# VI. Approximation Algorithms: Travelling Salesman Problem

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Easter 2015

UNIVERSITY OF CAMBRIDGE

# Outline

Introduction

General TSP

Metric TSP

# The Traveling Salesman Problem (TSP)

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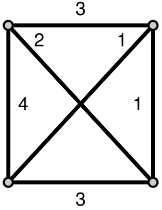

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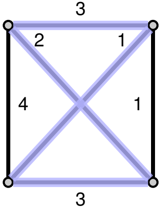


$3 + 2 + 1 + 3 = 9$

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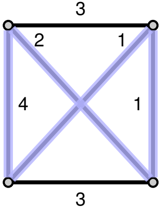


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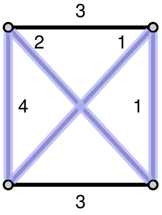


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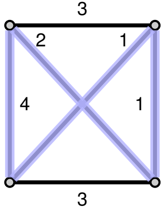


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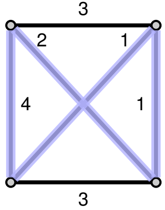


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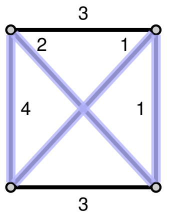


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- Metric TSP: costs satisfy triangle inequality:

$$\underline{\forall u, v, w \in V:} \qquad c(u, w) \leq c(u, v) + c(v, w).$$

↳ this "enforces" complete graph

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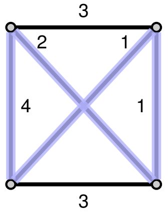


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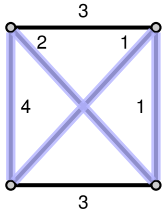


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Even this version is NP hard (Ex. 35.2-2)

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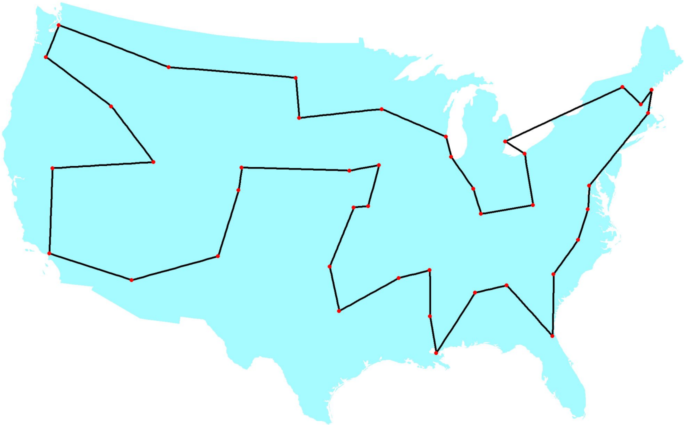
# History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



**http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html**

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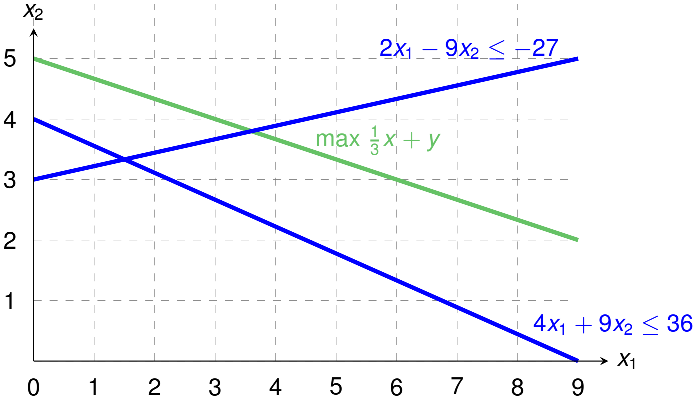

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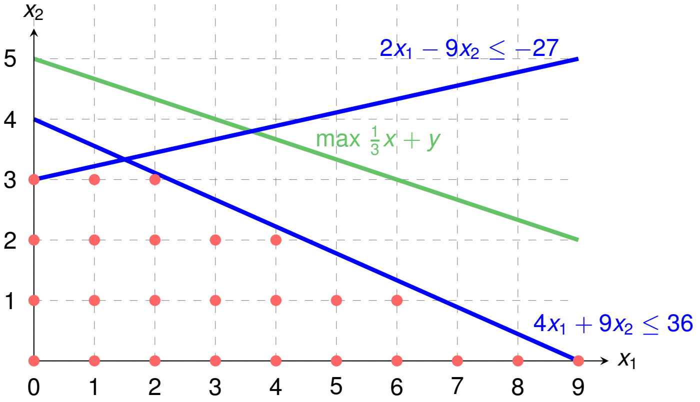

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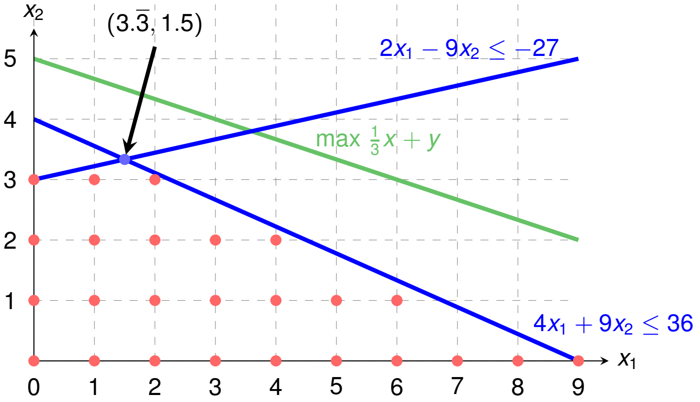

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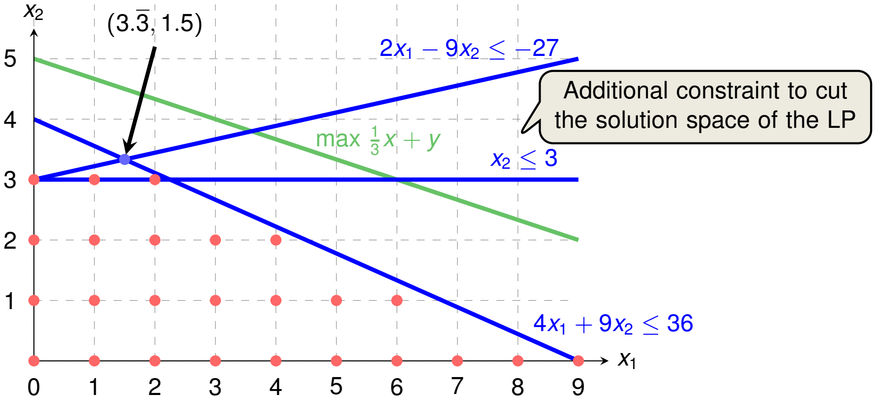

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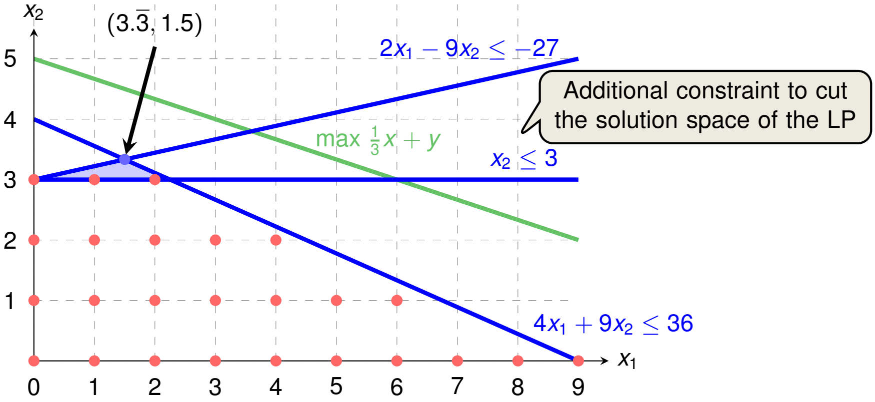

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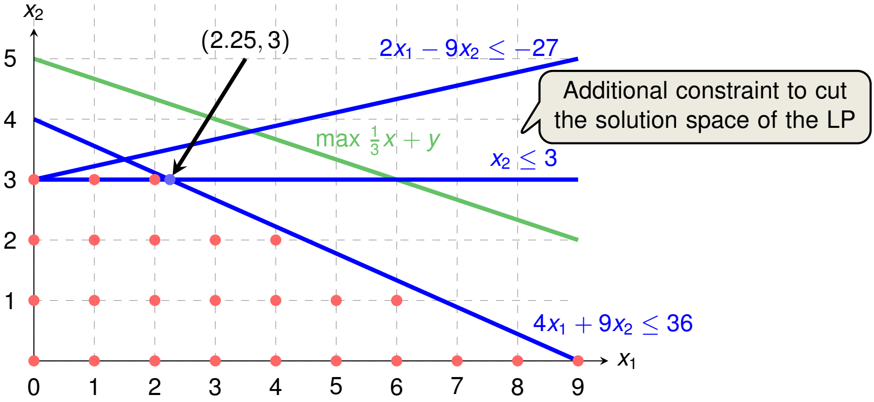

