I. Sorting Networks

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

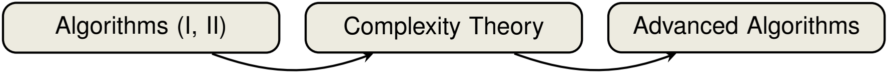
Outline of this Course

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

(Tentative) List of Topics



Algorithms (I, II)

Complexity Theory

Advanced Algorithms

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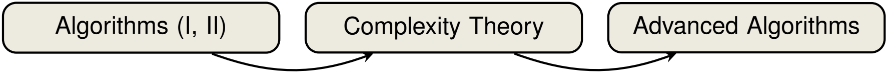

Algorithms (I, II)

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Advanced Algorithms

- I. Sorting Networks (Sorting, Counting, Load Balancing) → 2
- II. Matrix Multiplication (Serial and Parallel)
- III. Linear Programming (Formulating, Applying and Solving) → 2
- IV. Approximation Algorithms: Covering Problems
- V. Approximation Algorithms via Exact Algorithms
- VI. Approximation Algorithms: Travelling Salesman Problem
- VII. Approximation Algorithms: Randomisation and Rounding
- VIII. Approximation Algorithms: MAX-CUT Problem

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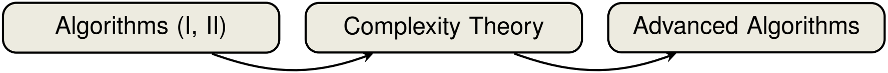

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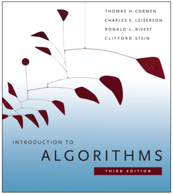

Closely follow the book and use the same numberring of theorems/lemmas etc.

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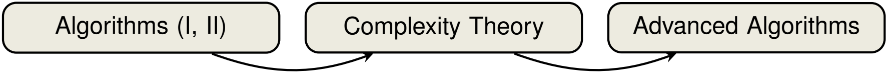
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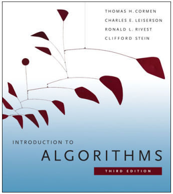

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(Serial) Sorting Algorithms

- we already know several (comparison-based) sorting algorithms: Insertion sort, Bubble sort, Merge sort, Quick sort, Heap sort
- execute one operation at a time
- can handle arbitrarily large inputs
- sequence of comparisons is not set in advance

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Sorting Networks

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Simple concept, but surprisingly deep and complex theory!

Comparison Networks

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  - comparator is a device with, on given two inputs, x and y, returns two outputs x' and y'

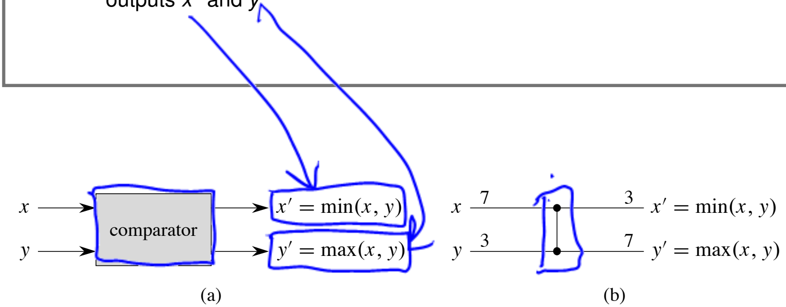

Figure 27.1 **(a)** A comparator with inputs x and y and outputs x' and y'. **(b)** The same comparator, drawn as a single vertical line. Inputs $x = 7$, $y = 3$ and outputs $x' = 3$, $y' = 7$ are shown.

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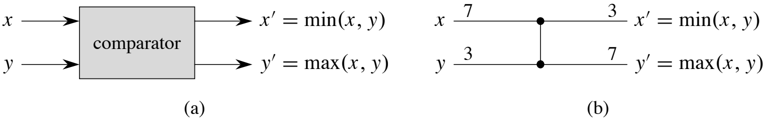
operates in $O(1)$



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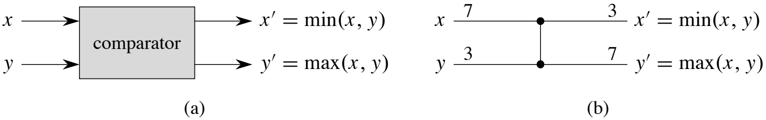

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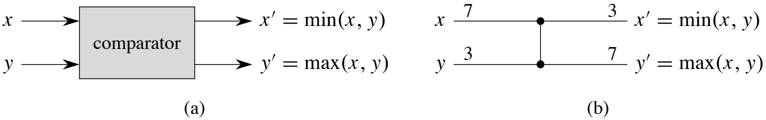

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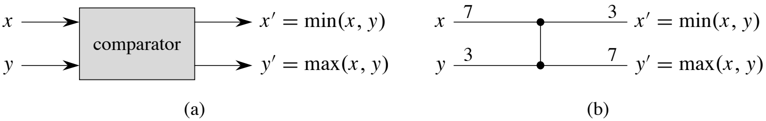
Convention: use the same name for both a wire and its value.



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A sorting network is a comparison network which works correctly (that is, it sorts every input)

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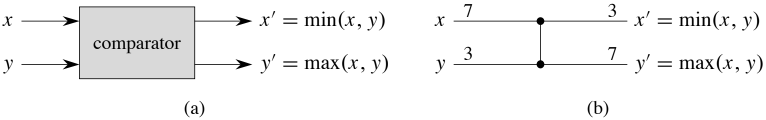

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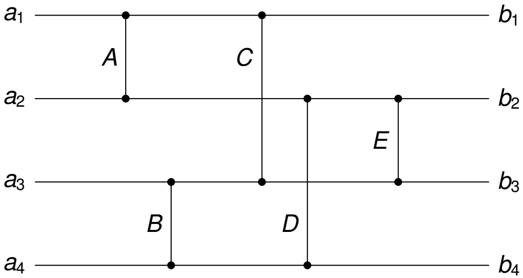
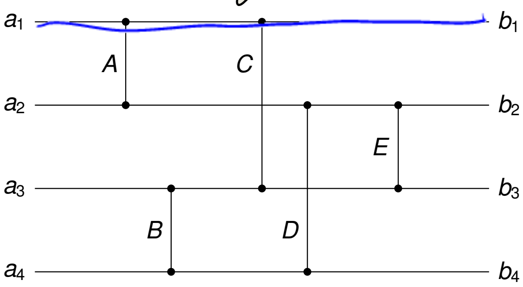


Example of a Comparison Network (Figure 27.2)

A horizontal line represents a sequence of distinct wires



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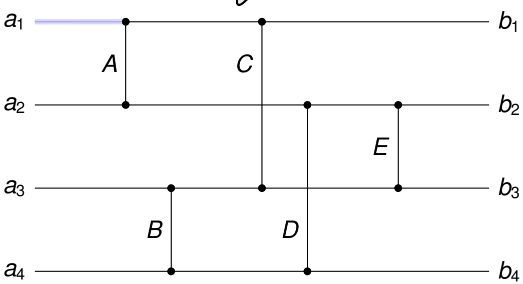

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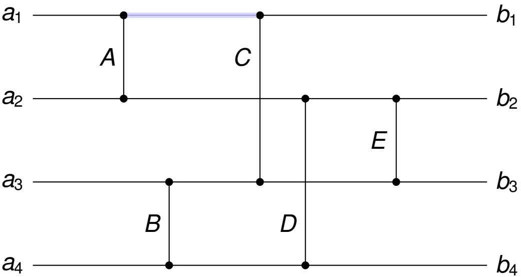

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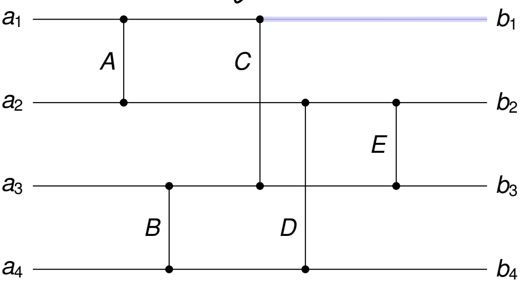
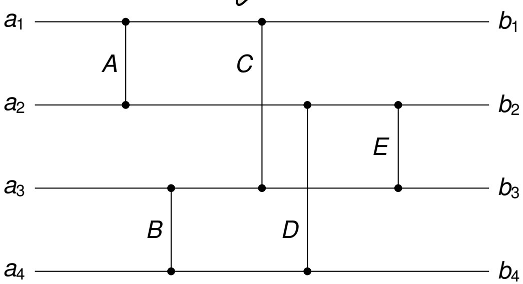


Example of a Comparison Network (Figure 27.2)

Interconnections between comparators must be acyclic



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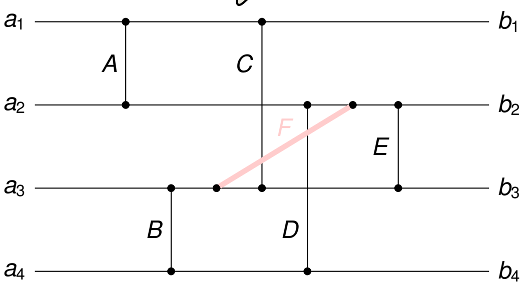

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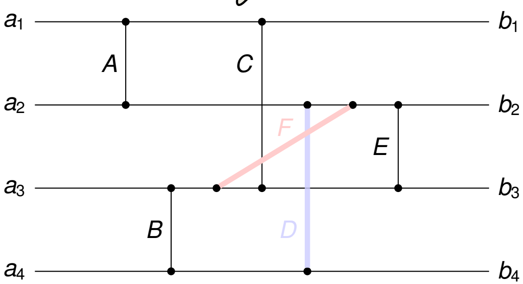

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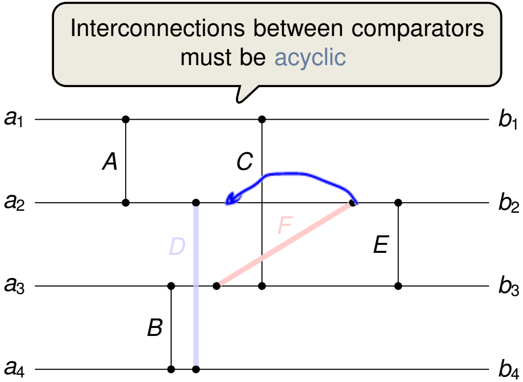

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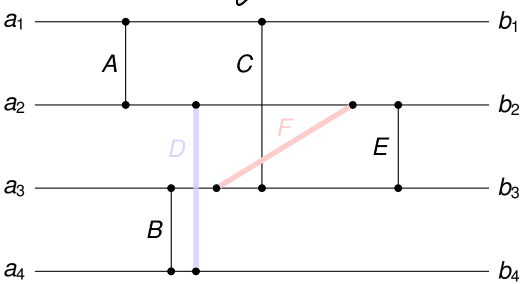

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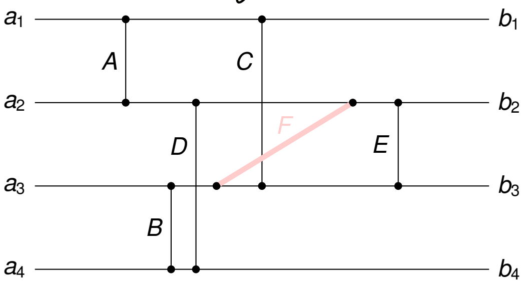

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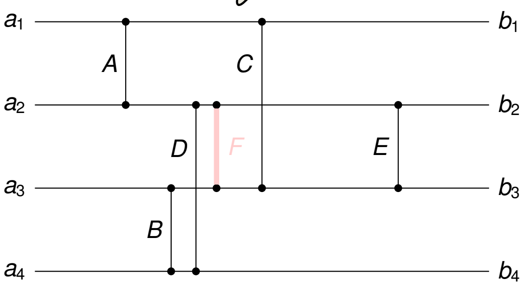

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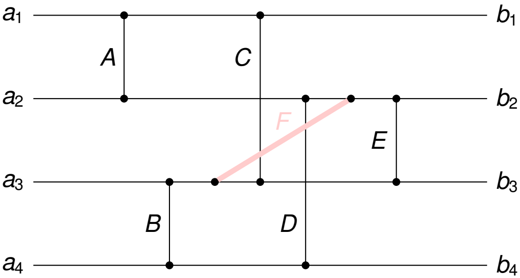

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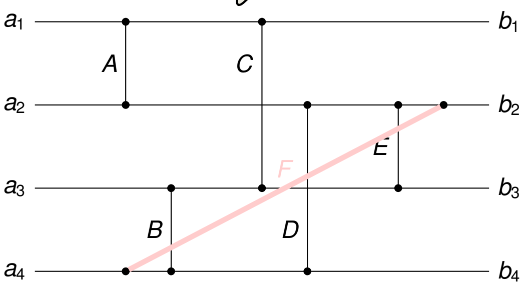

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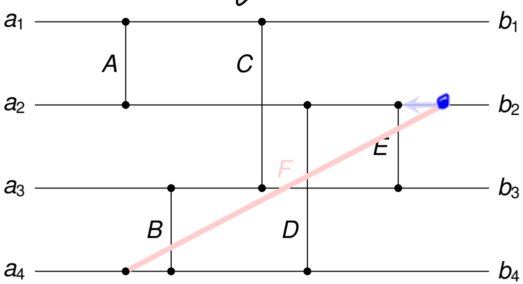

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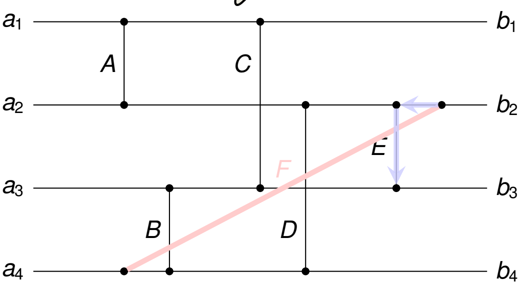

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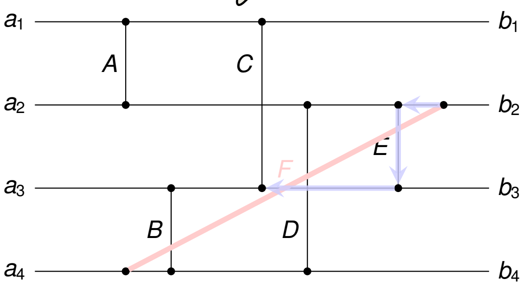

