VII. Approximation Algorithms: Randomisation and Rounding

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UNIVERSITY OF CAMBRIDGE

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

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Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n.

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extends in the natural way to randomised algorithms

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- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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Idea: What about assigning each variable independently at random?

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Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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$7 \cdot \frac{7}{8} > 6$

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Follows from the previous Corollary.

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid \underline{x_1 = 1}\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid \underline{x_1 = 0}\,].$$

Y is defined as in the previous proof.

$$E[Y] = \sum_{y \in Y} y \cdot Pr[Y = y]$$

$$E[Y \mid x_1 = 1] = \sum_{y \in Y} y \cdot Pr[Y = y \mid x_1 = 1]$$

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One of the two conditional expectations is greater than $\mathbf{E}[\,Y\,]$!

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Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is greater than $\mathbf{E}[\,Y\,]$

Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

Expected Approximation Ratio

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GREEDY-3-CNF(ϕ, n, m)

```
for $\boxed{j}$ = 1, 2, ..., n
    Compute E[ Y | x_1 = v_1 ..., x_{j-1} = v_{j-1}, $\boxed{x_j = 1}$ ]
    Compute E[ Y | x_1 = v_1, ..., x_{j-1} = v_{j-1}, $\boxed{x_j = 0}$ ]
    Let x_j = v_j so that the conditional expectation is maximized
return the assignment v_1, v_2, ..., v_n
```

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

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computable in $O(1)$

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↓

random assignment to 3 (or less) literals in one clause

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$$\mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, \underline{x_j = v_j} \,\right] \geq \mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1} \,\right]$$

x_j is still random

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&\geq \mathbf{E}\left[\, Y \,\right]
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&\geq \mathbf{E}\left[\, Y \,\right] = \frac{7}{8} \cdot m.
\end{aligned}$$

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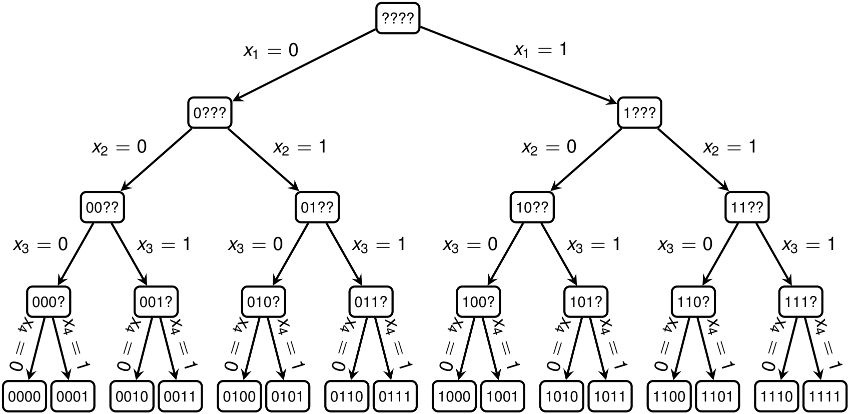
&\geq \mathbf{E}\left[\, Y \,\right] = \frac{7}{8} \cdot m. \qquad \square
\end{aligned}$$

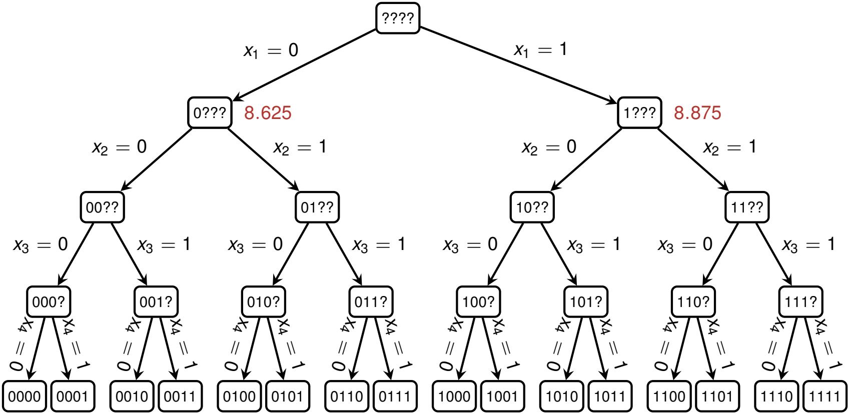
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

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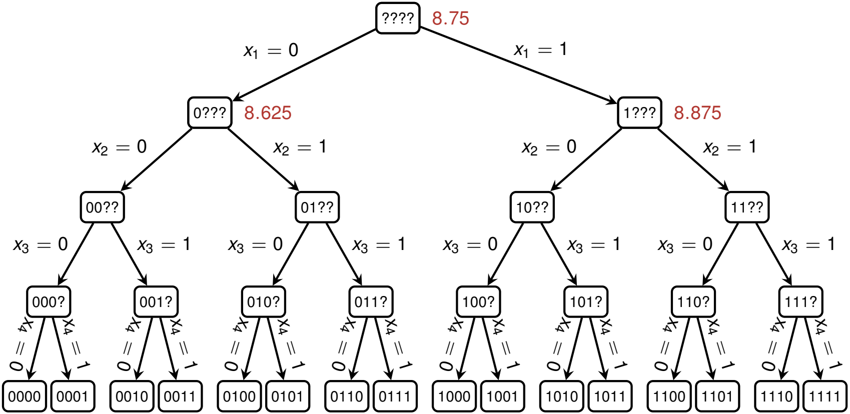
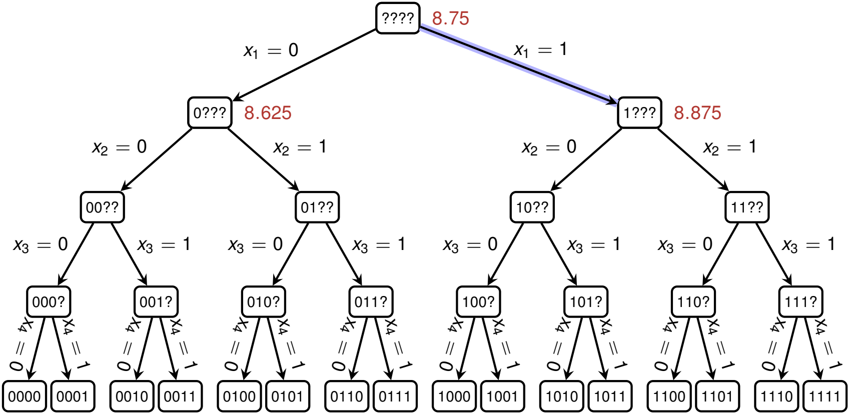


