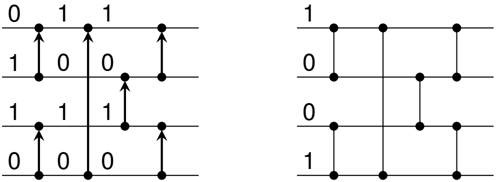

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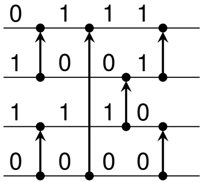

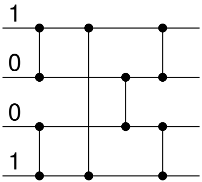

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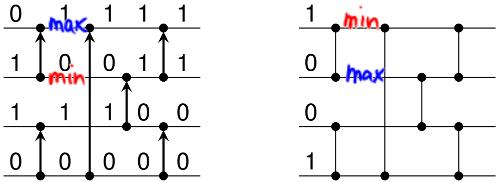

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- C is a counting network $\Rightarrow$ all ones will be routed to the lower wires
- S corresponds to C $\Rightarrow$ all zeros will be routed to the lower wires

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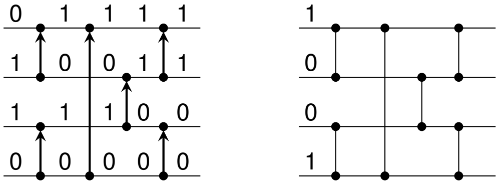

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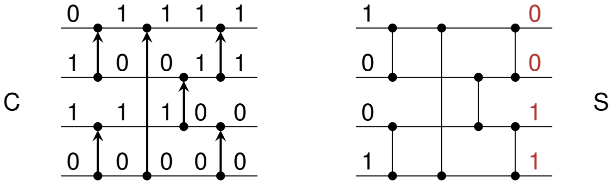

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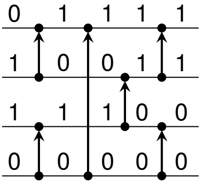
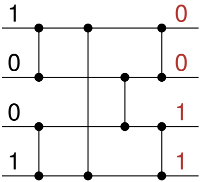
- S corresponds to C $\Rightarrow$ all zeros will be routed to the lower wires
- By the Zero-One Principle, S is a sorting network. $\square$

C





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