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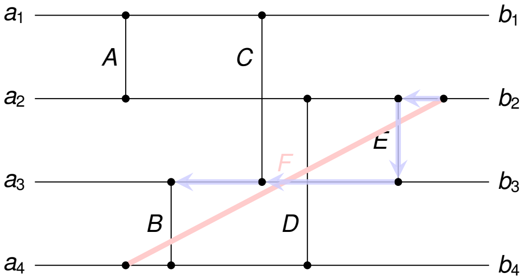

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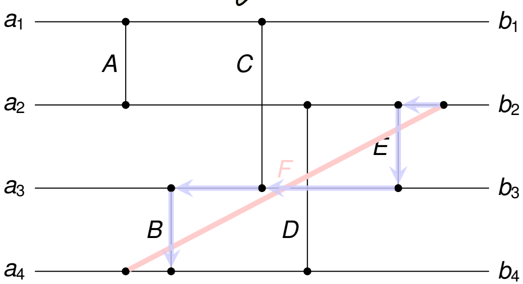

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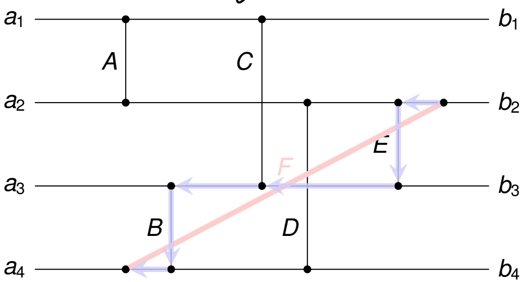

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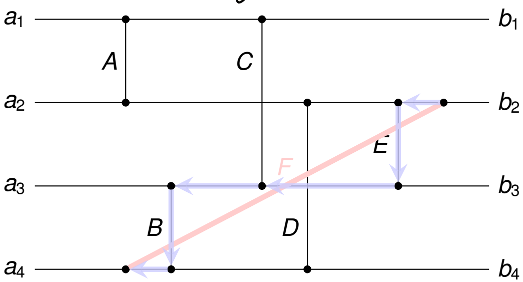
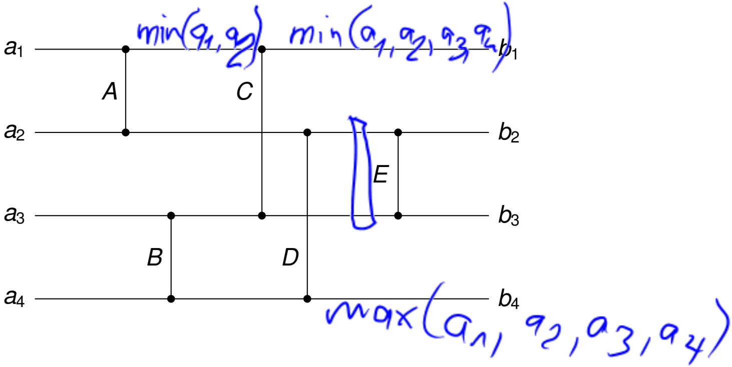


Tracing back a path must never cycle back on itself and go through the same comparator twice.

Example of a Comparison Network (Figure 27.2)

"Proof" that it is a Sorting Network:



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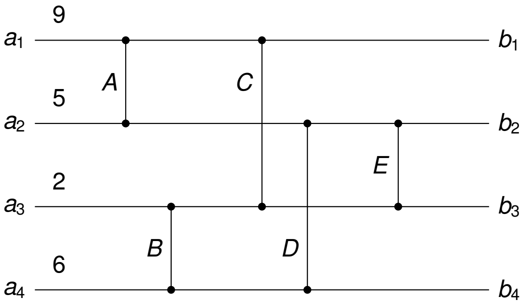

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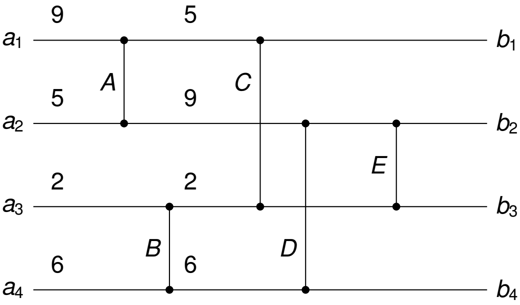

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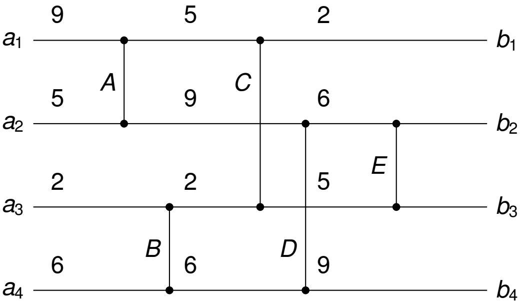

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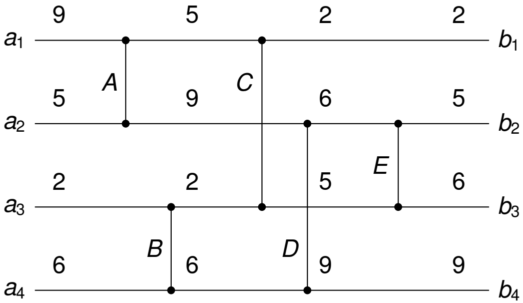

This network is in fact a sorting network!

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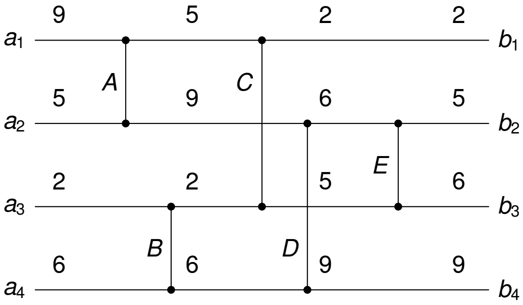

Depth of a wire:

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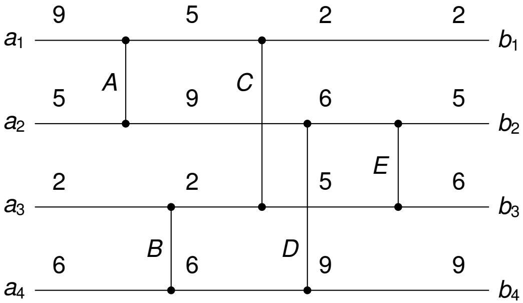

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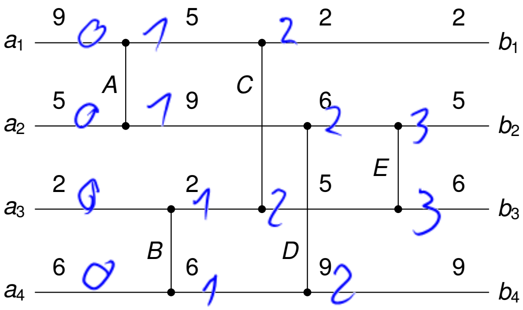

Depth of a wire:

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- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

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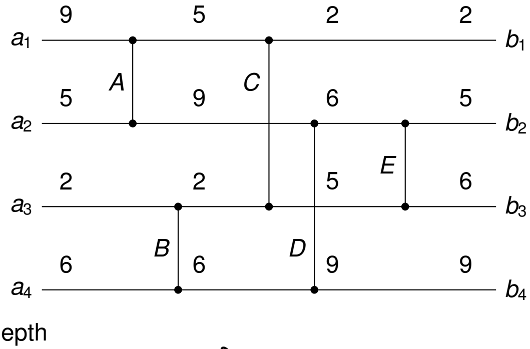

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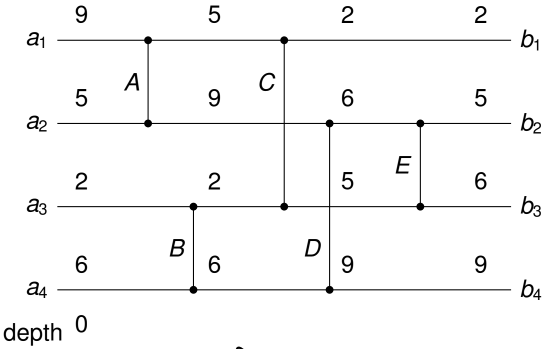

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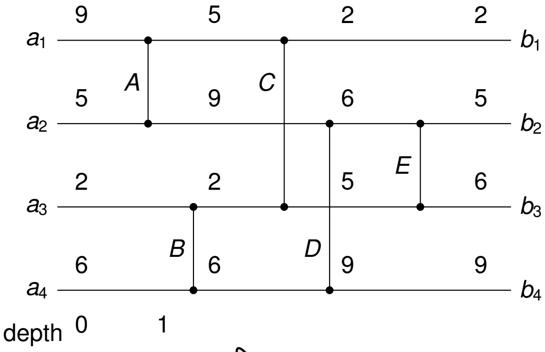

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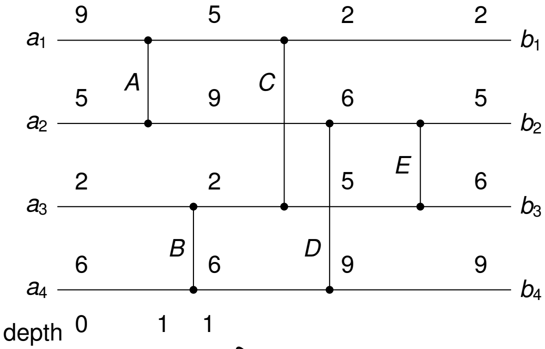

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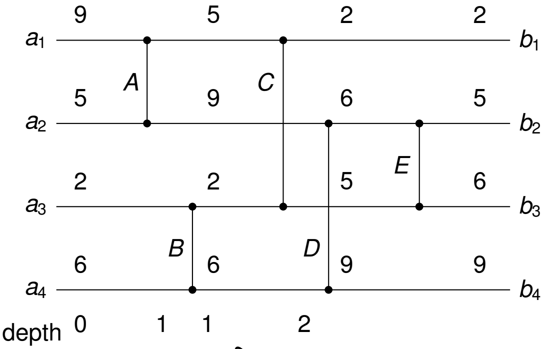

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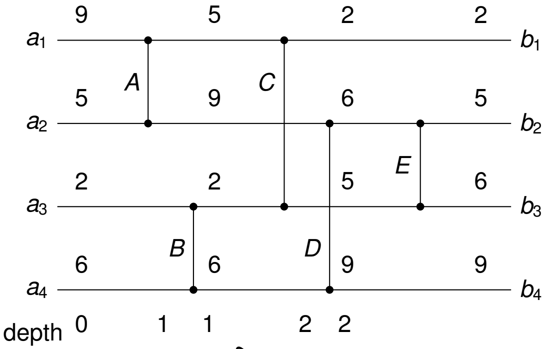

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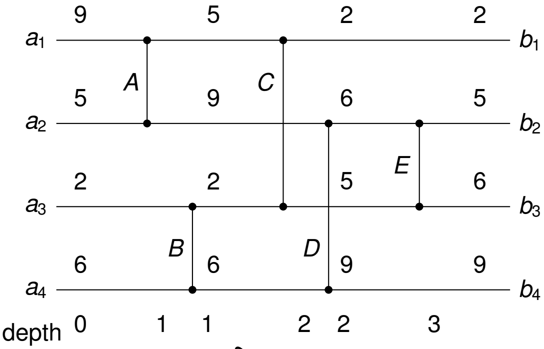
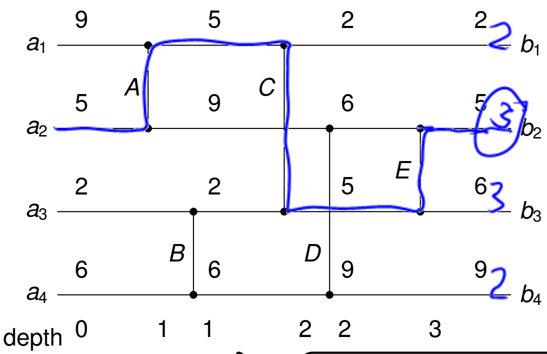


Depth of a wire:

- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

Example of a Comparison Network (Figure 27.2)



Maximum depth of an output wire equals total running time

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Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

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Lemma 27.1

If a comparison network transforms the input $a = \langle a_1, a_2, \ldots, a_n \rangle$ into the output $b = \langle b_1, b_2, \ldots, b_n \rangle$, then for any monotonically increasing function f, the network transforms $\underline{f(a)} = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle$ into $\underline{f(b)} = \langle f(b_1), f(b_2), \ldots, f(b_n) \rangle$.

$$\begin{tikzcd}
a \arrow[r, "c"] \arrow[d, "f"'] & b \arrow[d, "f"] \\
f(a) \arrow[r, "C"] & f(b)
\end{tikzcd}$$

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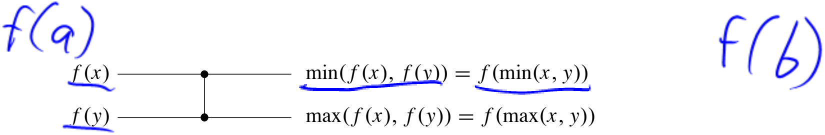
f(a)
f(b)
$f(x)$
$f(y)$
$\min(f(x), f(y)) = f(\min(x, y))$
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Figure 27.4 The operation of the comparator in the proof of Lemma 27.1. The function f is monotonically increasing.