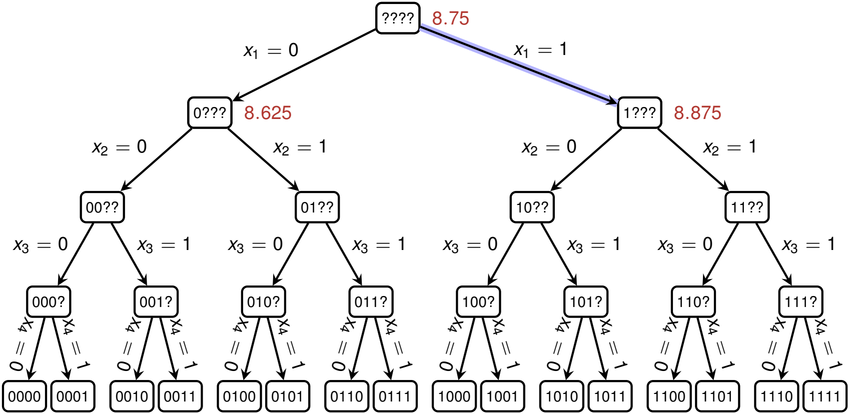
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



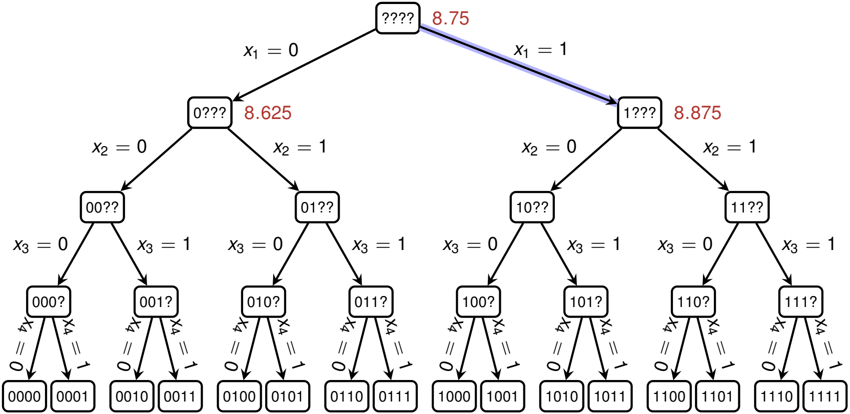
Run of GREEDY-3-CNF(φ, n, m)

$$\cancel{(x_1 \vee x_2 \vee x_3)} \wedge \cancel{(x_1 \vee \overline{x_2} \vee \overline{x_4})} \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_4})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_3} \vee x_4) \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_3})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\cancel{\overline{x_1}} \vee x_2 \vee x_3) \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee x_3) \wedge \cancel{(x_1 \vee x_3 \vee \overline{x_4})} \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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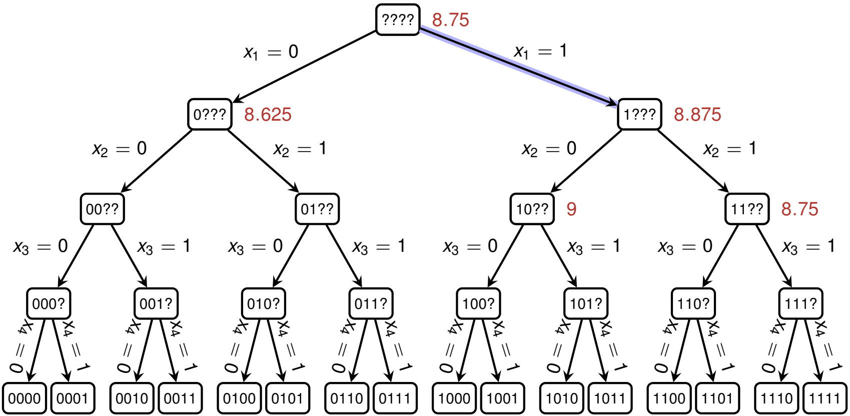

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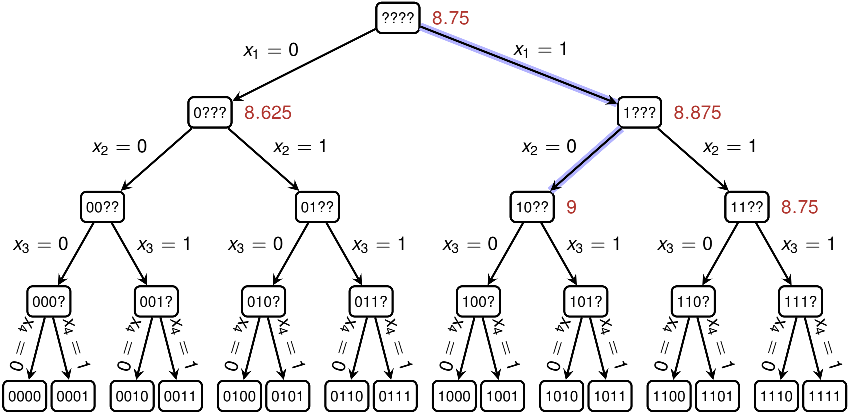