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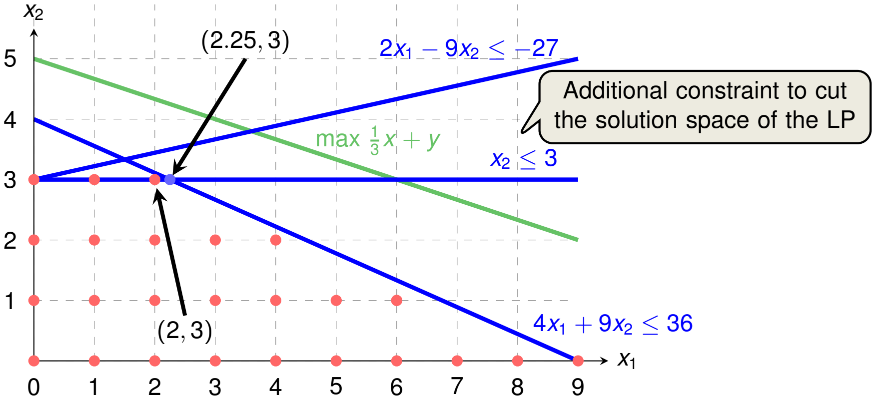

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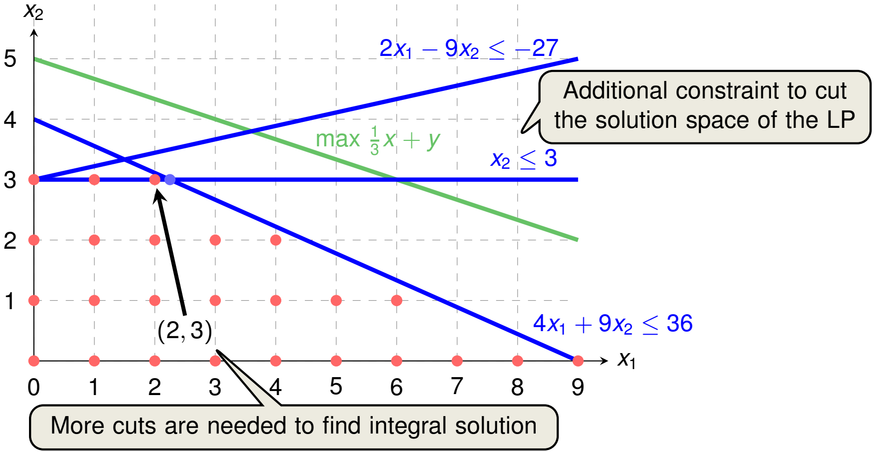

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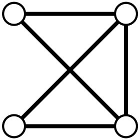

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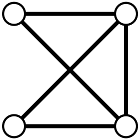

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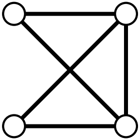
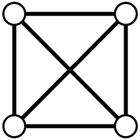
$G = (V, E)$ 

 $G' = (V, E')$

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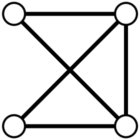

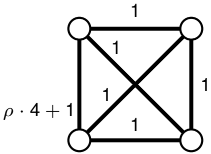


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Large weight will render this edge useless!

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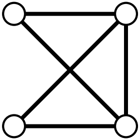

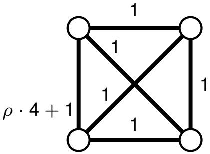


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Can create representations of $G'$ and $c$ in time polynomial in $|V|$ and $|E|$!

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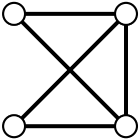

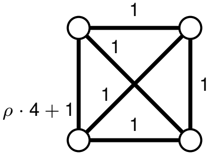


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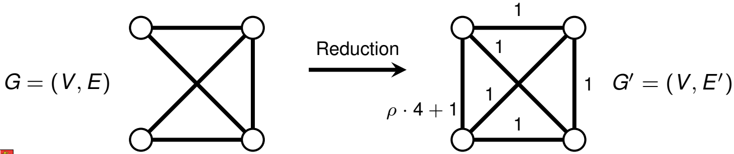

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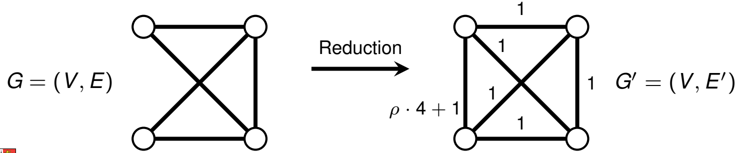

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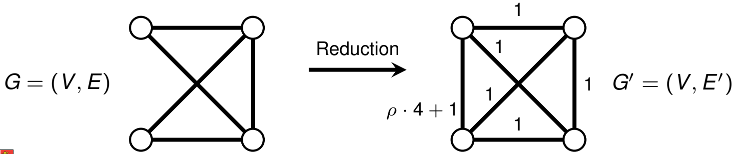

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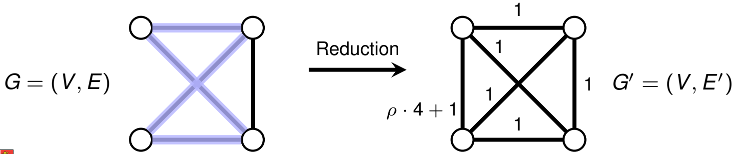

# Hardness of Approximation

**Theorem 35.3**

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio $\rho$ for the general TSP.

Proof: Idea: Reduction from the hamiltonian-cycle problem.

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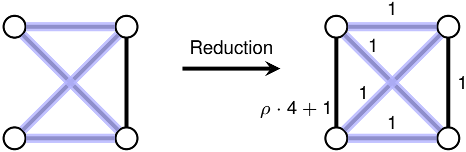


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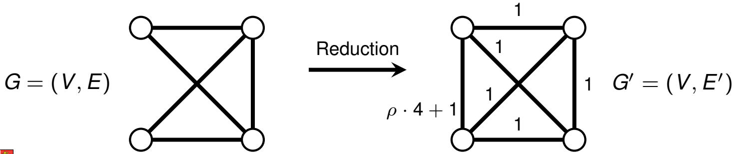

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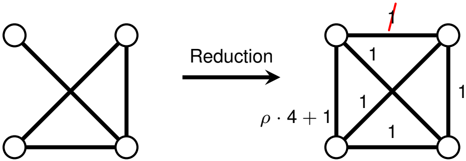


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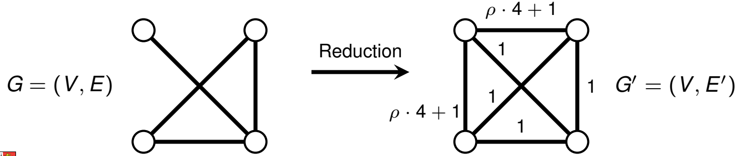

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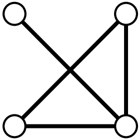
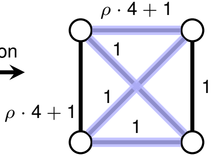


Reduction

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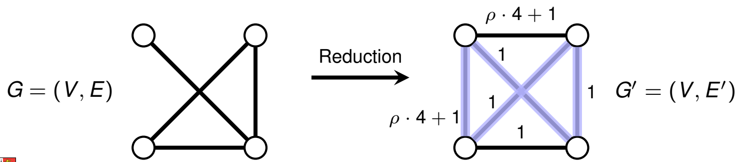

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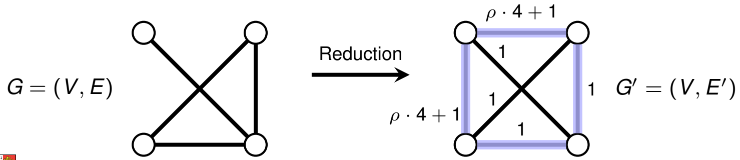

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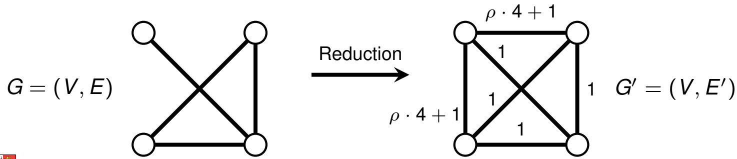

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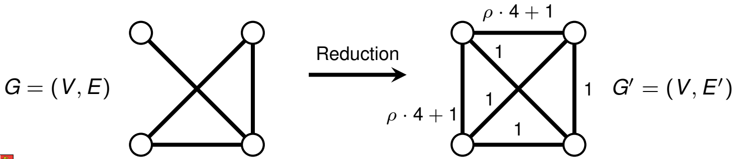