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Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

Proof of the Zero-One Principle

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Proof:

$i < j, \ i > j$

- For the sake of contradiction, suppose the network does not correctly sort.
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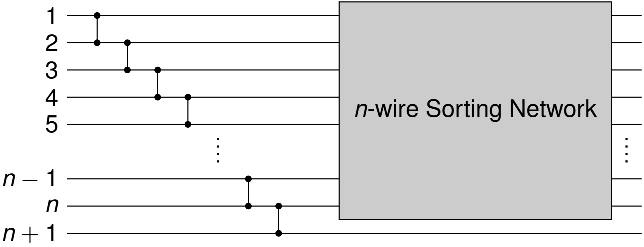
- Since the network places a_j before a_i, by the previous lemma $\Rightarrow f(a_j)$ is placed before $f(a_i)$
- But $f(a_j) = 1$ and $f(a_i) = 0$, which contradicts the assumption that the network sorts all sequences of 0's and 1's correctly □

Some Basic (Recursive) Sorting Networks



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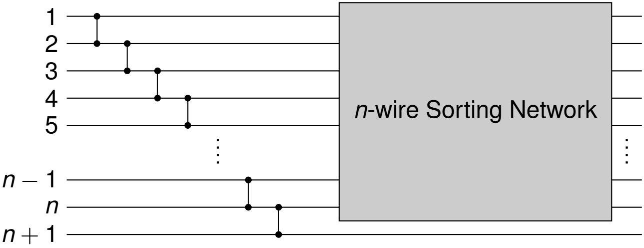
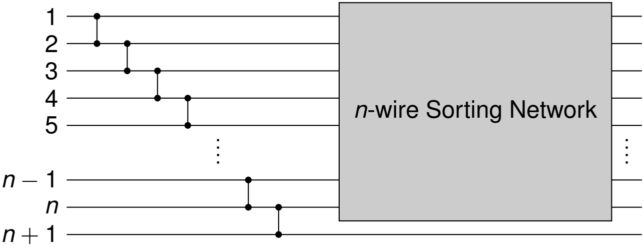


Bubble Sort

Some Basic (Recursive) Sorting Networks



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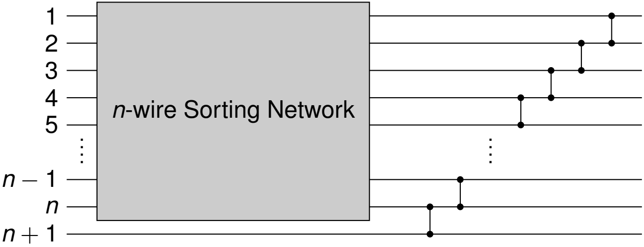

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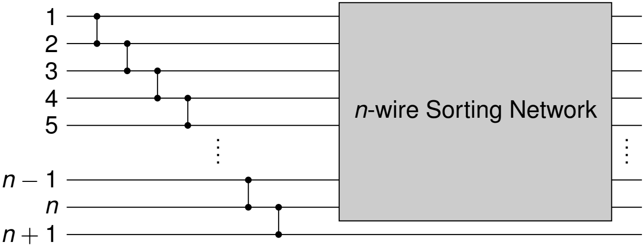

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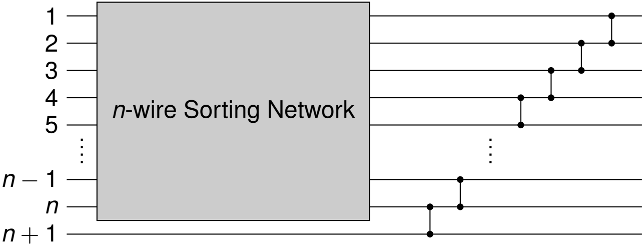
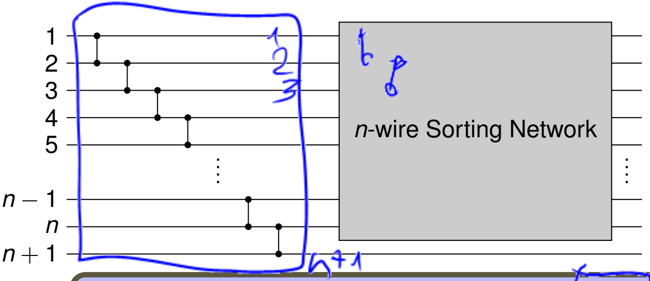


Insertion Sort

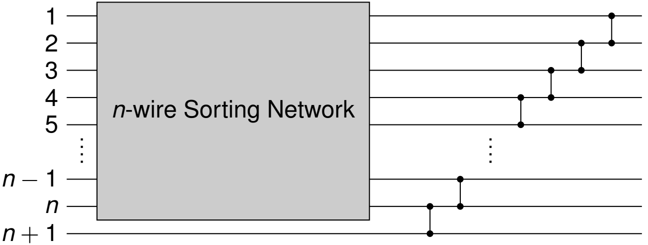
Some Basic (Recursive) Sorting Networks



Bubble Sort

$T(n) = n + T(n-1)$

These are Sorting Networks, but with depth $\Theta(n)$.



Insertion Sort

Outline

Outline of this Course

Introduction to Sorting Networks

Batcher’s Sorting Network

Counting Networks

Bitonic Sequences

Bitonic Sequence

A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

Sequences of one or two numbers are defined to be bitonic.

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Examples:

- $\langle 1, 4, 6, 8, 3, 2 \rangle$?

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- $\langle 6, 9, 4, 2, 3, 5 \rangle$?

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- $\langle 6, 9, 4, 2, 3, 5 \rangle$ ✓
- $\langle 9, 8, 3, 2, 4, 6 \rangle$ ✓
- $\langle \underbrace{4, 5, 7}_{}, \underbrace{1, 2, 6}_{} \rangle$?

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- binary sequences: ?

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Examples:

- $\langle 1, 4, 6, 8, 3, 2 \rangle$ ✓
- $\langle 6, 9, 4, 2, 3, 5 \rangle$ ✓
- $\langle 9, 8, 3, 2, 4, 6 \rangle$ ✓
- ~~$\langle 4, 5, 7, 1, 2, 6 \rangle$~~
- binary sequences: $0^i 1^j 0^k$, or, $1^i 0^j 1^k$, for $i, j, k \geq 0$.

Towards Bitonic Sorting Networks

Half-Cleaner

A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

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We always assume that n is even.

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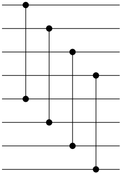

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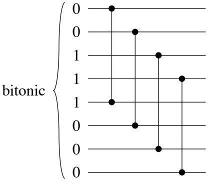

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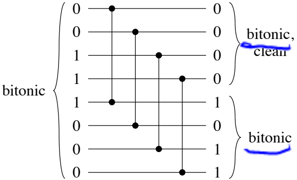

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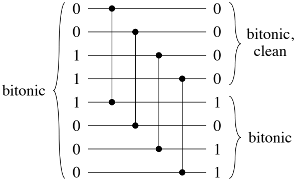

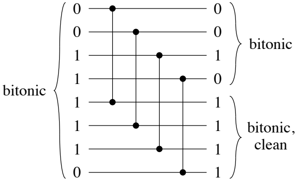

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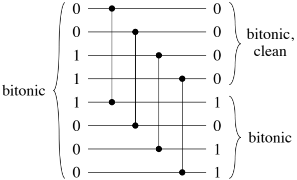
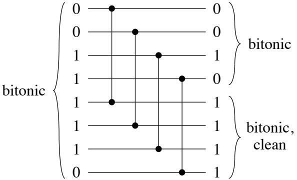
Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.





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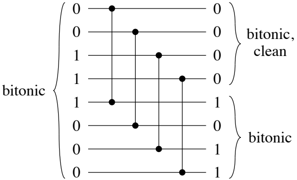

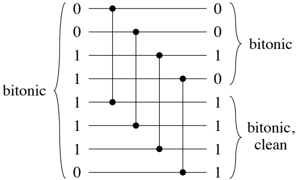

Proof of Lemma 27.3

W.l.o.g. assume that the input is of the form $0^i 1^j 0^k$, for some $i, j, k \geq 0$.

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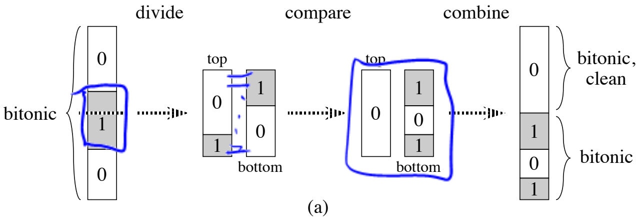

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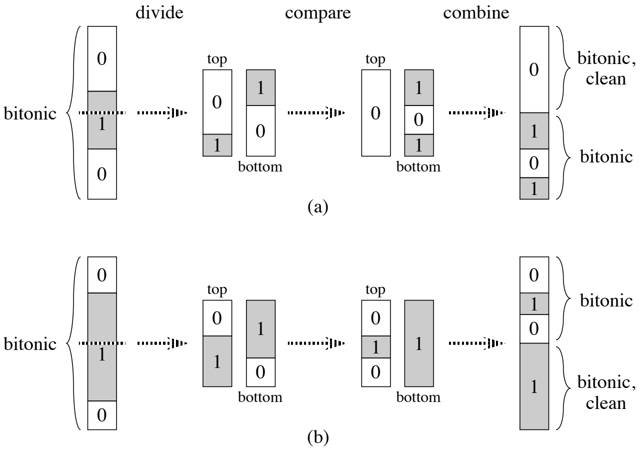

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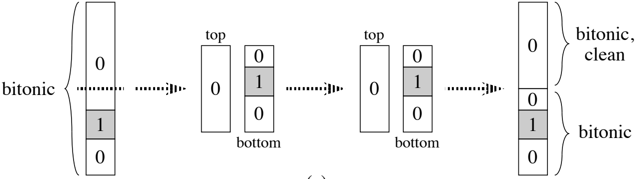
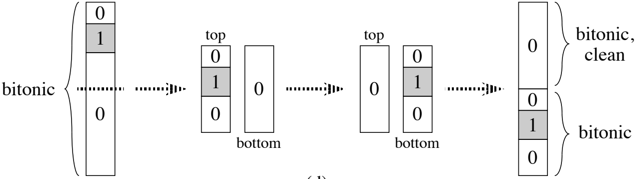


(c)



(d)

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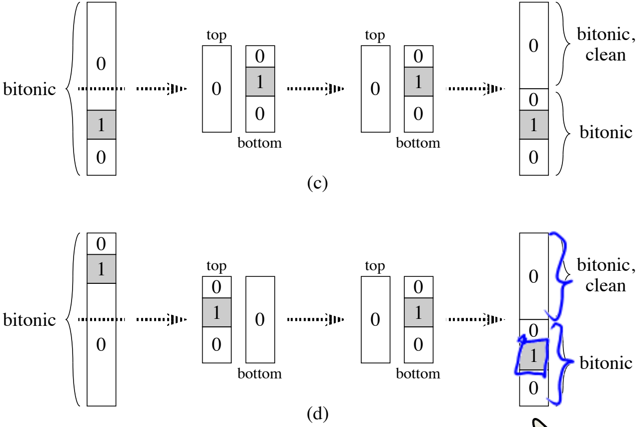

This suggests a recursive approach, since it now suffices to sort the top and bottom half separately.

The Bitonic Sorter

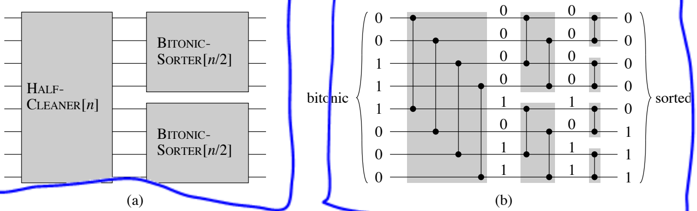

Figure 27.9 The comparison network BITONIC-SORTER[n], shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER[n] followed by two copies of BITONIC-SORTER[$n/2$] that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

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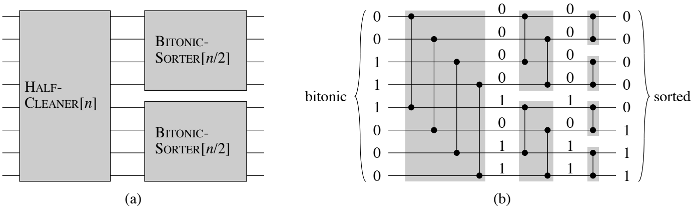

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Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

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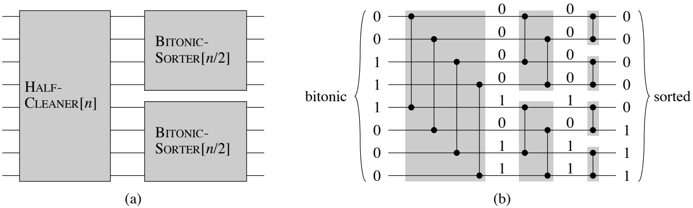

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Henceforth we will always assume that n is a power of 2.

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The Bitonic Sorter

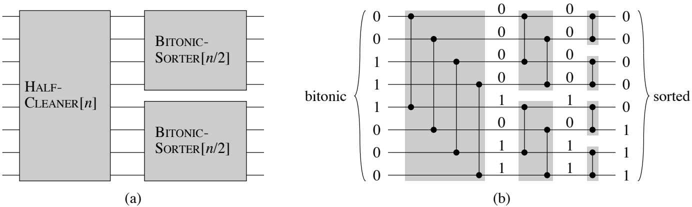

Figure 27.9 The comparison network BITONIC-SORTER[n], shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER[n] followed by two copies of BITONIC-SORTER[$n/2$] that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

Henceforth we will always assume that n is a power of 2.

Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

BITONIC-SORTER[n] has depth $\log n$ and sorts any zero-one bitonic sequence.

Merging Networks

Merging Networks

- can merge two sorted input sequences into one sorted output sequences
- will be based on a modification of BITONIC-SORTER[n]

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- consider two given sequences $X = 00000111$, $Y = 00001111$

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- consider two given sequences $X = 00000111$, $Y = 00001111$
- concatenating X with Y^R (the reversal of Y) $\Rightarrow$ 0000011111110000

This sequence is bitonic!

Hence in order to merge the sequences X and Y, it suffices to perform a bitonic sort on X concatenated with Y^R.

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$

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- Recall: first half-cleaner of BITONIC-SORTER[n] compares i and $n/2 + i$

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- Recall: first half-cleaner of Bitonic-Sorter[n] compares i and $n/2 + i$

$\Rightarrow$ First part of Merger[n] compares inputs $\underline{i}$ and $\underline{n - i}$ for $i = 1, 2, \ldots, n/2$

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2}\rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n\rangle$
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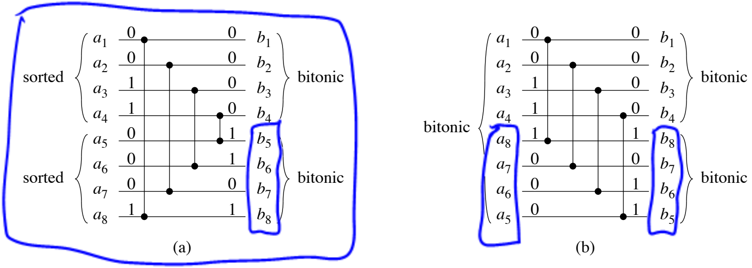

Figure 27.10 Comparing the first stage of MERGER[n] with HALF-CLEANER[n], for $n = 8$. **(a)** The first stage of MERGER[n] transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2}\rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n\rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n\rangle$. **(b)** The equivalent operation for HALF-CLEANER[n]. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1}\rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1}\rangle$.