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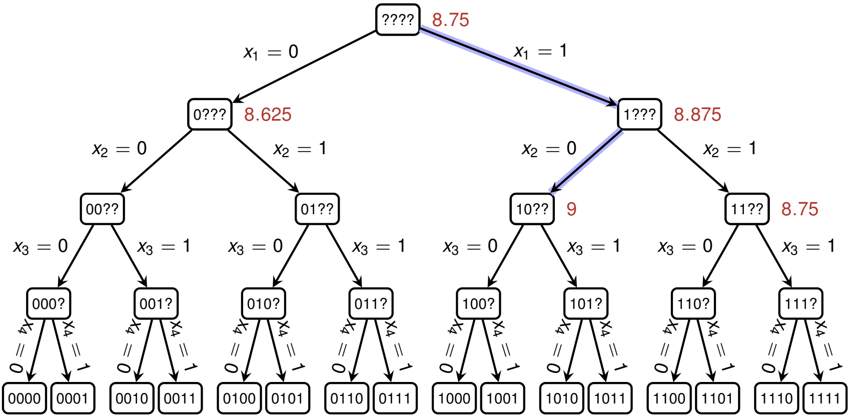

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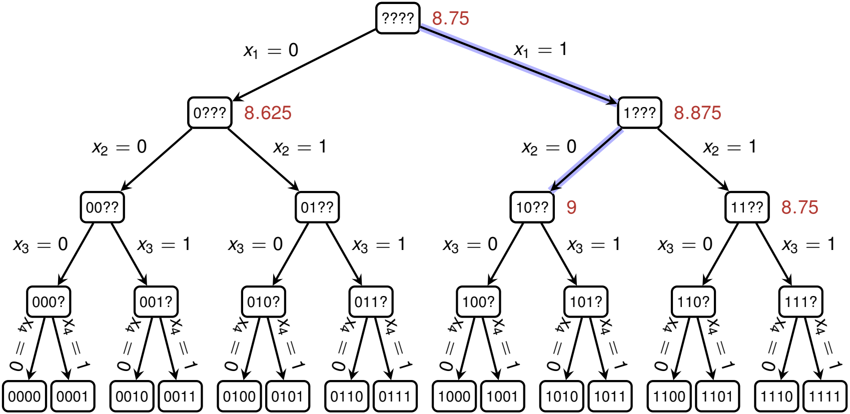

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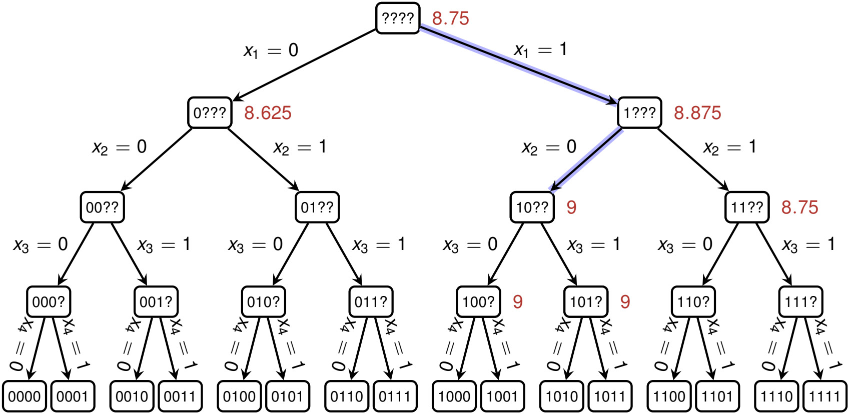

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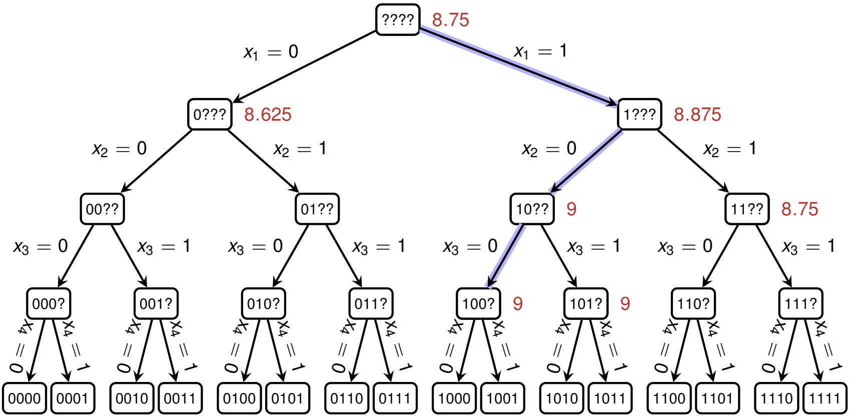
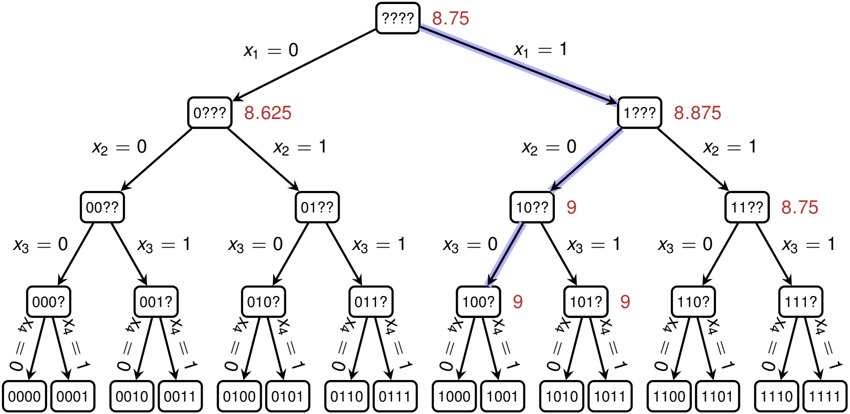