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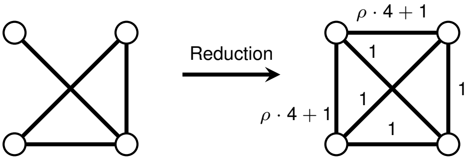


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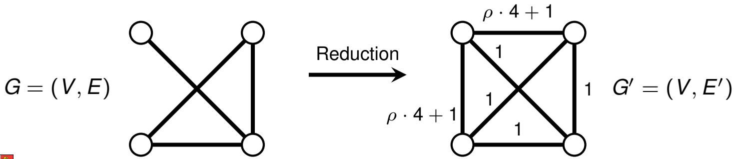

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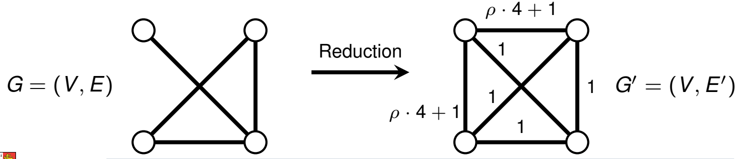

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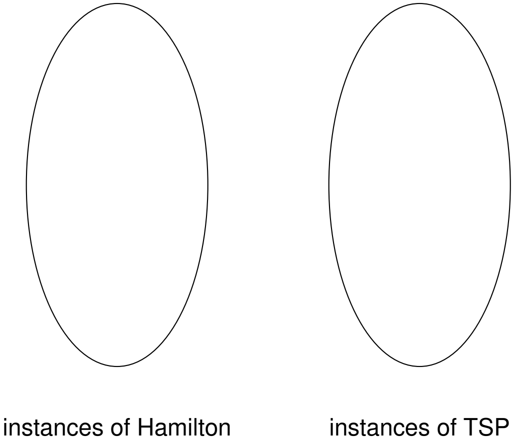

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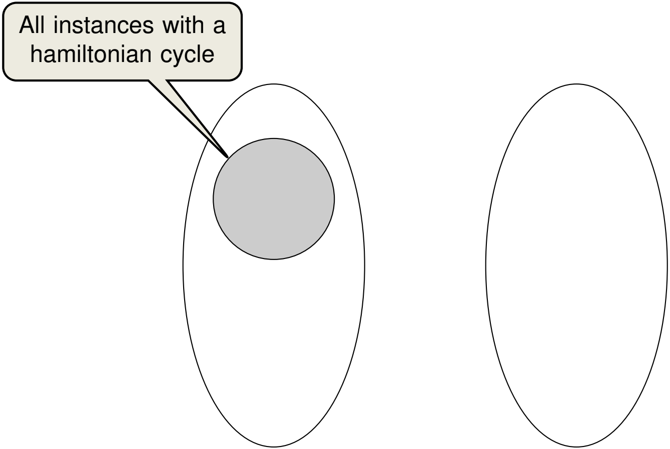


instances of Hamilton　　　　instances of TSP

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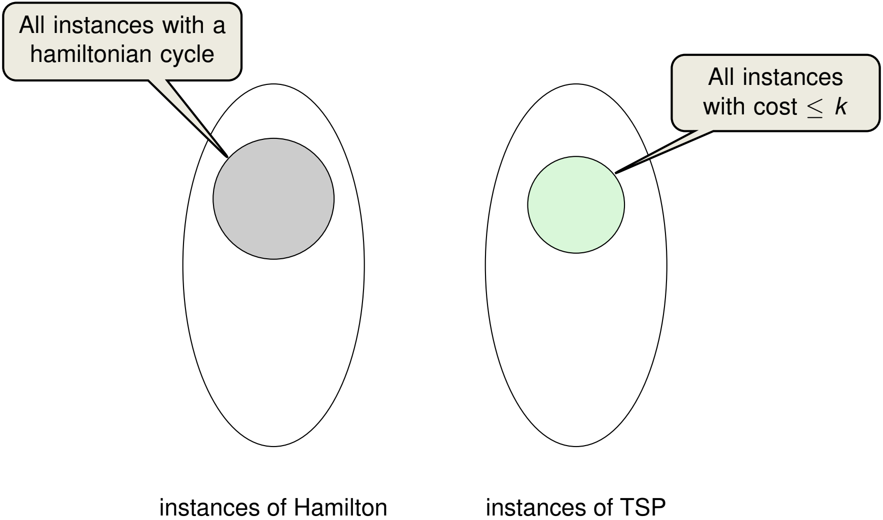

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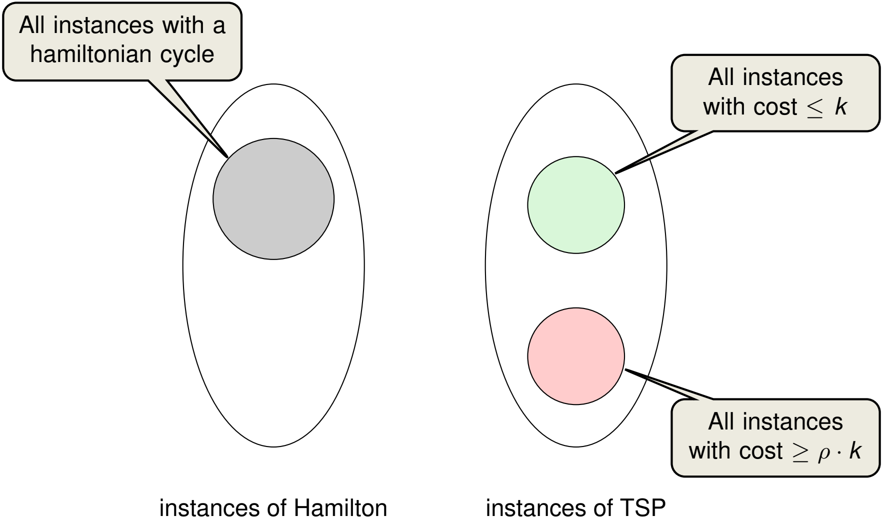

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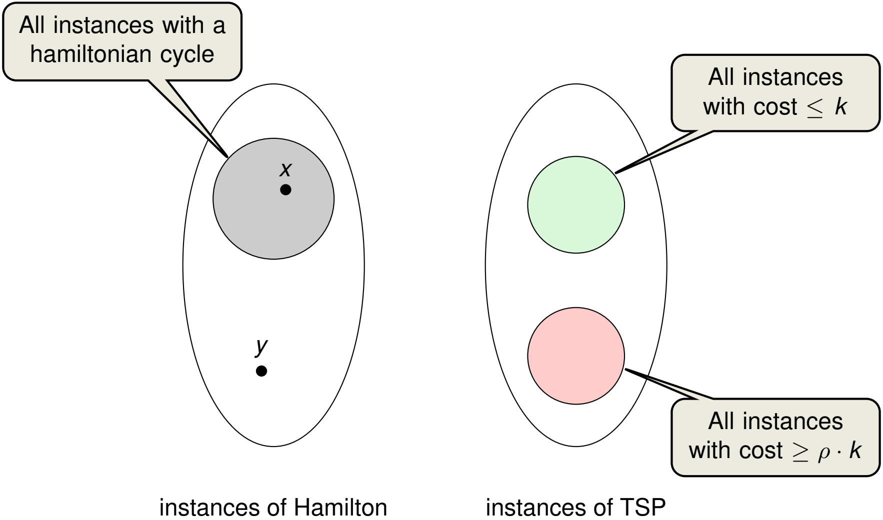

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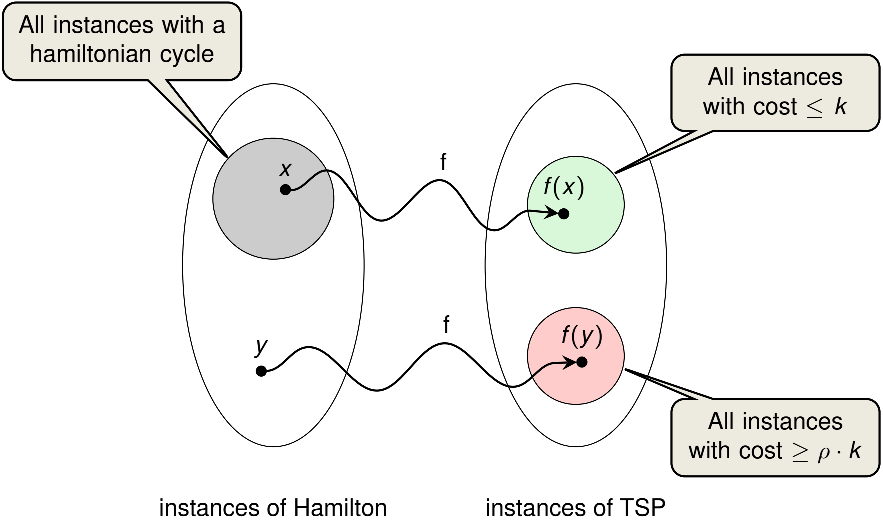