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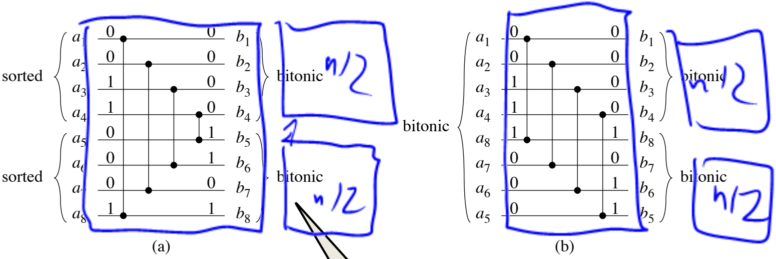
- Recall: first half-cleaner of BITONIC-SORTER$[n]$ compares i and $n/2 + i$

⇒ First part of MERGER$[n]$ compares inputs i and $n - i$ for $i = 1, 2, \ldots, n/2$



sorted
sorted
a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8
0 0 1 1 0 0 0 1
0 0 0 0 1 1 0 1
b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8
bitonic
bitonic
n/2
n/2
(a)
bitonic
a_1 a_2 a_3 a_4 a_8 a_7 a_6 a_5
0 0 1 1 1 0 0 0
0 0 0 0 1 0 1 1
b_1 b_2 b_3 b_4 b_8 b_7 b_6 b_5
bitonic
bitonic
n/2
n/2
(b)

Lemma 27.3 still applies, since the reversal of a bitonic sequence is bitonic.

Figure 27.10 Comparing the first stage of MERGER$[n]$ with HALF-CLEANER$[n]$, for $n = 8$. **(a)** The first stage of MERGER$[n]$ transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n \rangle$. **(b)** The equivalent operation for HALF-CLEANER$[n]$. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1} \rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1} \rangle$.

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$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i$ for $i = 1, 2, \ldots, n/2$

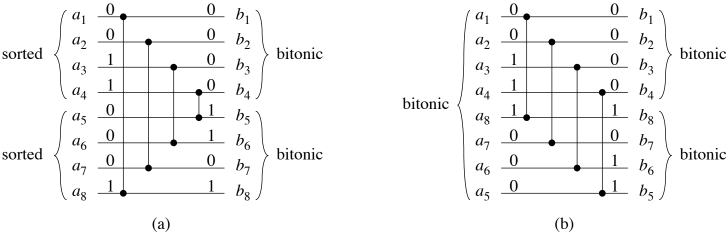

Figure 27.10 Comparing the first stage of MERGER[n] with HALF-CLEANER[n], for $n = 8$. **(a)** The first stage of MERGER[n] transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n \rangle$. **(b)** The equivalent operation for HALF-CLEANER[n]. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1} \rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1} \rangle$.

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- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$
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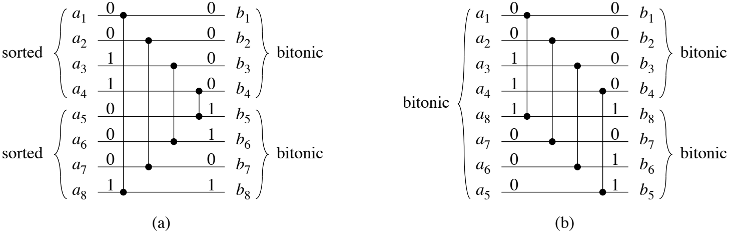
- Recall: first half-cleaner of BITONIC-SORTER[n] compares i and $n/2 + i$

$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i$ for $i = 1, 2, \ldots, n/2$

- Remaining part is identical to BITONIC-SORTER[n]



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Construction of a Merging Network (2/2)

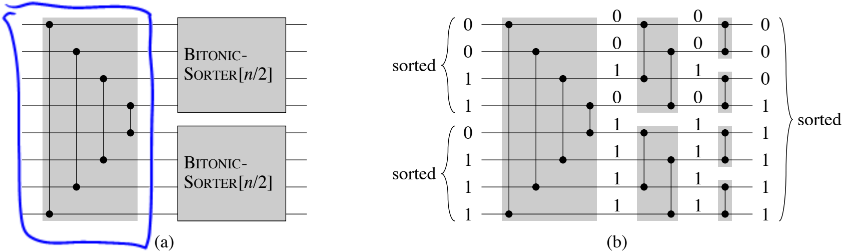


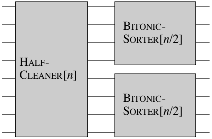
Figure 27.11 A network that merges two sorted input sequences into one sorted output sequence. The network MERGER[n] can be viewed as BITONIC-SORTER[n] with the first half-cleaner altered to compare inputs i and $n - i + 1$ for $i = 1, 2, \ldots, n/2$. Here, $n = 8$. **(a)** The network decomposed into the first stage followed by two parallel copies of BITONIC-SORTER[$n/2$]. **(b)** The same network with the recursion unrolled. Sample zero-one values are shown on the wires, and the stages are shaded.

Construction of a Sorting Network

Main Components

1. BITONIC-SORTER[n]
   - sorts any bitonic sequence
   - depth $\log n$

1.



Construction of a Sorting Network

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2. MERGER[n]
   - merges two sorted input sequences
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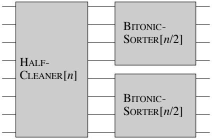

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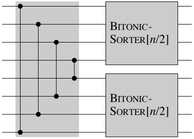

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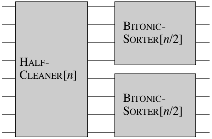

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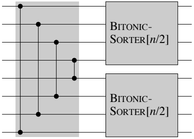
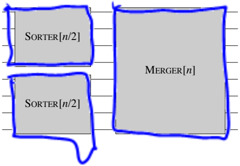


Batcher's Sorting Network

- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.



Construction of a Sorting Network

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   - depth $\log n$
2. MERGER[n]
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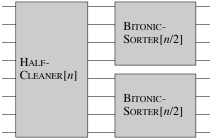
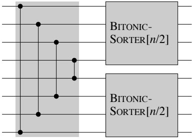
   - depth $\log n$

1\.



2\.



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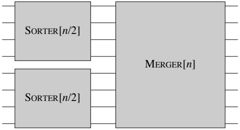
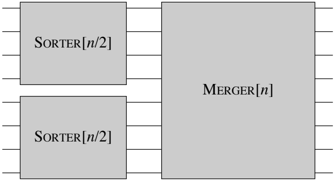
can be seen as a parallel version of merge sort



Unrolling the Recursion (Figure 27.12)



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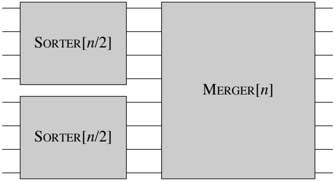

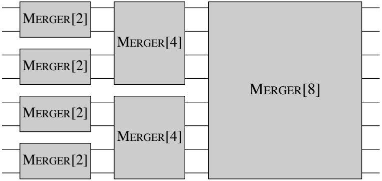

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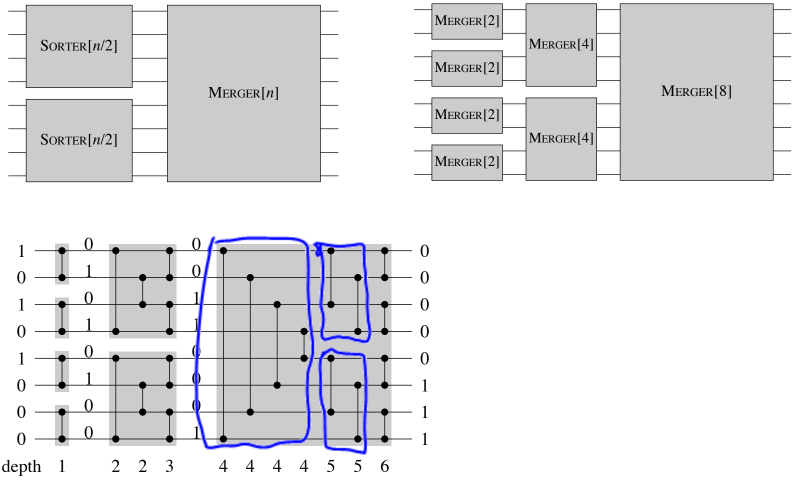

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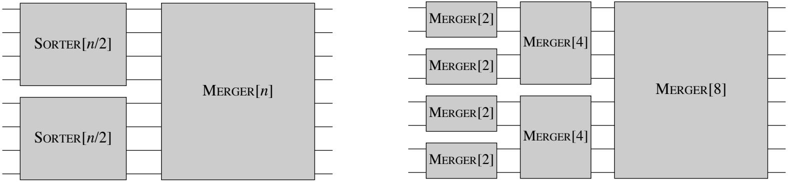
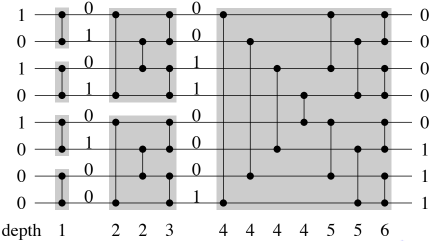




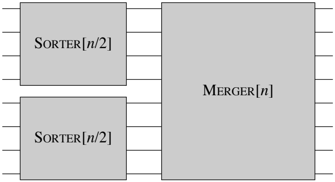
Recursion for $D(n)$:

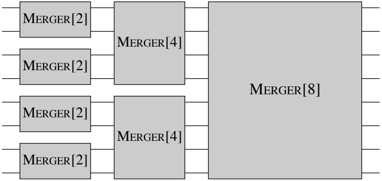
$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ \underline{D(n/2)} + \underline{\log n} & \text{if } n = 2^k. \end{cases}$$

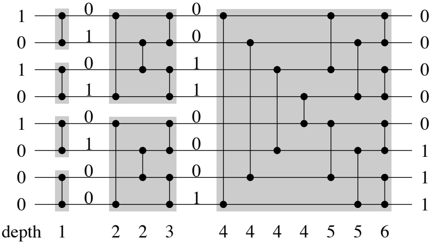
$$D(n) = \log n + \log\left(\frac{n}{2}\right) + \log\left(\frac{n}{4}\right) + \ldots + \log(2)$$

$$= \log n + (\log n - 1) + (\log n - 2) + \ldots + 1$$

Unrolling the Recursion (Figure 27.12)







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$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + \log n & \text{if } n = 2^k. \end{cases}$$

Solution: $D(n) = \Theta(\log^2 n)$.

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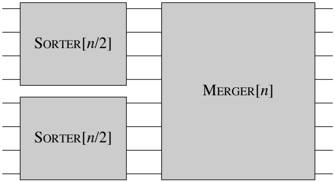

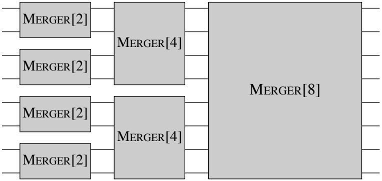

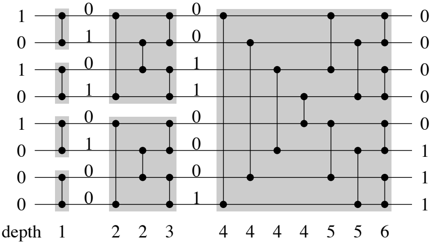

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Solution: $D(n) = \Theta(\log^2 n)$.

SORTER[n] has depth $\Theta(\log^2 n)$ and sorts any input.

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $\boxed{O(\log n)}$.

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

Quite elaborate construction, and involves huges constants.

A Glimpse at the AKS Network

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Perfect Halver

A perfect halver is a comparator network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\underline{\log_2 n}$ exist $\rightsquigarrow$ yields sorting networks of depth $\boxed{\Theta((\log n)^2)}$.

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Approximate Halver

An (n, ϵ)-approximate halver, $\underline{\epsilon < 1}$, is a comparator network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

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We will prove that such networks can be constructed in constant depth!

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

Quite elaborate construction, and involves huges constants.

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Perfect Halver

A perfect halver is a comparator network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\log_2 n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

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Approximate Halver

$\varepsilon = \frac{1}{100}$

An $\underline{(n, \epsilon)}$-approximate halver, $\epsilon < 1$, is a comparator network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

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Expander Graphs

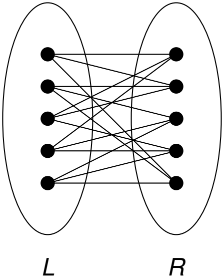
Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is the union of d matchings (perfect)
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| \geq \min\{\mu \cdot |S|, n/2 - |S|\}$$

n=10



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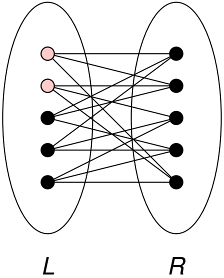

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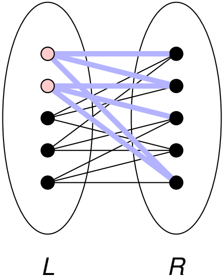

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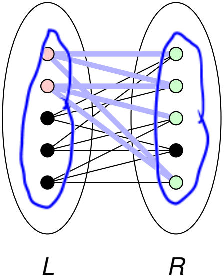

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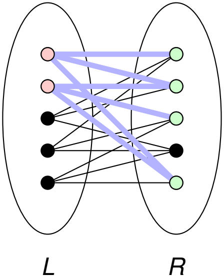
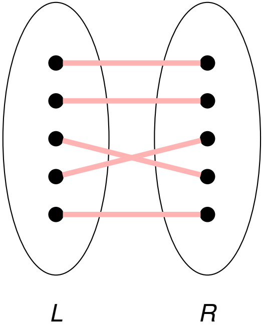


Expander Graphs:

- **probabilistic construction** "easy": take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers



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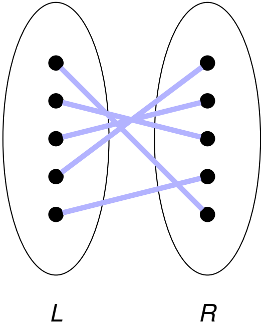

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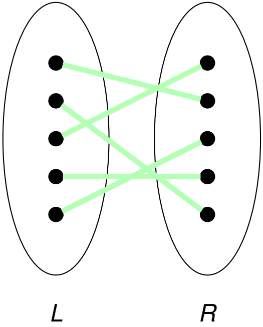

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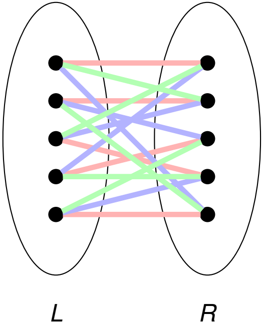