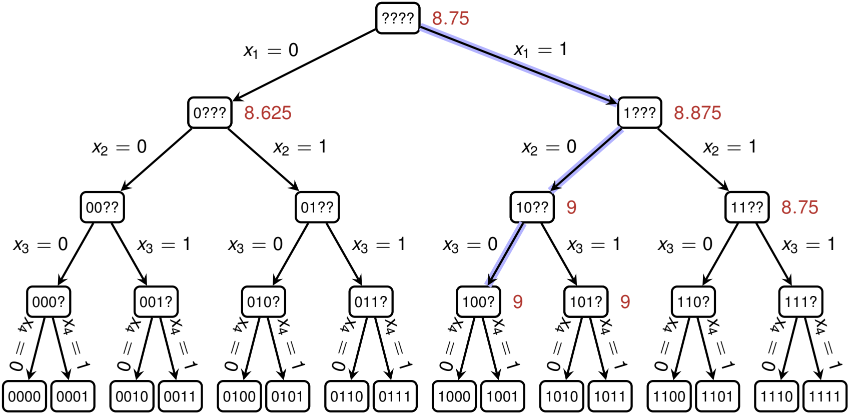
Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\cancel{\overline{x_3} \vee x_4}) \wedge 1 \wedge 1 \wedge (\cancel{x_3}) \wedge 1 \wedge 1 \wedge (\cancel{\overline{x_3}} \vee \overline{x_4})$$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$$



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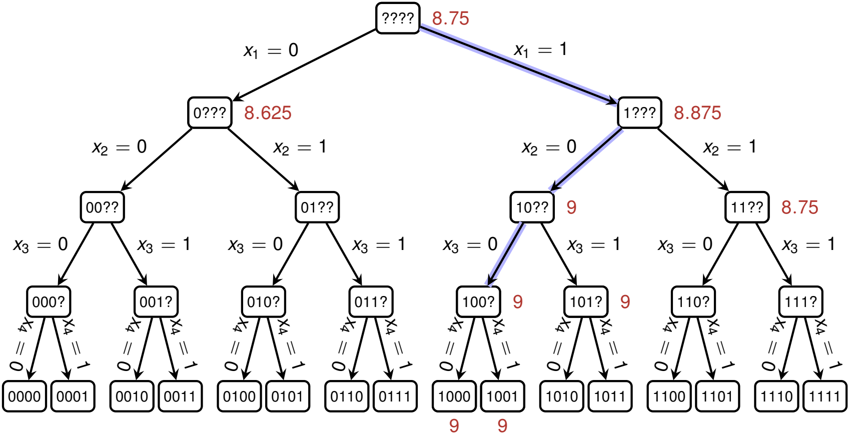

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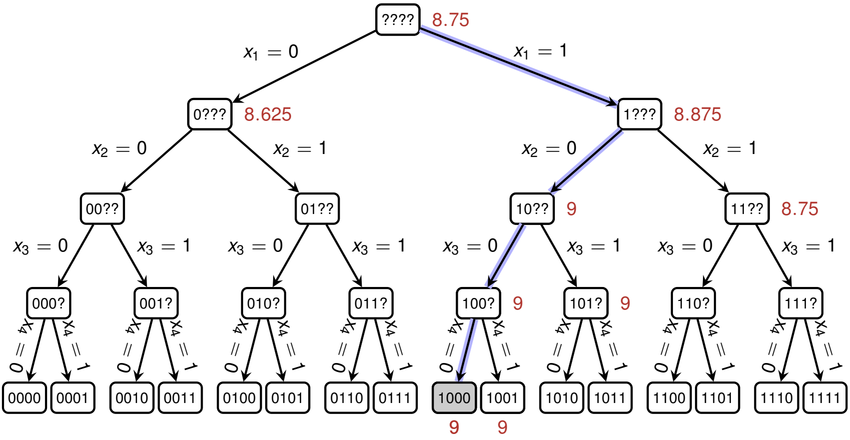

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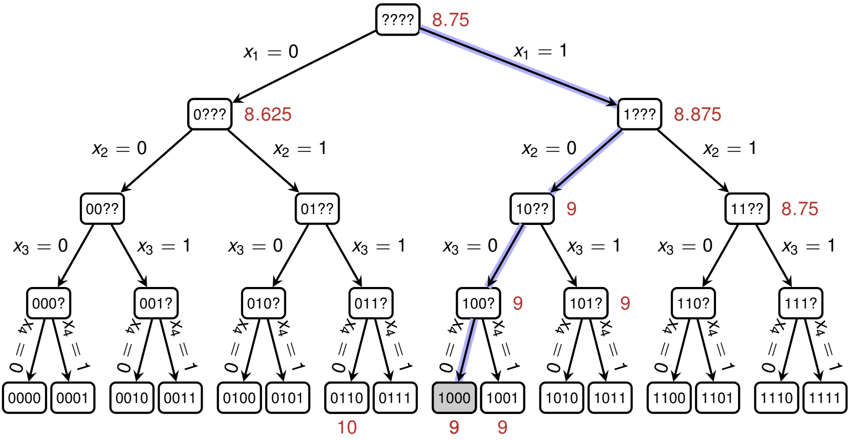