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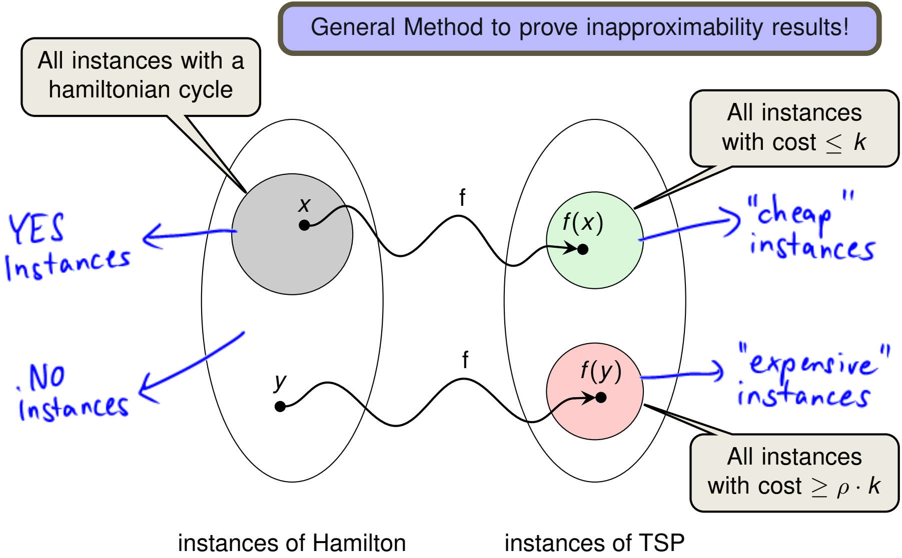

## Outline

Introduction

General TSP

Metric TSP

## The TSP Problem with the Triangle Inequality

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR$(G, c)$

1 select a vertex $r \in G.V$ to be a "root" vertex
2 compute a minimum spanning tree $T$ for $G$ from root $r$
  using MST-PRIM$(G, c, r)$
3 let $H$ be a list of vertices, ordered according to when they are first visited
  in a preorder tree walk of $T$
4 **return** the hamiltonian cycle $H$

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return the hamiltonian cycle H
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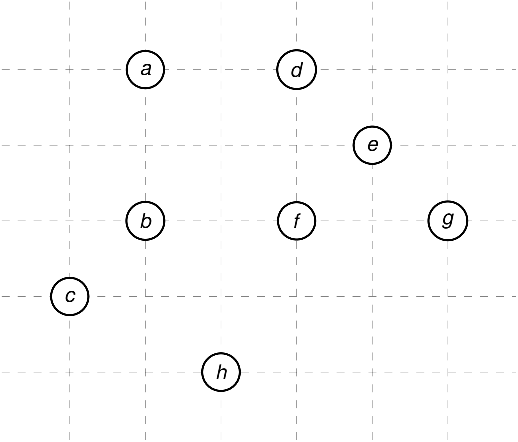
Runtime is dominated by MST-PRIM, which is $\Theta(V^2)$.

number of edges is $V^2$, as $G$ is a complete graph.

# Run of APPROX-TSP-TOUR



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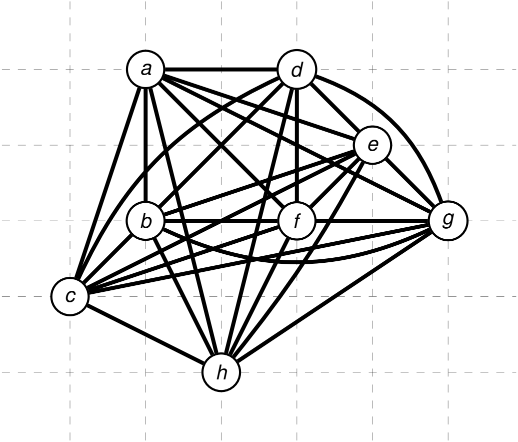


1. Compute MST

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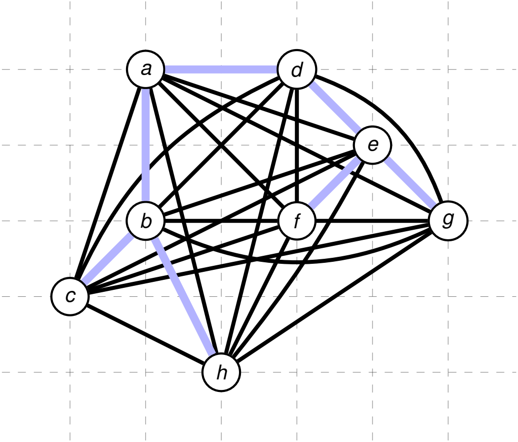


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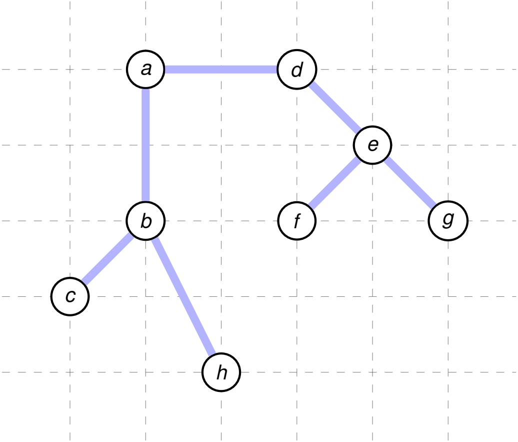


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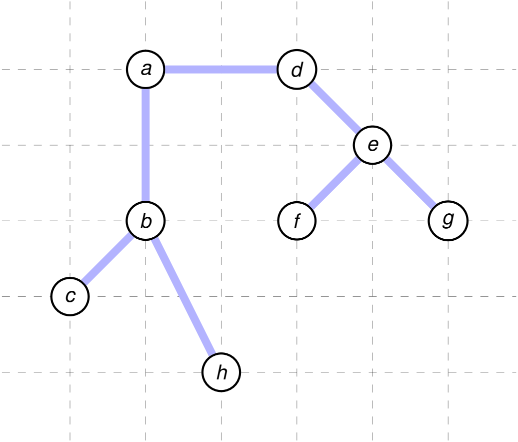


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2. Perform preorder walk on MST

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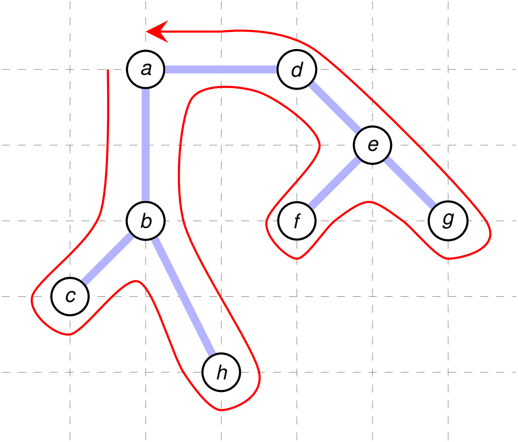


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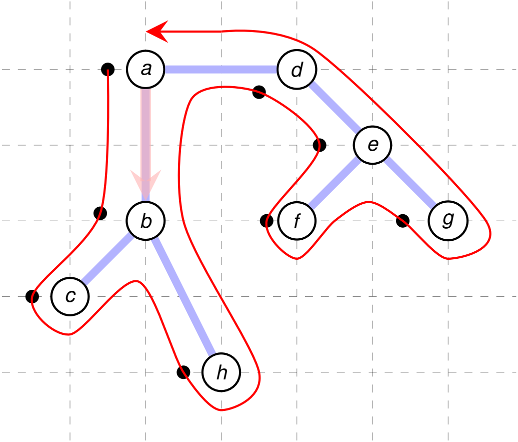


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3. Return list of vertices according to the preorder tree walk

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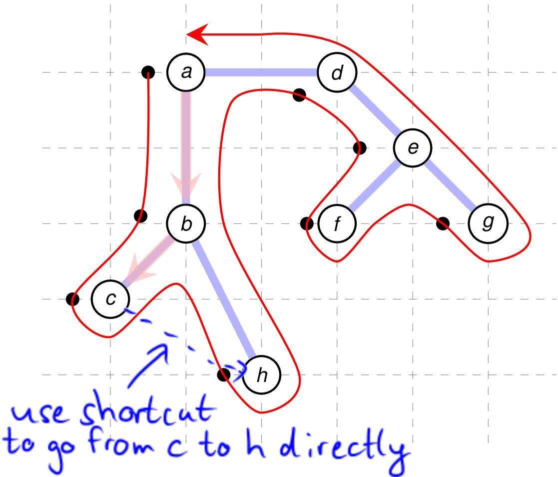