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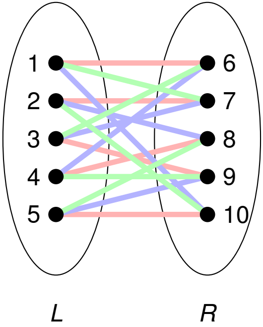

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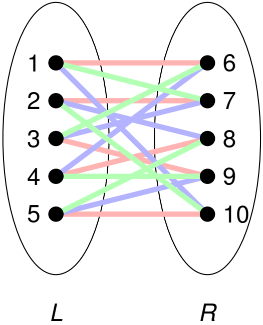

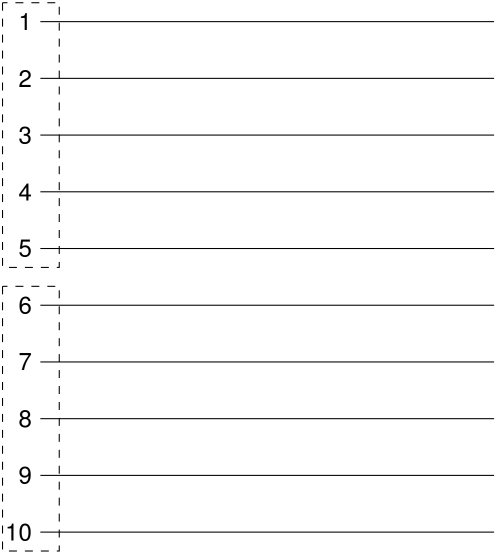

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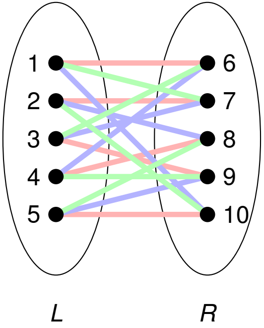

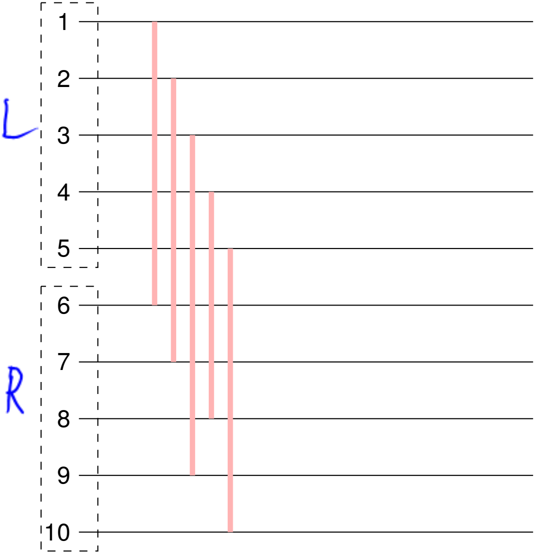

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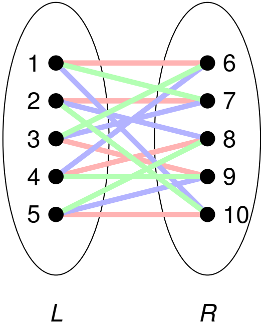

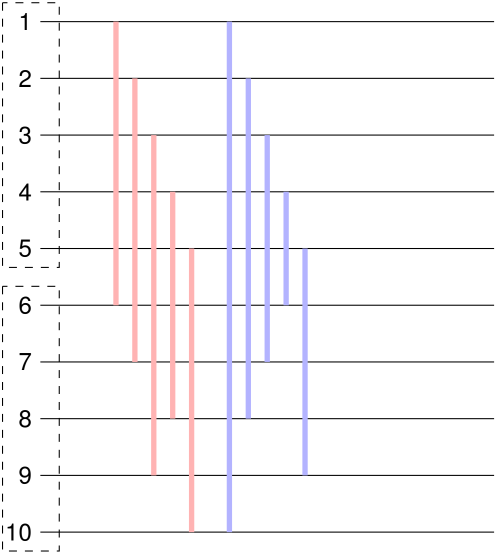

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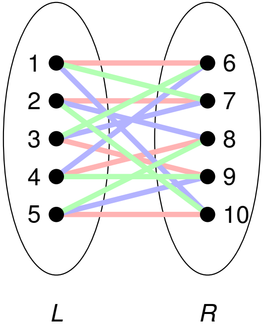
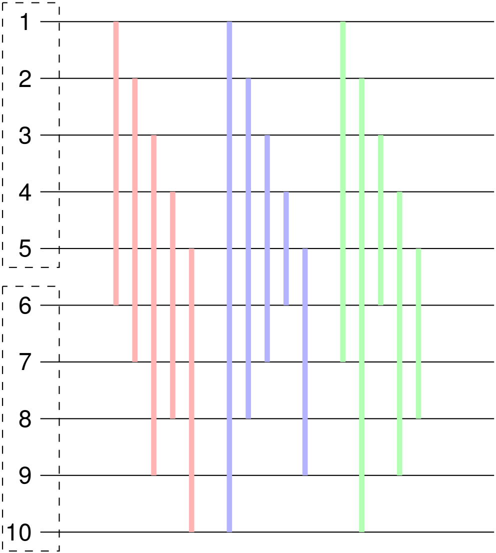
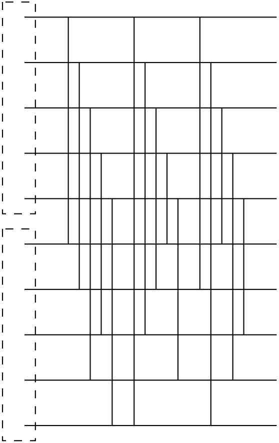




Existence of Approximate Halvers

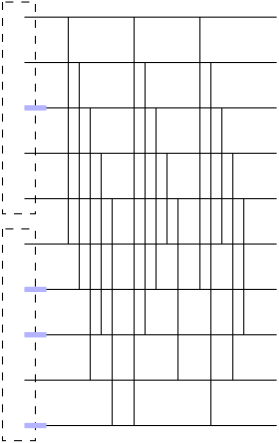
Proof:



Existence of Approximate Halvers

Proof:

- $X :=$ wires with the k smallest inputs



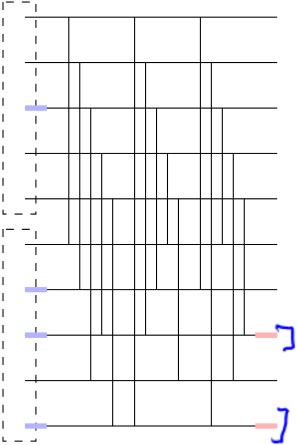
Existence of Approximate Halvers

Proof Strategy:

1. $|N(Y)| \gg |Y|$
2. $|N(Y)| \leq |X| = k$

Proof: keys

- X := wires with the k smallest inputs
- Y := wires in lower half with k smallest outputs



Existence of Approximate Halvers

Proof:

- X := wires with the k smallest inputs
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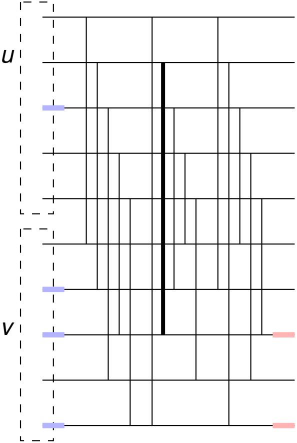
- For every $u \in N(Y)$: $\exists$ comparator (u, v), $v \in Y$



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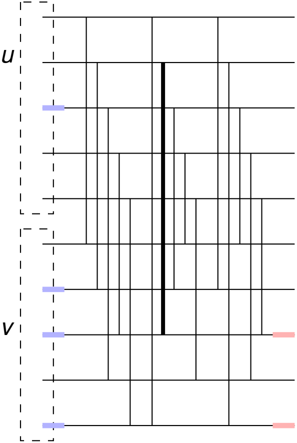

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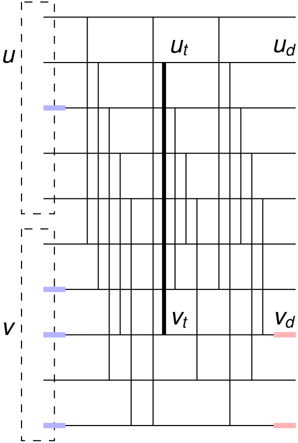

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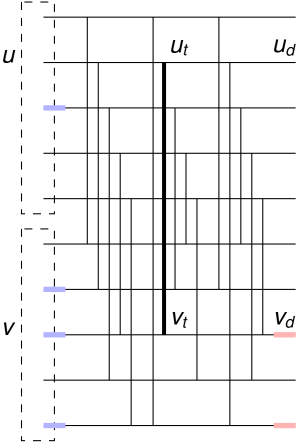

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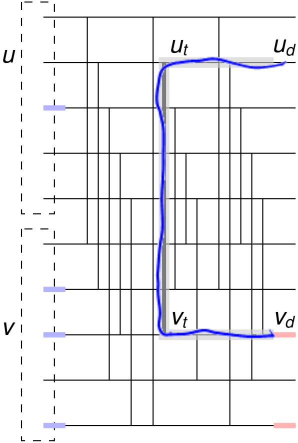

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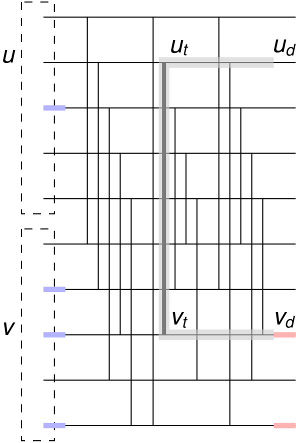
$$\boxed{|Y| + |N(Y)| \leq k.}$$



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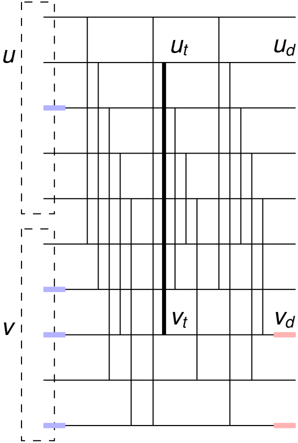

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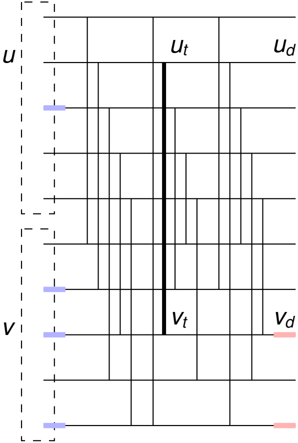

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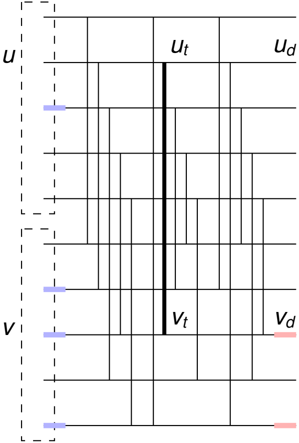

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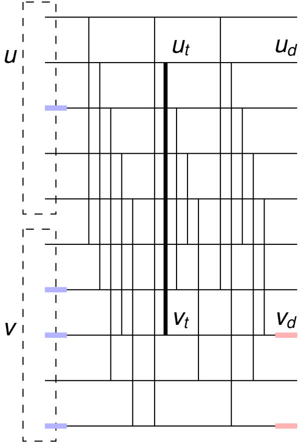

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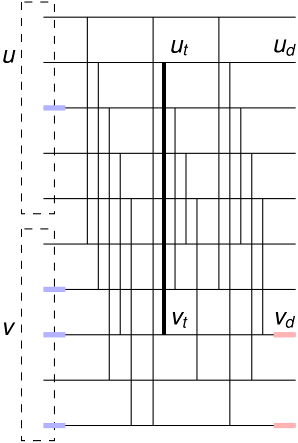

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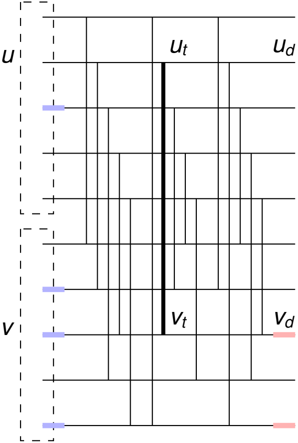
Here we used that $\boxed{k \leq n/2}$



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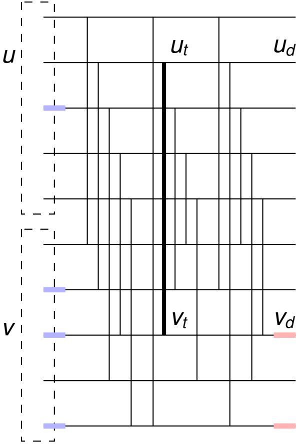
- The same argument shows that at most $\epsilon \cdot k$, $\epsilon := 1/(\mu+1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. □



AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

> *"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"*

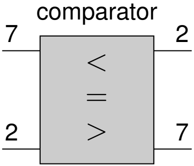
Richard J. Lipton (Georgia Tech)

> *"The AKS sorting network is* ***galactic****: it needs that n be larger than* 2^{78} *or so to finally be smaller than Batcher's network for n items."*

Siblings of Sorting Network

Sorting Networks

- sorts any input of size *n*
- special case of Comparison Networks



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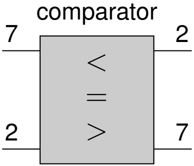
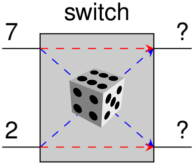


Switching (Shuffling) Networks

- creates a random permutation of *n* items
- special case of Permutation Networks



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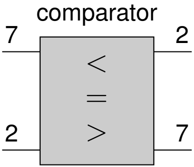
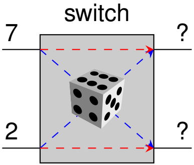
- sorts any input of size *n*
- special case of Comparison Networks



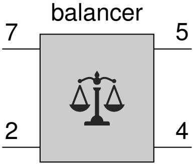
Switching (Shuffling) Networks

- creates a random permutation of *n* items
- special case of Permutation Networks



Counting Networks

- balances any stream of tokens over *n* wires
- special case of Balancing Networks



Outline

Outline of this Course

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

Values could represent addresses in memories or destinations on an interconnection network

Counting Network

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Balancing Networks

- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,. . .)