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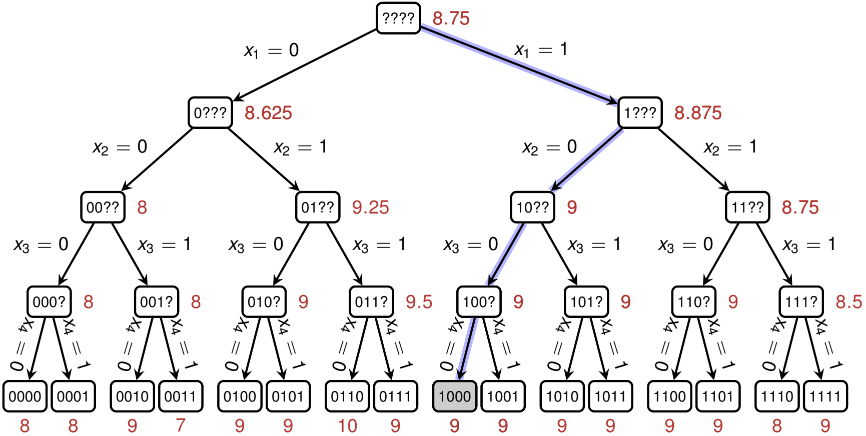

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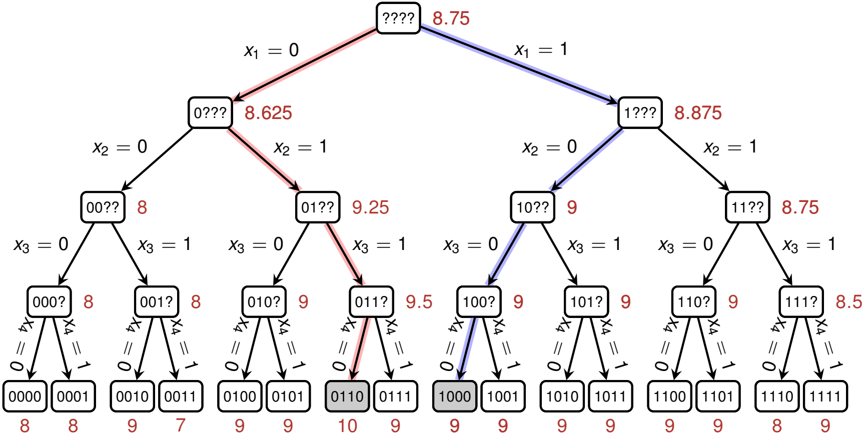

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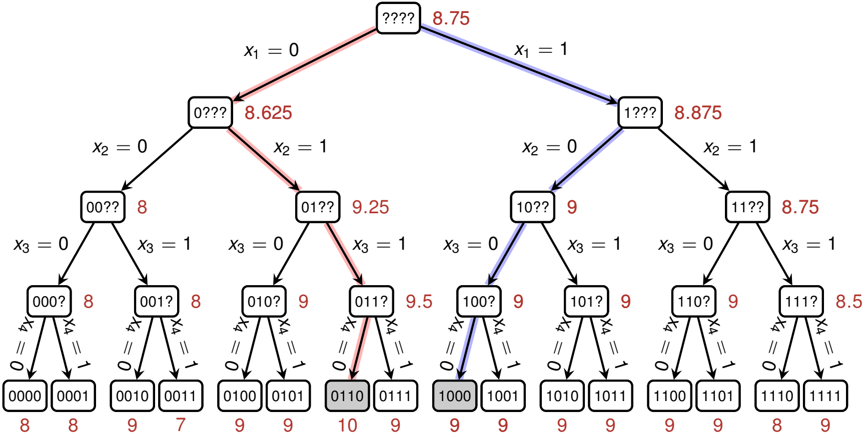

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

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Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-SAT unless P=NP.

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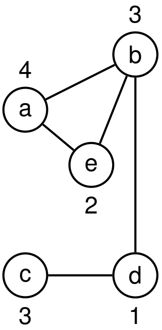
Roughly speaking, there is nothing smarter than just guessing.

Outline

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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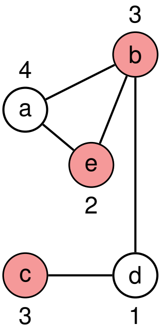

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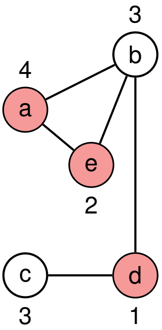

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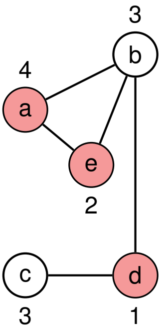

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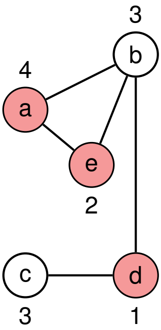

Applications:

The Weighted Vertex-Cover Problem

Vertex Cover Problem

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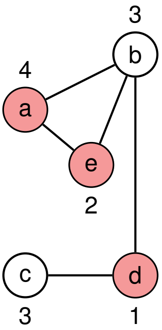

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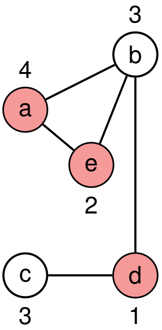

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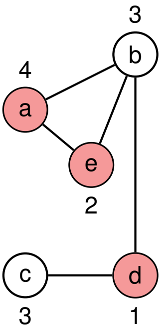