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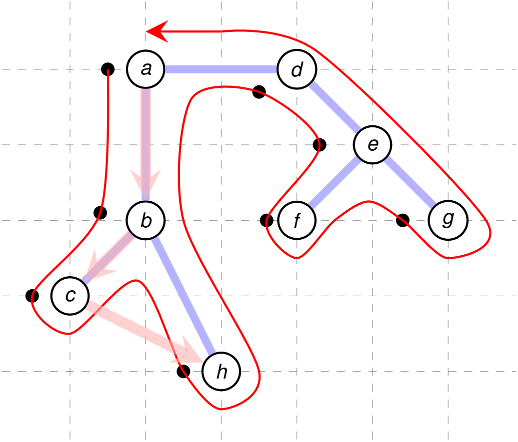


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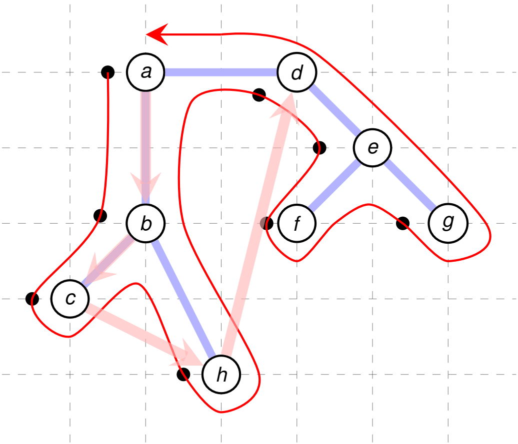


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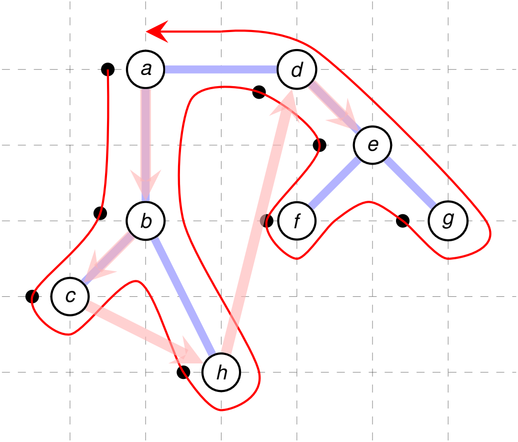


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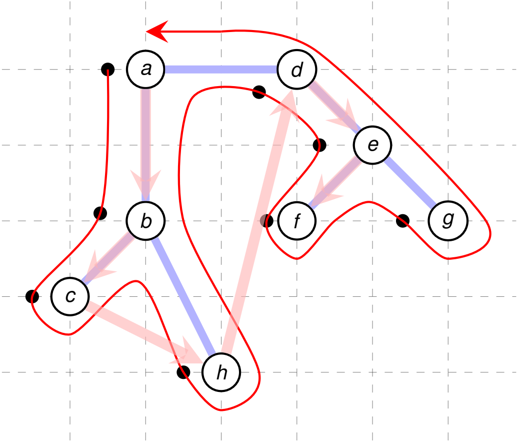


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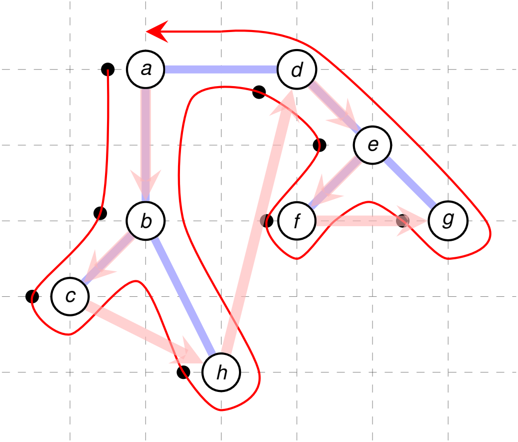


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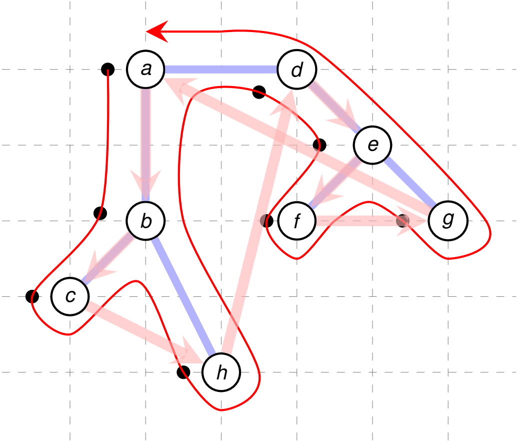


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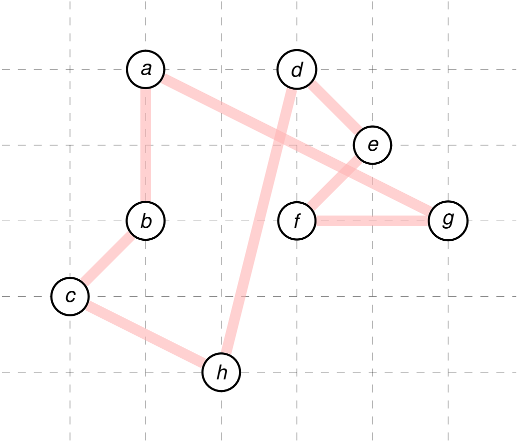


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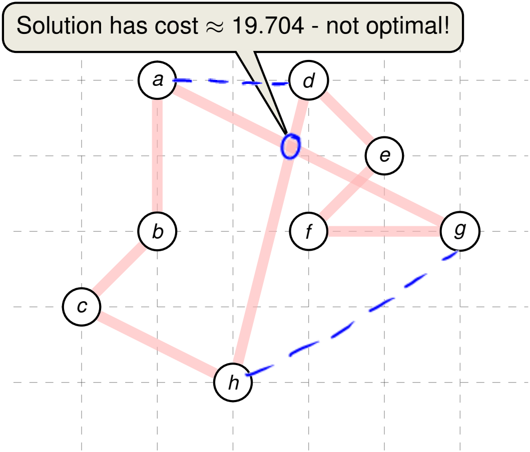


1. Compute MST ✓
2. Perform preorder walk on MST ✓
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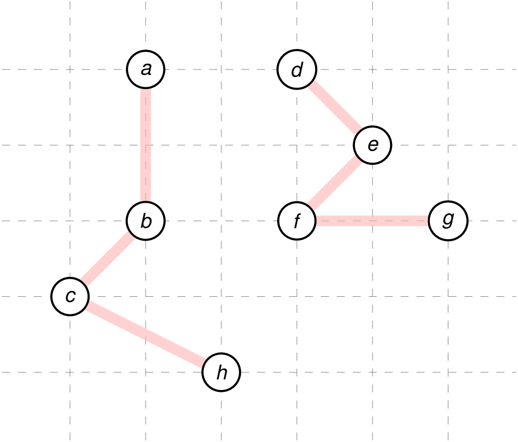


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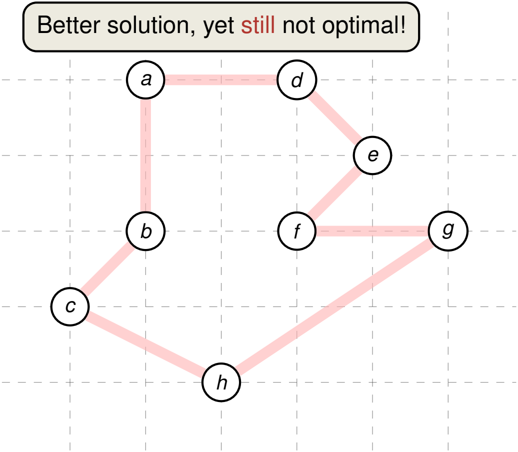


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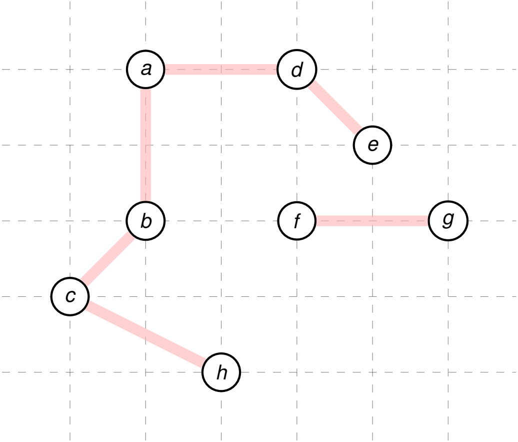


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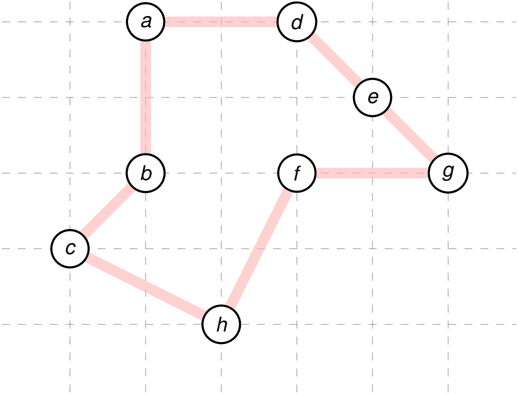
## Run of APPROX-TSP-TOUR

This is the optimal solution (cost $\approx$ 14.715).