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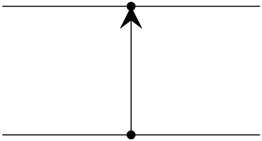

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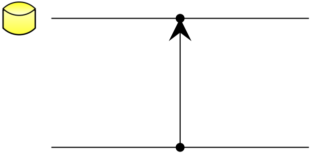

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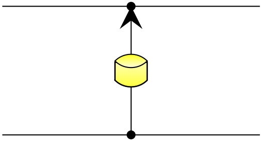

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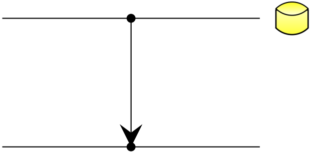

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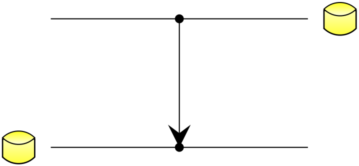

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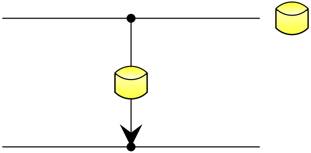

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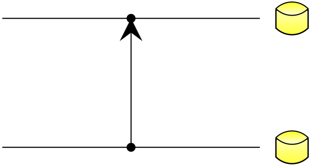

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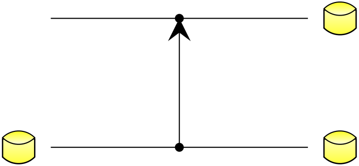

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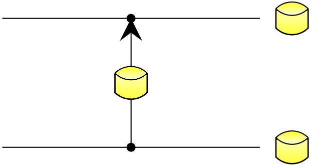

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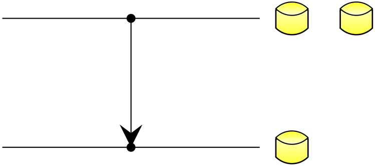

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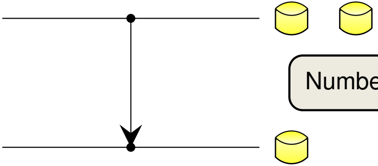

Number of tokens differs by at most one

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires

Bitonic Counting Network

Counting Network (Formal Definition)

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3. In a quiescent state: $\underline{\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i}$
4. A counting network is a balancing network with the **step-property:**

$$\boxed{0 \leq y_i - y_j \leq 1 \text{ for any } i < j.}$$

$(4,4,4,3,3,3,3,3)$

Bitonic Counting Network

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$$0 \leq y_i - y_j \leq 1 \text{ for any } i < j.$$

Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\underline{\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i}$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\underline{\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1}$, then $\underline{\exists!\, j = 1, 2, \ldots, n}$ with $\underline{x_j = y_j + 1}$ and $x_i = y_i$ for $j \neq i$.

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Key Lemma

Consider a MERGER[n]. Then if the inputs $x_1, \ldots, x_{n/2}$ and $x_{n/2+1}, \ldots, x_n$ have the step property, then so does the output $y_1, \ldots, y_n$.

Proof (by induction on n) → power of 2

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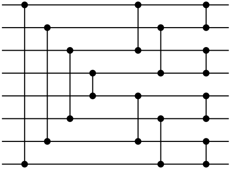

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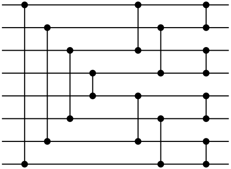

Proof (by induction on n)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer

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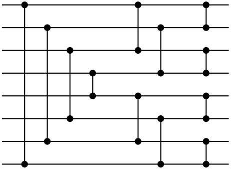

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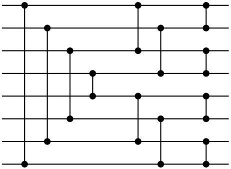

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- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER$[n/2]$ subnetworks

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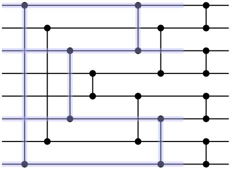

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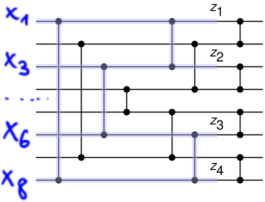

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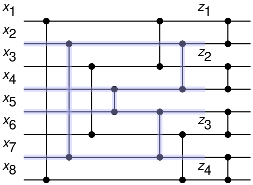

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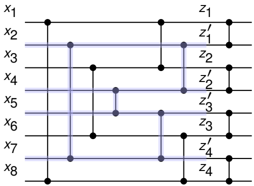

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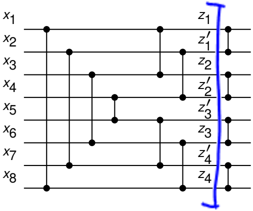

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- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property

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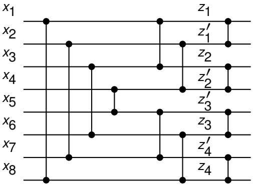

Proof (by induction on n)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \lceil \frac{1}{2} \sum_{i=1}^{n} x_i \rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
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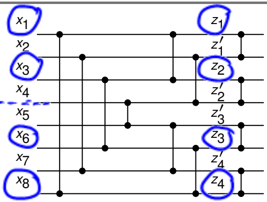
$$Z = \left\lceil \tfrac{1}{2} \cdot (x_1 + x_2 + x_3 + x_4) \right\rceil + \left\lfloor \tfrac{1}{2} \cdot (x_5 + x_6 + x_7 + x_8) \right\rfloor$$



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- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$
- F1 $\Rightarrow$ $Z = \lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \rceil + \lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rfloor$ and $Z' = \lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \rfloor + \lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rceil$

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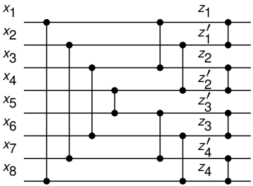

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Correctness of the Bitonic Counting Network

Facts

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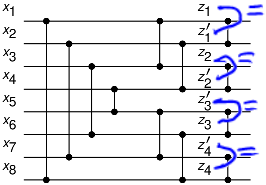

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- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓

Correctness of the Bitonic Counting Network

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

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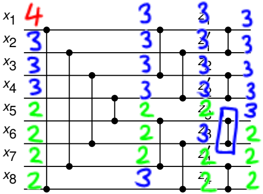

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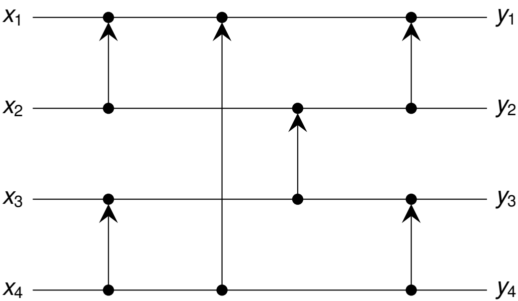
- F1 $\Rightarrow$ $Z = \lceil \frac{1}{2} \sum_{i=1}^{n/2} x_i \rceil + \lfloor \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rfloor$ and $Z' = \lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \rfloor + \lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rceil$
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$.
  Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action



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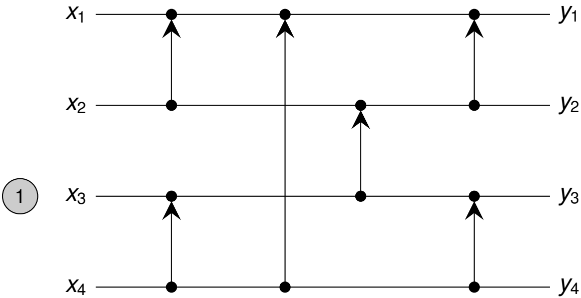

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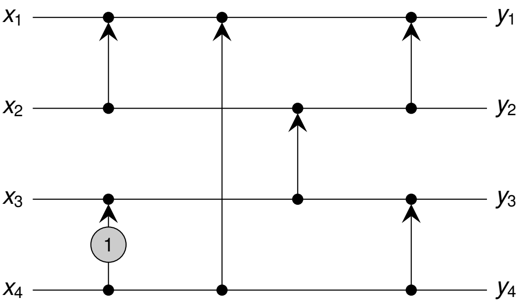

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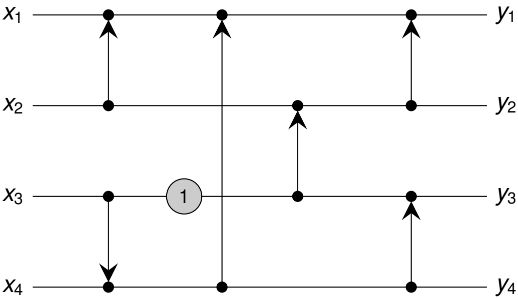

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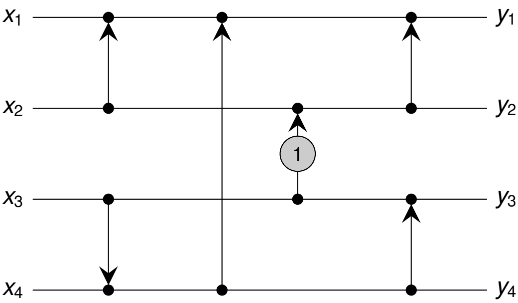

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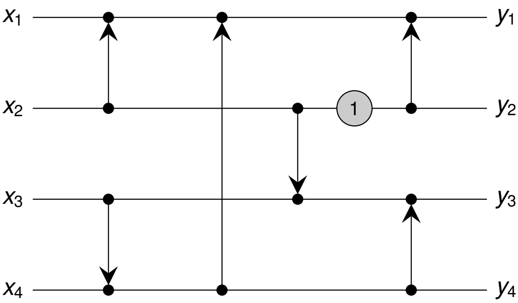

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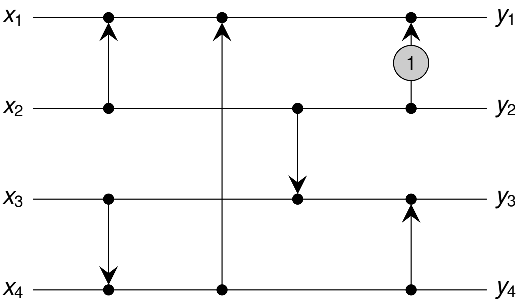

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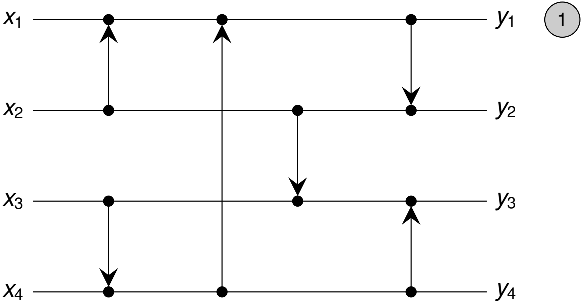

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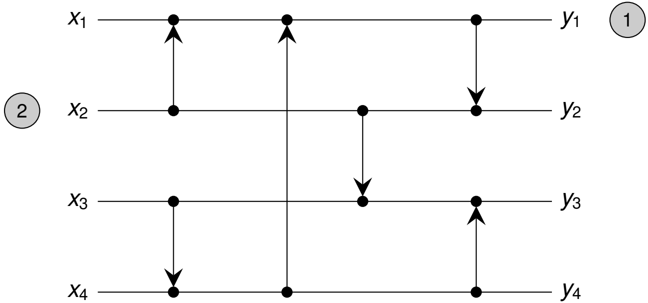

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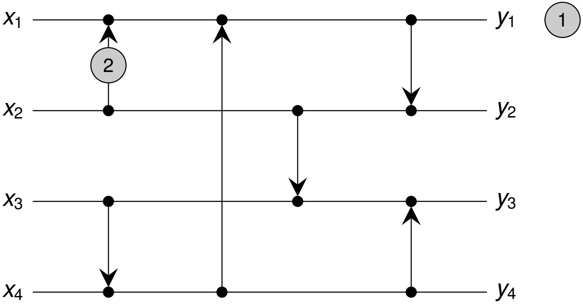

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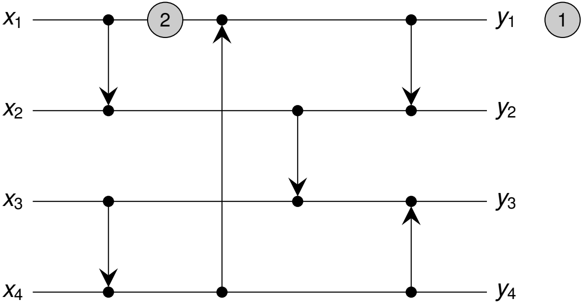

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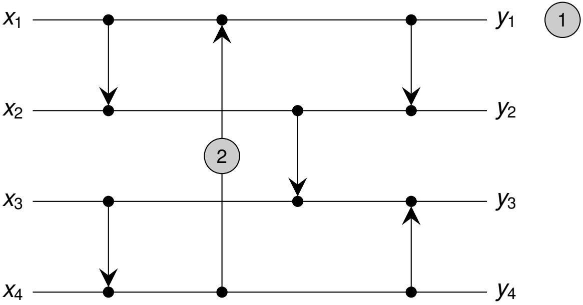

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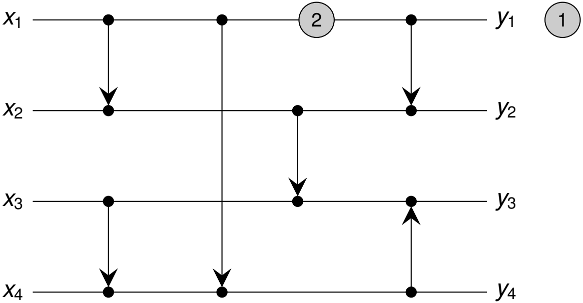

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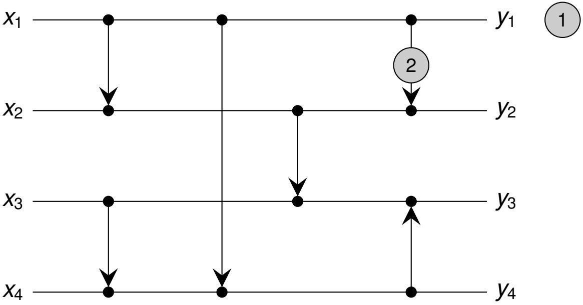