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

```
APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
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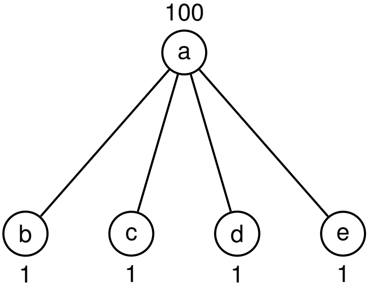

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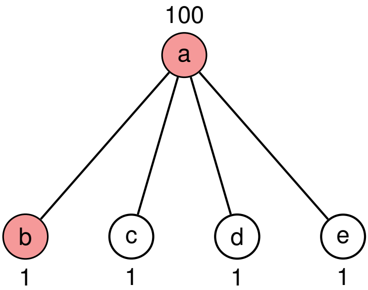
ignores all vertex-weights



Computed solution has weight 101.

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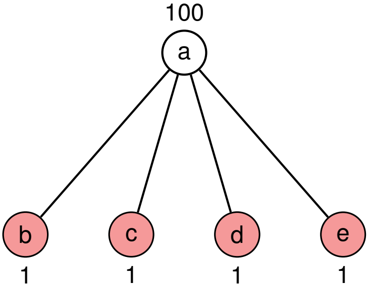

Optimal solution has weight 4.

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & & \\
\text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u,v) \in E \\
 & x(v) & \in & \{0,1\} & \text{for each } v \in V
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Linear Program

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\end{array}
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$\hookrightarrow$ more formally, $0 \leq x(v) \leq 1$

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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return C

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

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C = ∅
compute x̄, an optimal solution to the linear program
for each v ∈ V
    if x̄(v) ≥ 1/2
        C = C ∪ {v}
return C
```

same as the
Greedy for unweighted

Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

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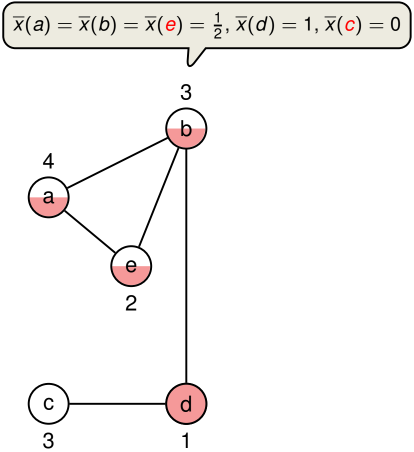
Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC



fractional solution of LP
with weight $= 5.5$

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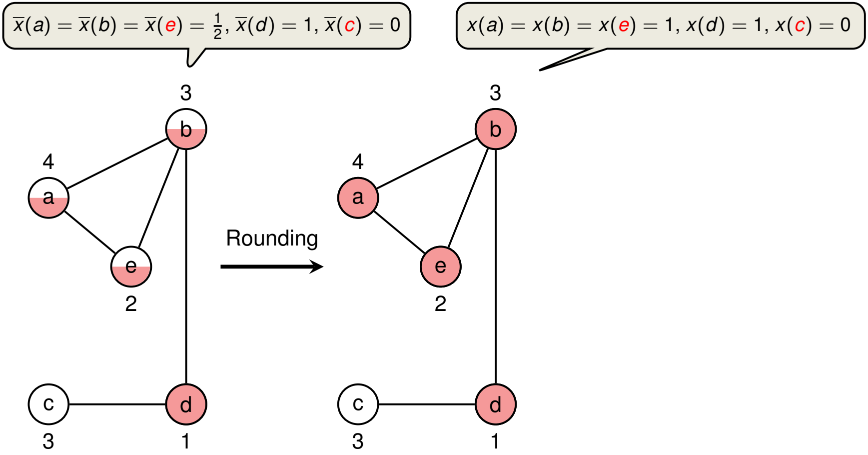

fractional solution of LP
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rounded solution of LP
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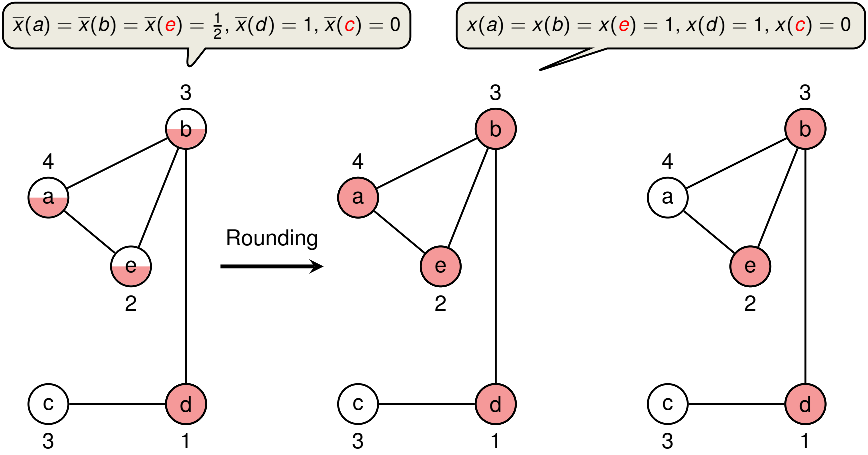

fractional solution of LP
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rounded solution of LP
with weight $= 10$

optimal solution
with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2):

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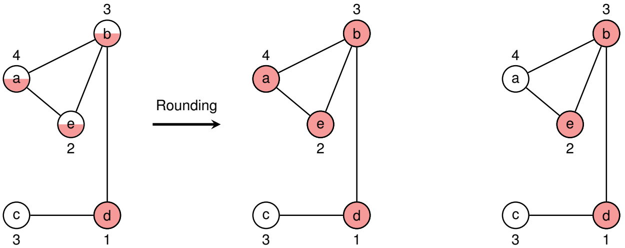

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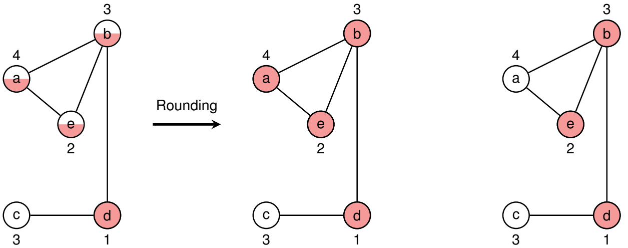