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## Proof of the Approximation Ratio

**Theorem 35.2**

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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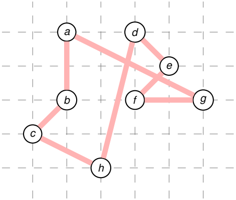


solution $H$ of APPROX-TSP

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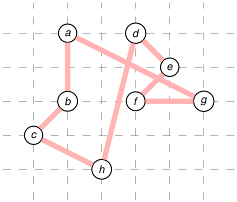


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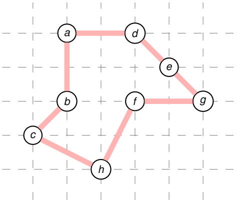


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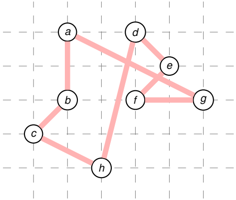


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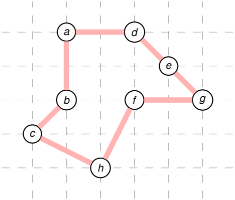


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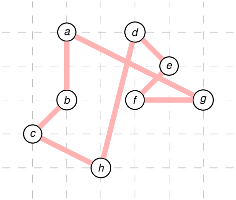


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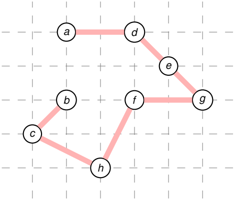


spanning tree as a subset of $H^*$

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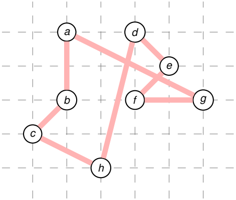


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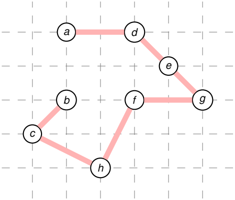


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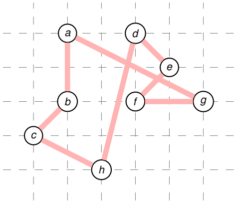


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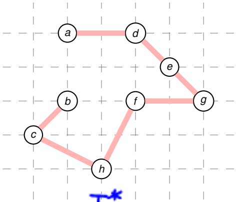


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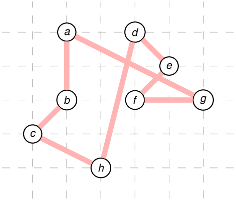


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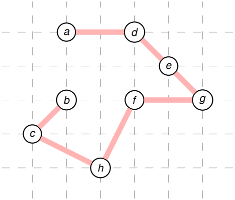


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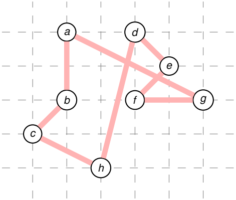


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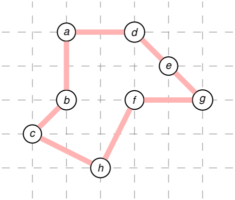


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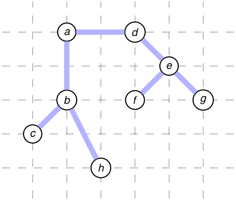
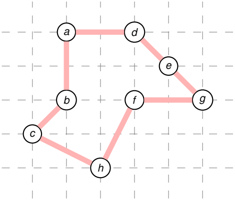



minimum spanning tree $T$

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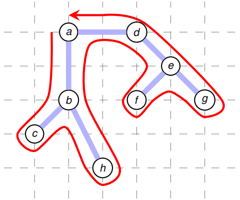
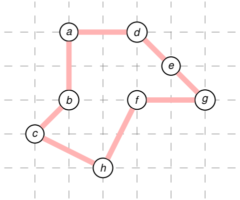



Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$

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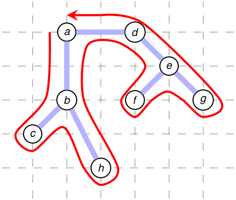


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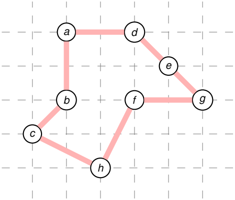


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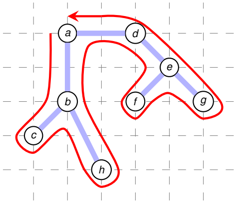


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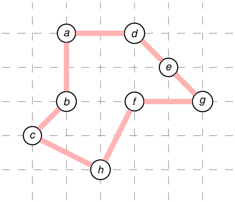


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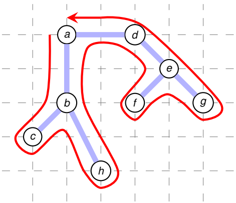


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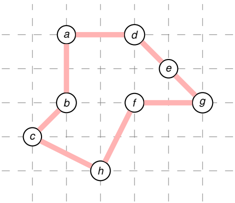


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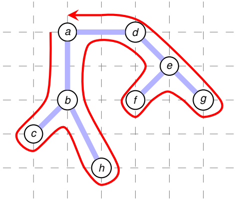


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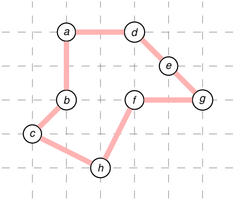


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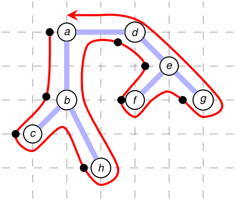


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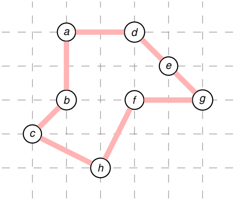


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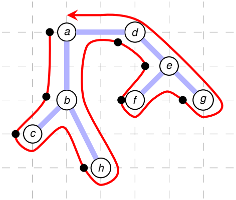


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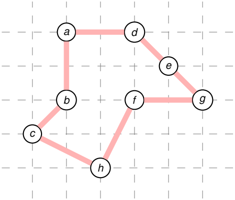


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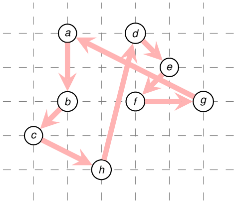


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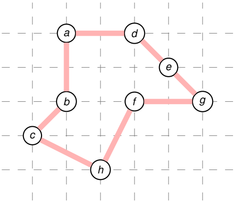


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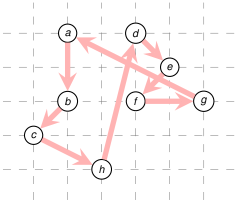


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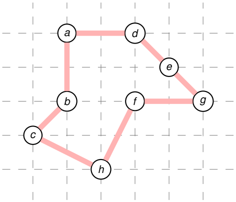


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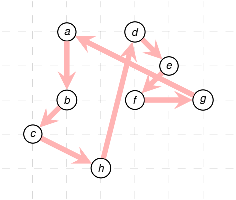


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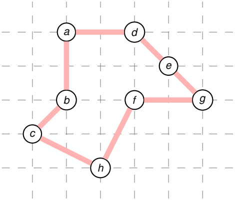


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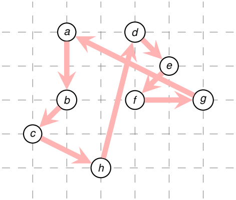


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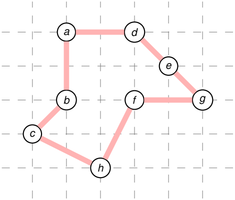


optimal solution $H^*$

## Proof of the Approximation Ratio

**Theorem 35.2**

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour $H^*$ and remove one edge

$\Rightarrow$ yields a spanning tree and therefore $c(T) \leq c(H^*)$

exploiting that all edge costs are non-negative!