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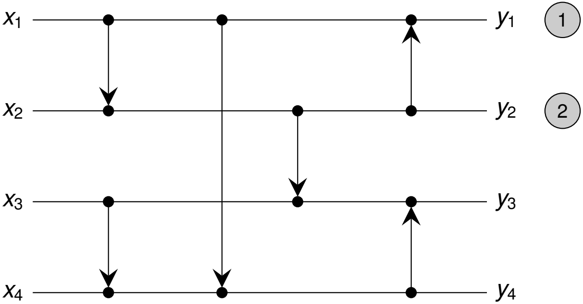

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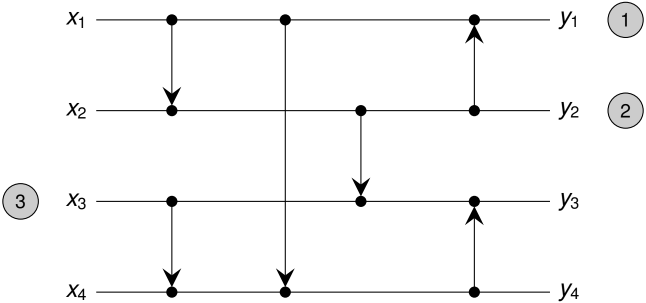

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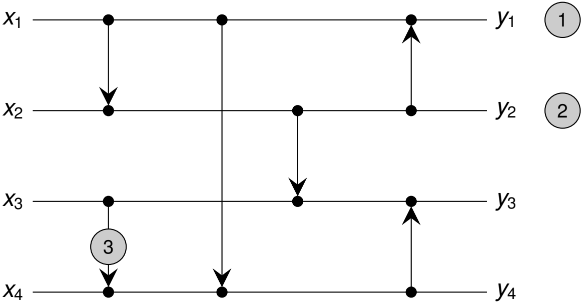

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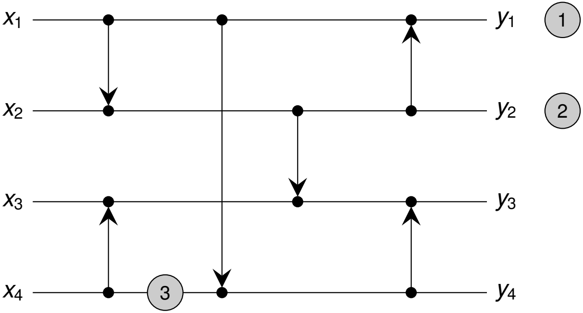

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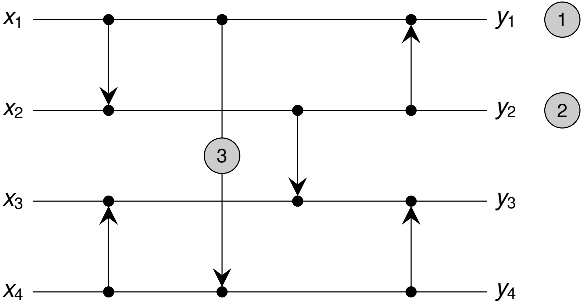

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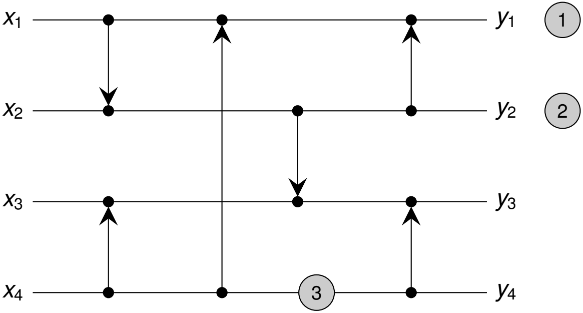

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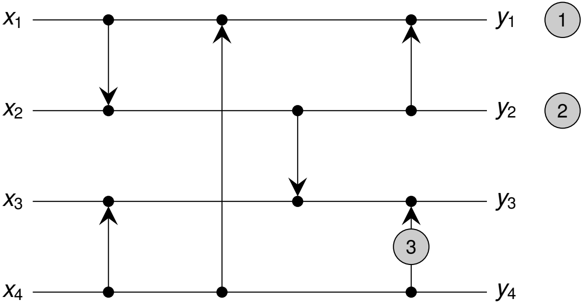

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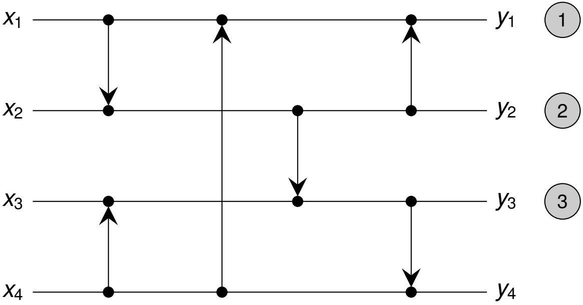

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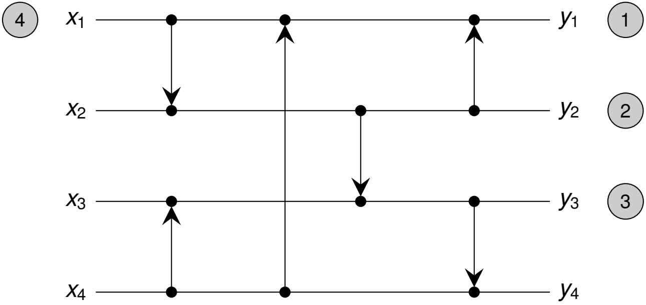

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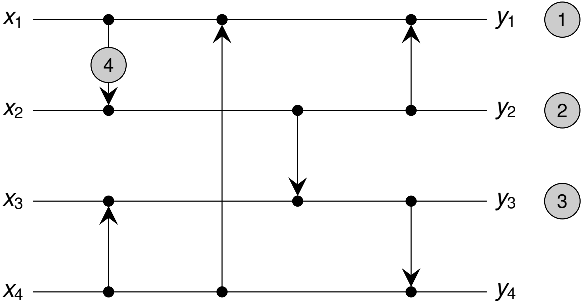

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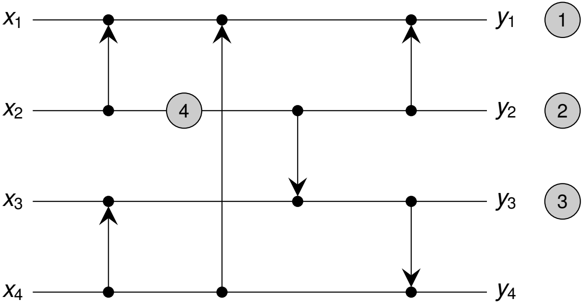

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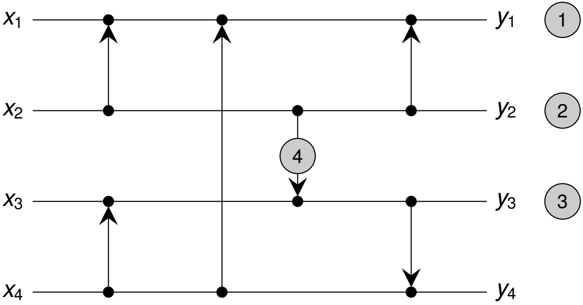

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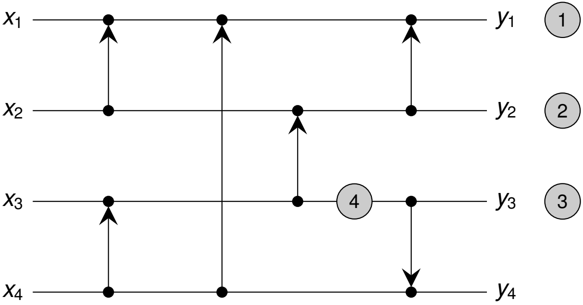

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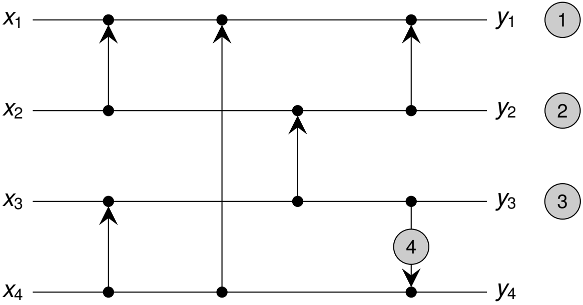

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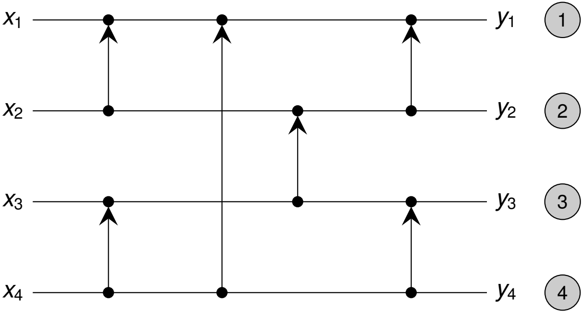

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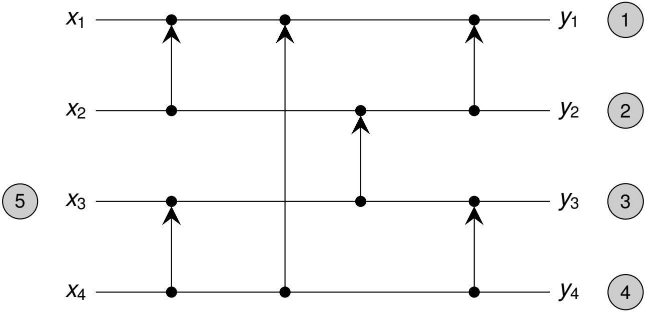

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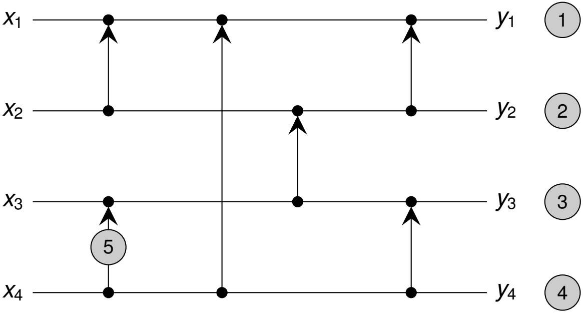

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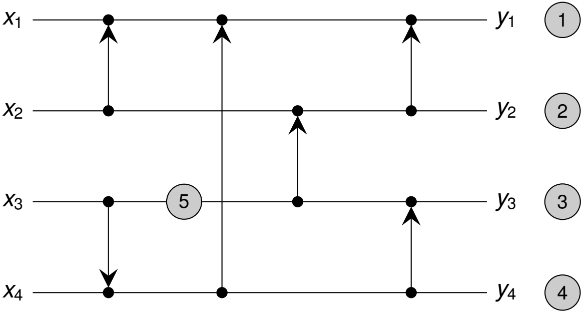

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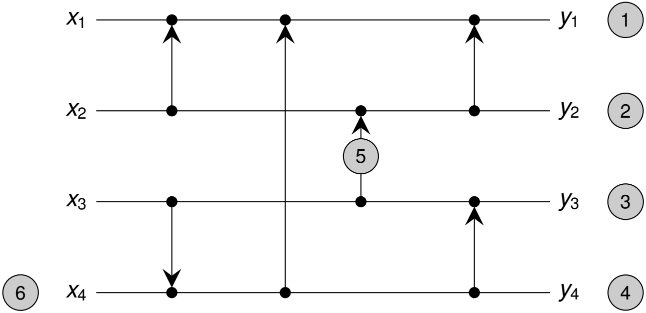

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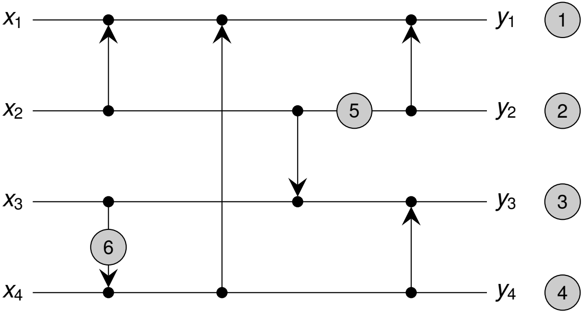

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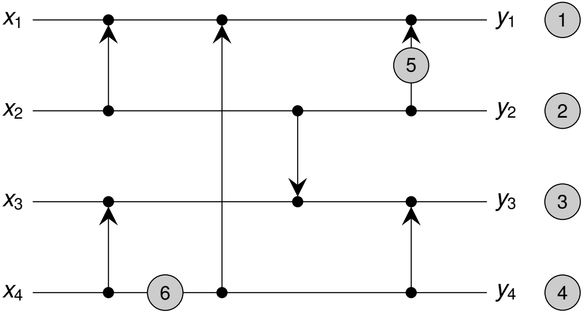

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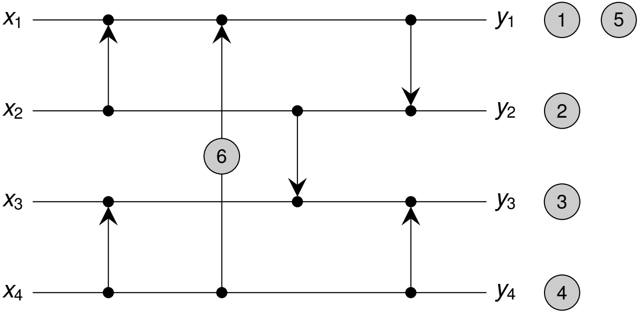

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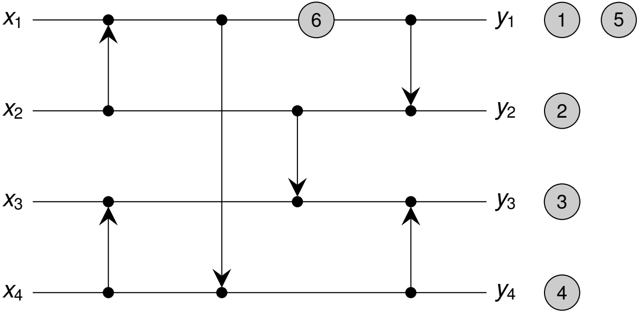

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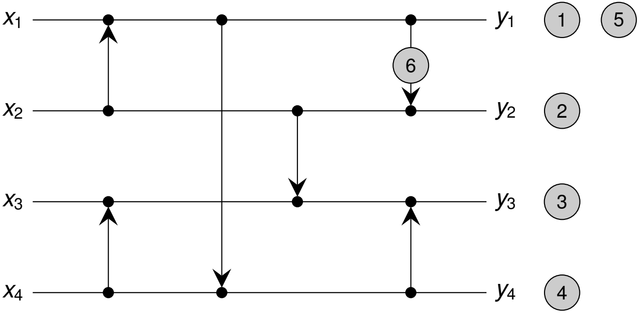