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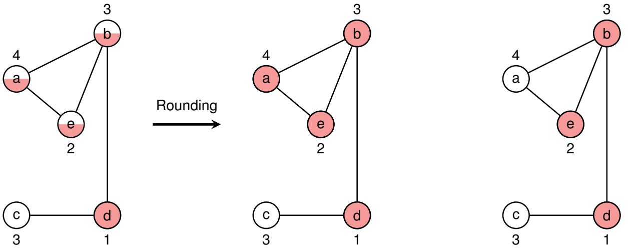
Proof (Approximation Ratio is 2):

- Let C^* be an optimal solution to the minimum-weight vertex cover problem
- Let z^* be the value of an optimal solution to the linear program, so



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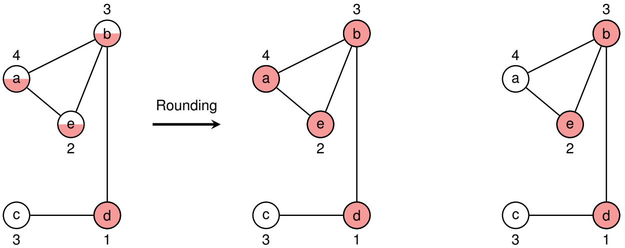

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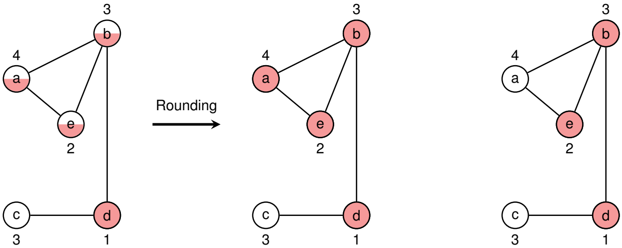

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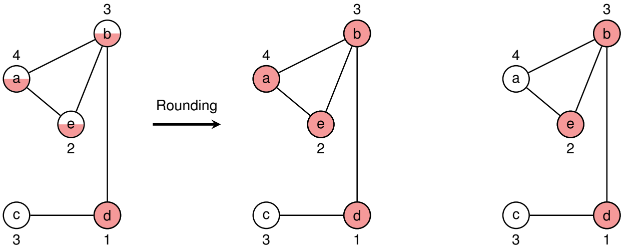

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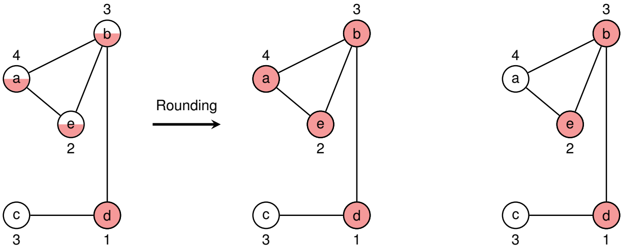

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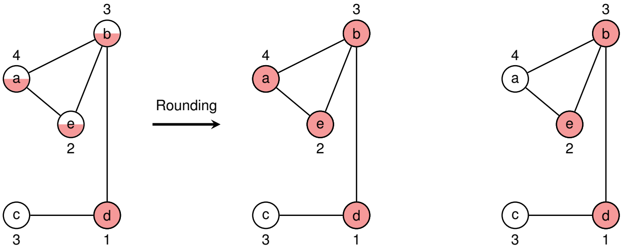

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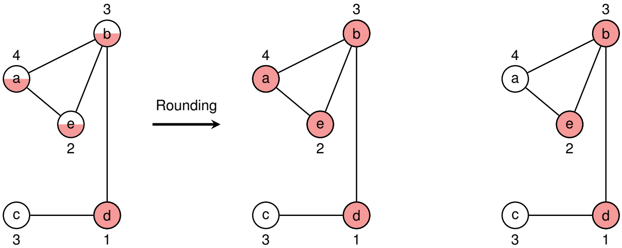

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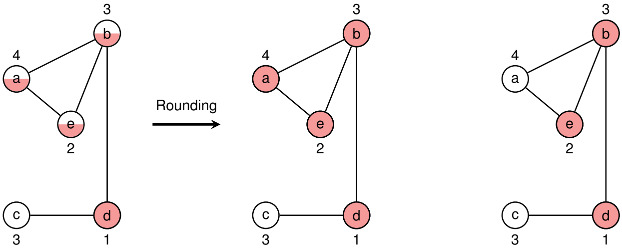

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Proof (Approximation Ratio is 2):

- Let C^* be an optimal solution to the minimum-weight vertex cover problem
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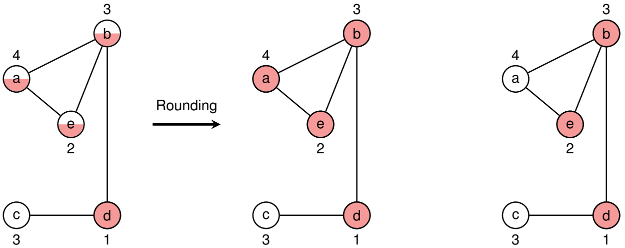

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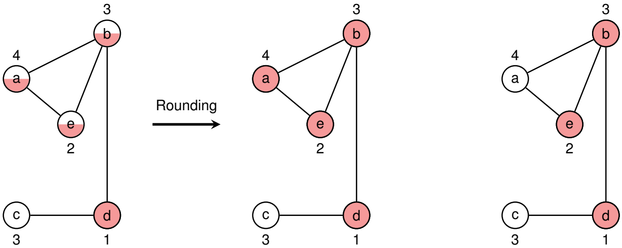

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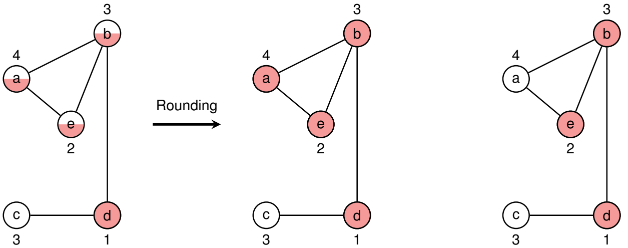