- Let $W$ be the full walk of the spanning tree $T$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

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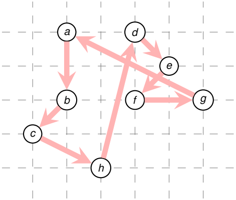


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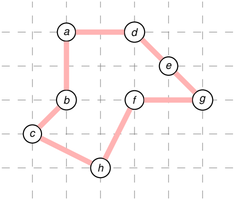


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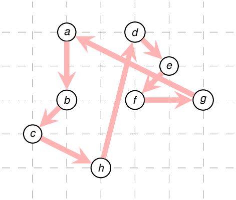
$$c(H) \leq c(W) \leq 2c(H^*) \qquad \square$$

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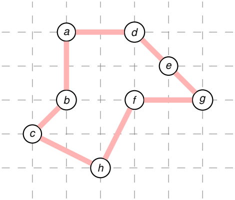


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# Christofides Algorithm

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CHRISTOFIDES$(G, c)$
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T$ for $G$ from root $r$
    using MST-PRIM$(G, c, r)$
compute a perfect matching $M$ with minimum weight in the complete graph
    over the odd-degree vertices in $T$
let $H$ be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T \cup M$
**return** $H$

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**Theorem (Christofides'76)**

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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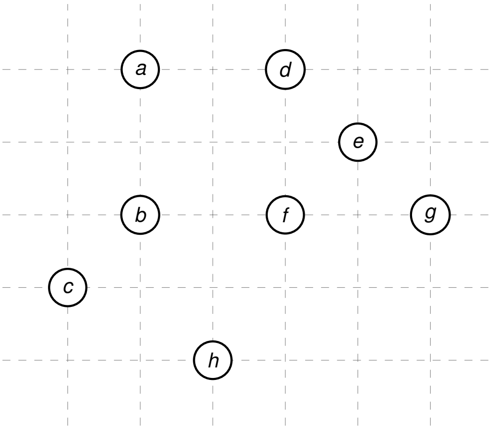

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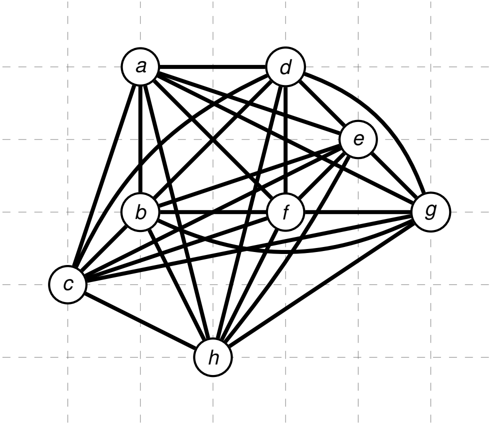
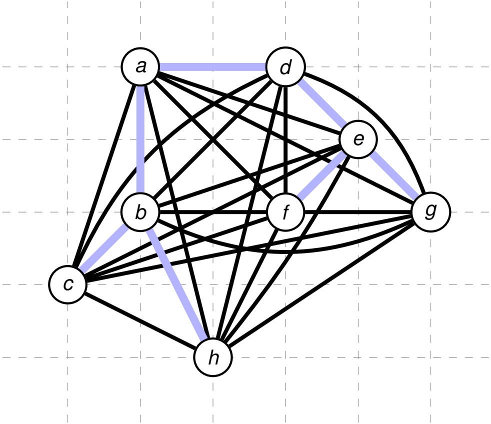



1. Compute MST

# Run of CHRISTOFIDES

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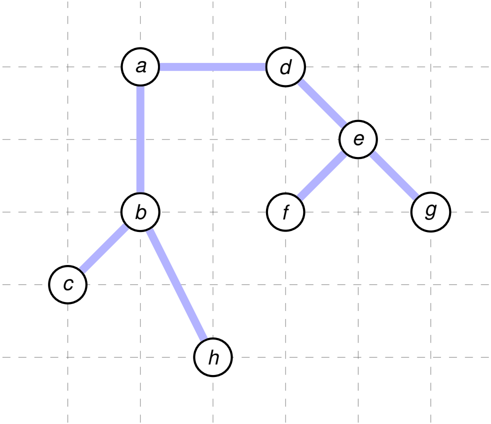
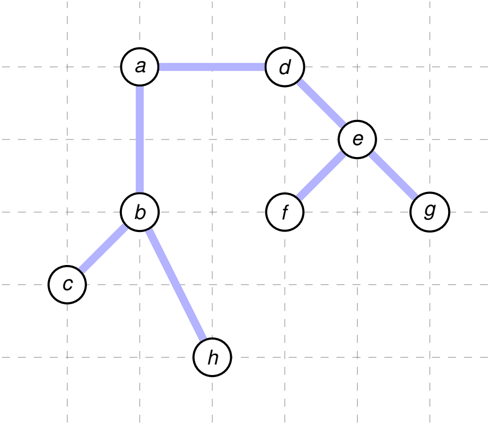


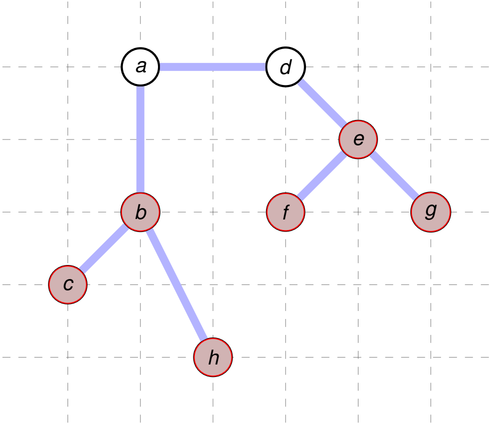

1. Compute MST ✓

## Run of CHRISTOFIDES




1. Compute MST ✓
2. Add a minimum-weight perfect matching $M$ of the odd vertices in $T$

## Run of CHRISTOFIDES

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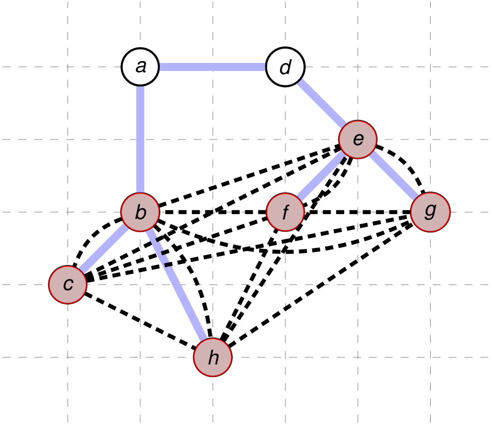


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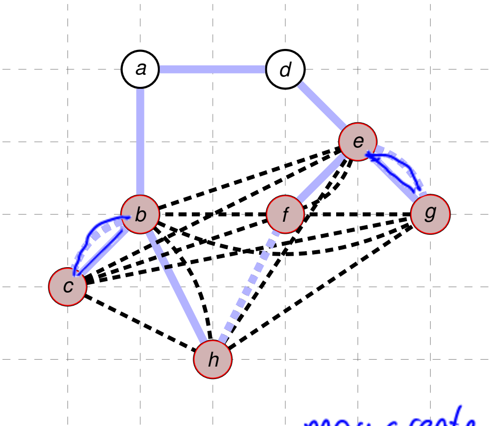


may create a multigraph!

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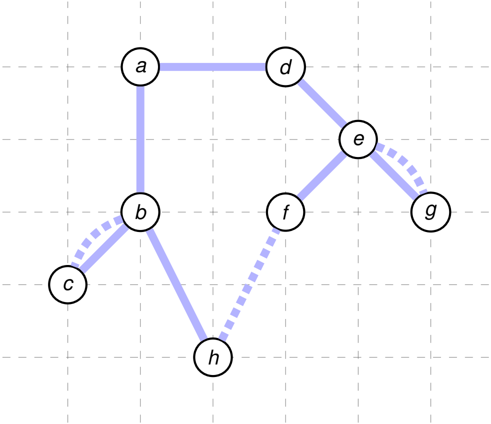


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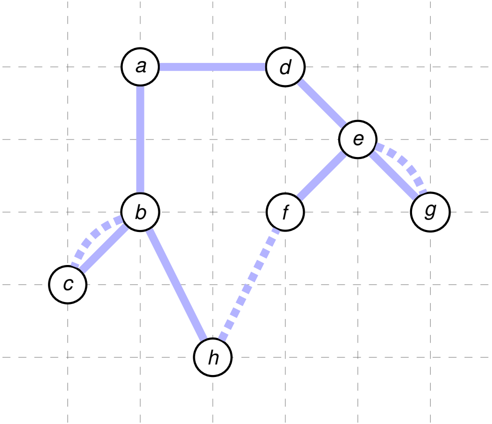
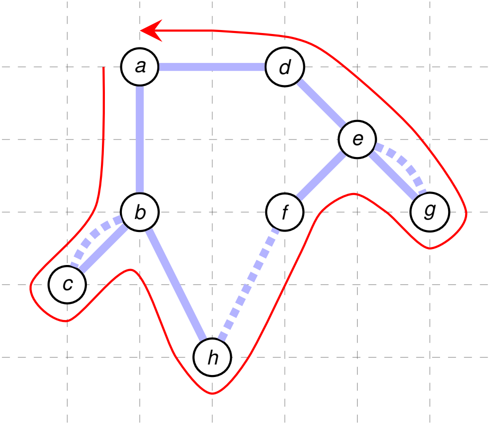



1. Compute MST ✓
2. Add a minimum-weight perfect matching $M$ of the odd vertices in $T$ ✓
3. Find an Eulerian Circuit (all vertices in $T \cup M$ have even degree)

# Run of CHRISTOFIDES

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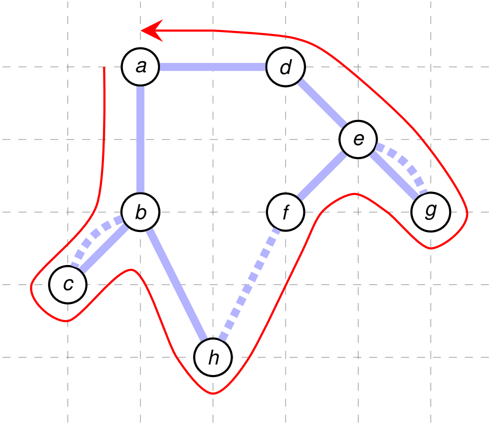


1. Compute MST ✓
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4. Transform the Circuit into a Hamiltonian Cycle

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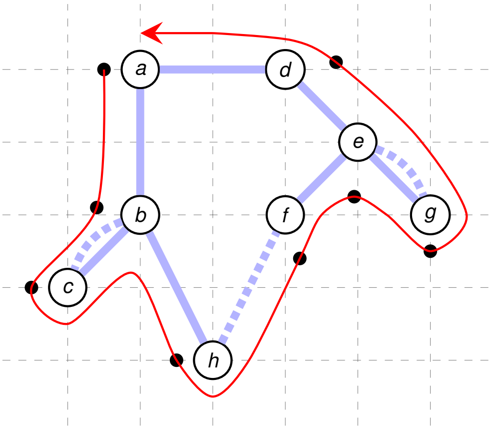


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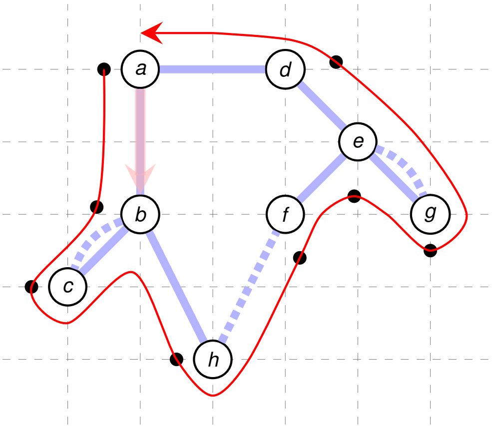


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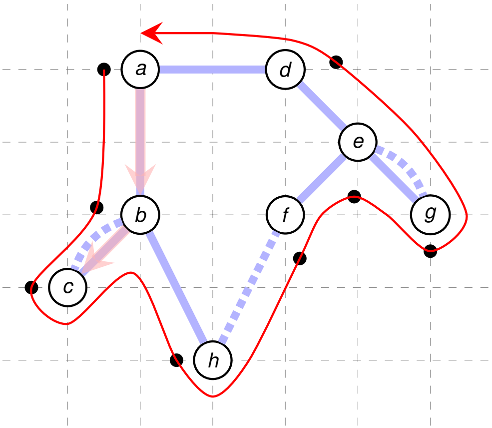


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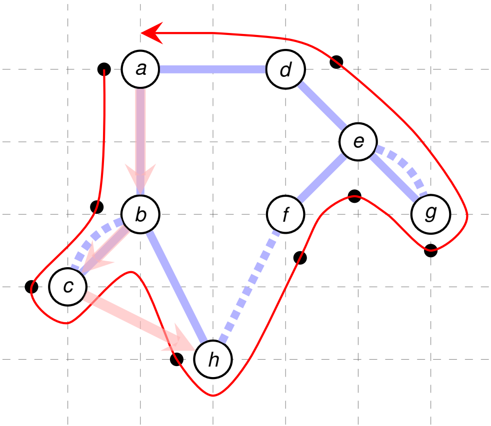


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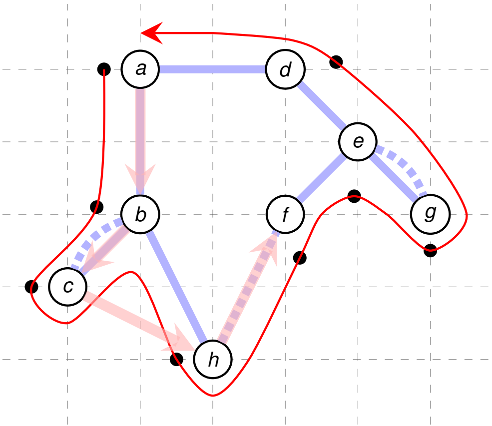


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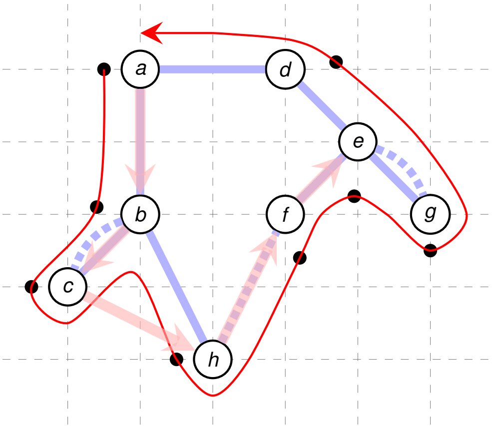


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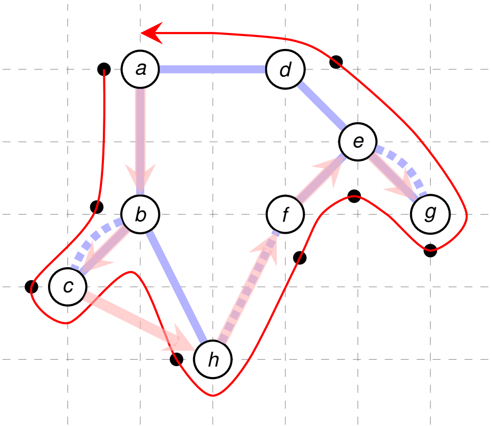


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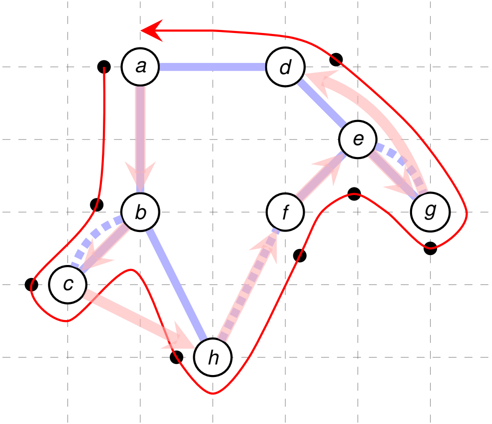


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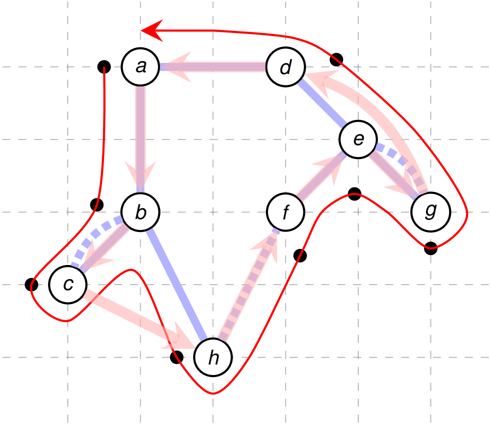


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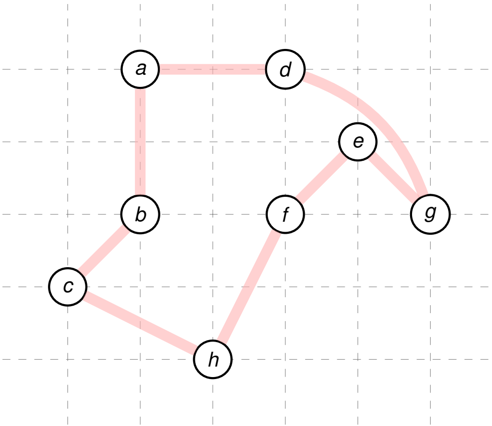


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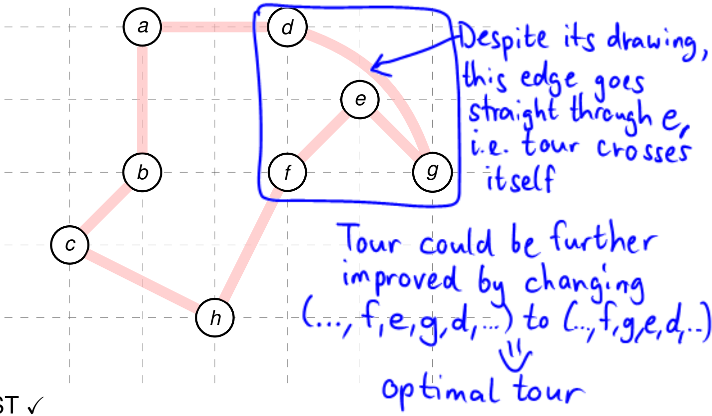
Solution has cost ≈ 15.54 - within 10% of the optimum!

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