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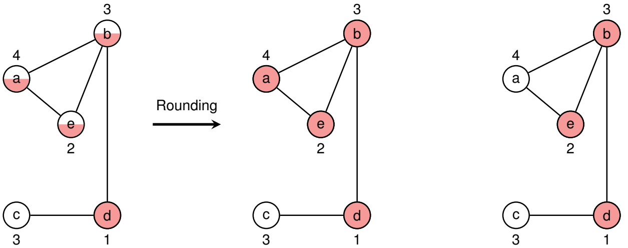

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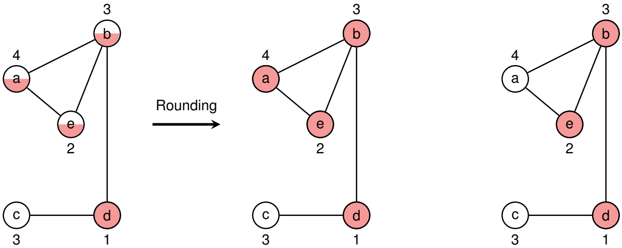

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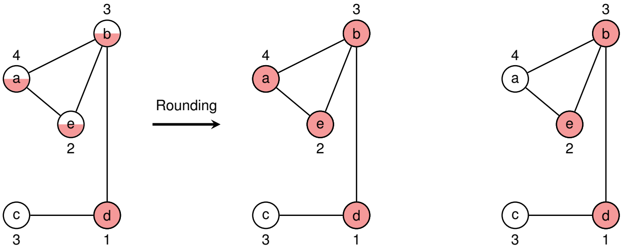

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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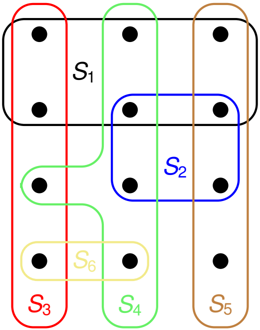

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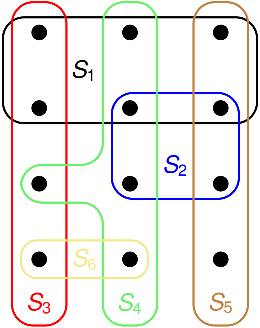

	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2

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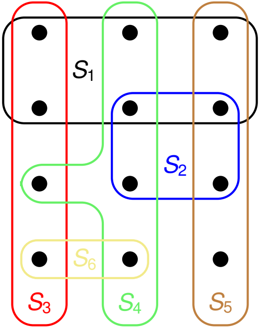

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

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0-1 Integer Program

$$\begin{array}{llcll} \text{minimize} & \displaystyle\sum_{S\in\mathcal{F}} c(S)y(S) & & & \\ \text{subject to} & \displaystyle\sum_{S\in\mathcal{F}:\, x\in S} y(S) & \geq & 1 & \text{for each } x \in X \\ & y(S) & \in & \{0,1\} & \text{for each } S \in \mathcal{F} \end{array}$$

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Linear Program

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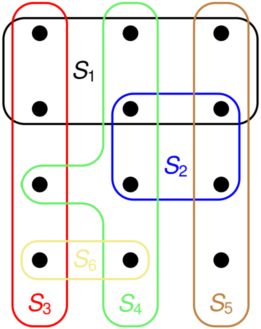
Back to the Example



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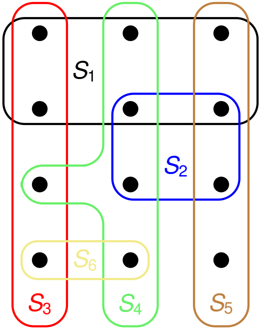

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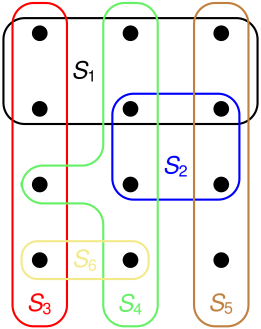

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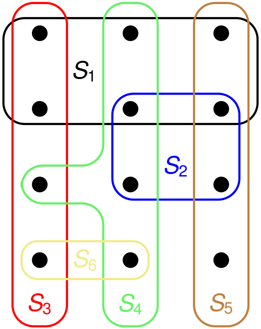

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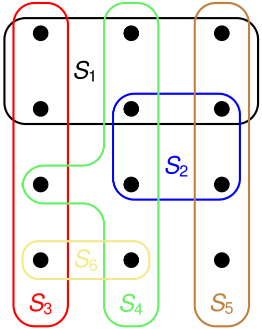

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Even worse: If all y's were below $1/2$, we would not even return a valid cover!

Randomised Rounding

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Idea: Interpret the y-values as probabilities for picking the respective set.

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Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $y(S)$.

→ may or may not be feasible

in other words, if $y(.)$ is the LP solution, then we obtain an IP solution $y'(.)$ by:

$$\forall S \in \mathcal{F}: \ y'(S) = \begin{cases} 1 & \text{with prob. } y(S) \\ 0 & \text{with prob. } 1 - y(S) \end{cases}$$

$$\Rightarrow E[y'(S)] = y(S)$$

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how we can apply linearity of expectations

$$= \sum_{S \in \mathcal{F}} \mathbf{Pr}[S \in \mathcal{C}] \cdot c(S)$$

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$$\mathbf{Pr}[\,x \notin \cup_{S \in \mathcal{C}} S\,]$$

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$$\mathbf{Pr}[\, x \notin \cup_{S \in \mathcal{C}} S \,] = \prod_{S \in \mathcal{F}\,:\, x \in S} \mathbf{Pr}[\, S \notin \mathcal{C} \,]$$

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clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

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Thank you and Best Wishes for the Exam!