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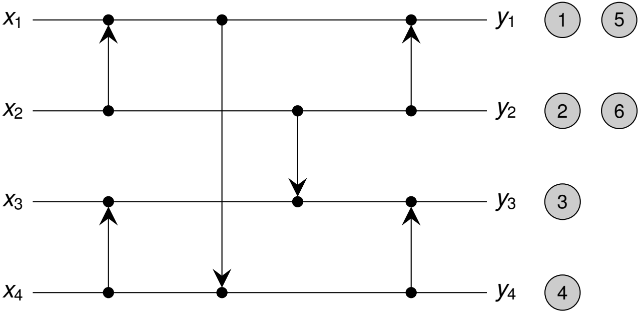
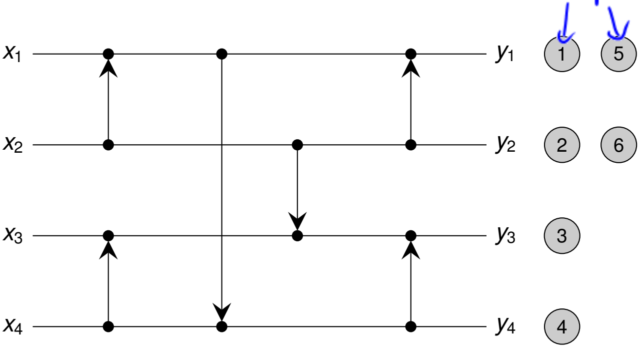


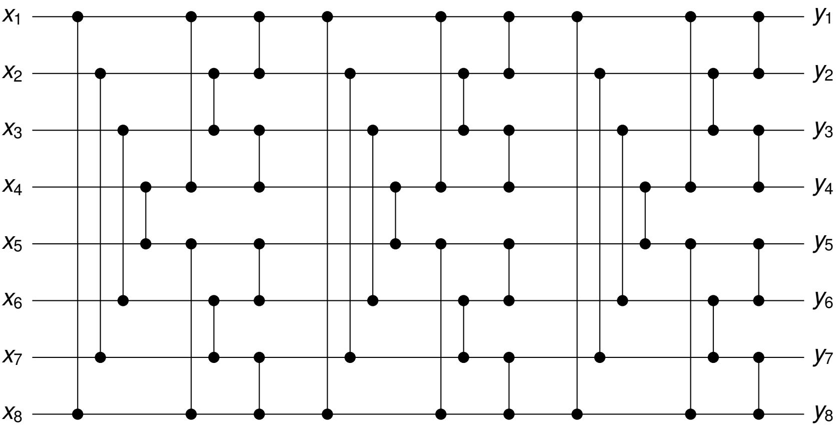
Bitonic Counting Network in Action

values are assigned at the output only



Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i + n, i + 2 \cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



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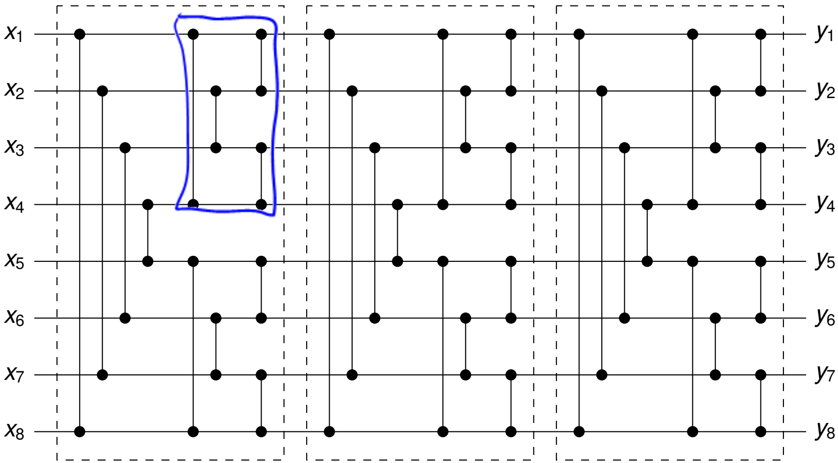

Consists of log n BLOCK[n] networks each of which has depth log n

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

From Counting to Sorting

The converse is not true!

Counting vs. Sorting

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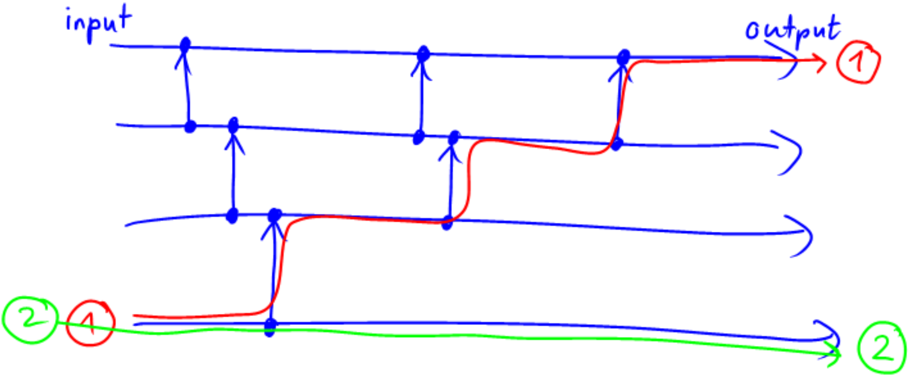

Bubble-Sort-Network [4]

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Proof.

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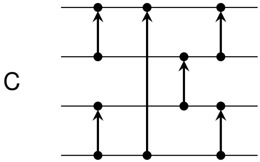

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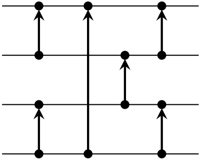

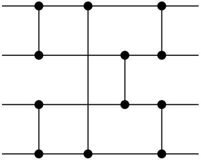

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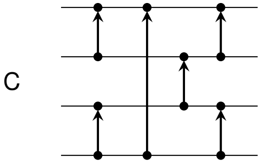

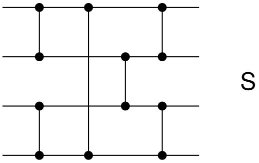

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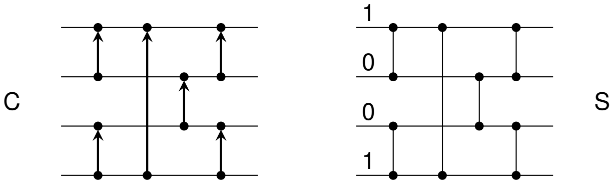

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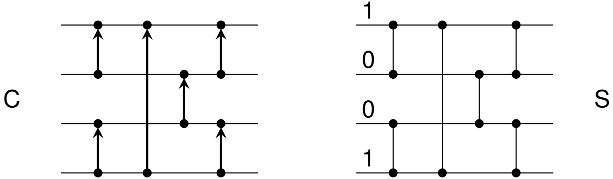

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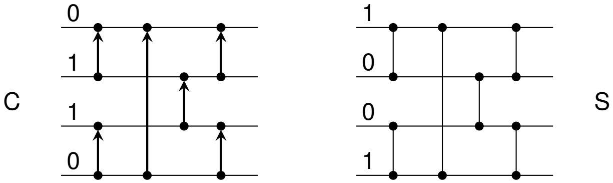

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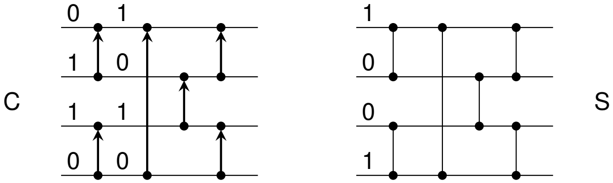

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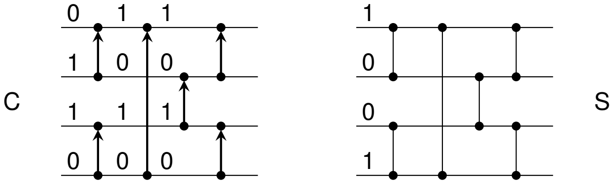

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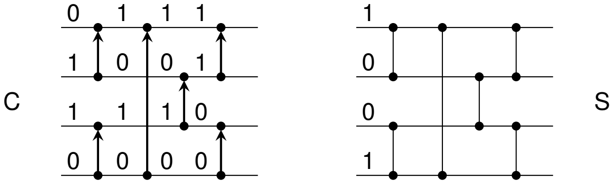

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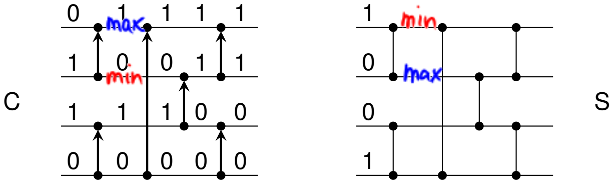

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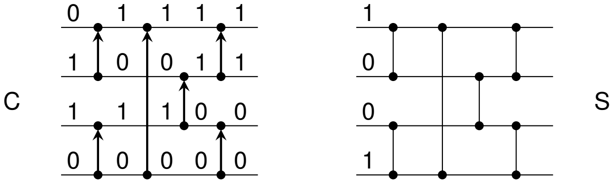

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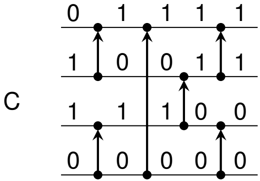

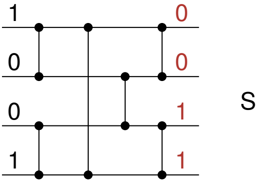

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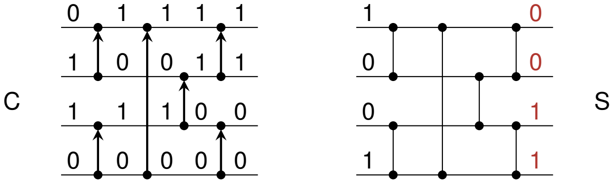
- By the Zero-One Principle, S is a sorting network. □



Outline

Introduction to Sorting Networks

Batcher's Sorting Network

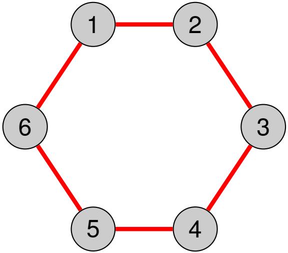
Counting Networks

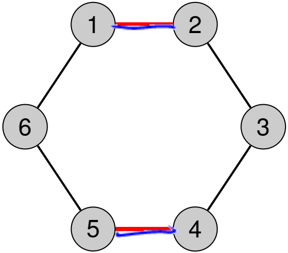
Load Balancing on Graphs

Introduction to Matrix Multiplication

Serial Matrix Multiplication

Communication Models: Diffusion vs. Matching

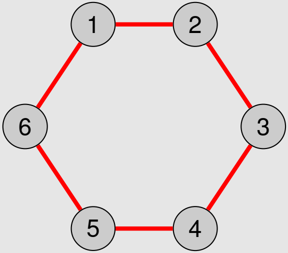




$$\underline{\mathbf{M}} = \begin{pmatrix} \frac{1}{3} & \underline{\frac{1}{3}} & 0 & 0 & 0 & \underline{\frac{1}{3}} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{M}^{(t)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Communication Models: Diffusion vs. Matching



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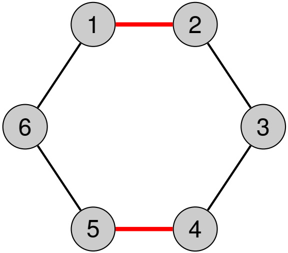

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Smoothness of the Load Distribution

- let $\underline{x} \in \mathbb{R}^n$ be a load vector
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Metrics

- ℓ_2-norm: $\Phi^t = \sqrt{\sum_{i=1}^n (x_i^t - \overline{x})^2}$
- makespan: $\max_{i=1}^n x_i^t$
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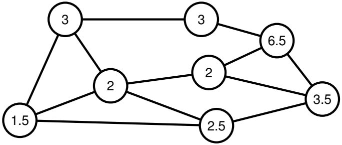

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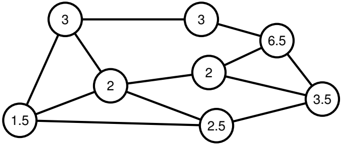

For this example:

- $\Phi^t = \sqrt{0^2 + 0^2 + 3.5^2 + 0.5^2 + 1^2 + 1^2 + 1.5^2 + 0.5^2} = \sqrt{17}$
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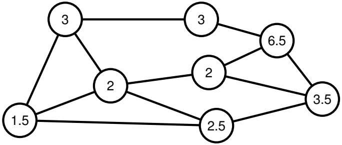

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Diffusion Matrix

Diffusion Matrix

Given an undirected, connected graph $G = (V, E)$ and a diffusion parameter $\alpha > 0$, the diffusion matrix M is defined as follows:

$$M_{ij} = \begin{cases} \alpha & \text{if } (i,j) \in E, \\ 1 - \alpha \deg(i) & \text{if } i = j, \\ \underline{0} & \text{otherwise.} \end{cases}$$

Diffusion Matrix

How to choose α for a d-regular graph?

- $\alpha = \frac{1}{d}$ may lead to oscillation (if graph is bipartite)
- $\alpha = \frac{1}{d+1}$ ensures convergence
- $\alpha = \frac{1}{2d}$ ensures convergence (and all eigenvalues of M are non-negative)

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This can be also seen as a random walk on G!

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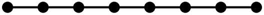
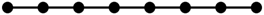
1D grid



$$\gamma(M) \approx 1 - \frac{1}{n^2}$$

1D grid



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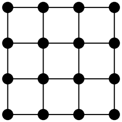

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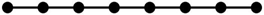

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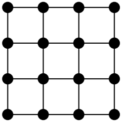

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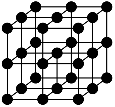
3D grid



$$\gamma(M) \approx 1 - \frac{1}{n^{2/3}}$$

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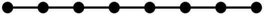

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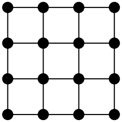
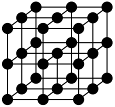


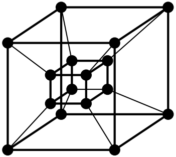
$\gamma(M) \approx 1 - \frac{1}{n}$

3D grid



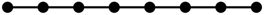
$\gamma(M) \approx 1 - \frac{1}{n^{2/3}}$

Hypercube



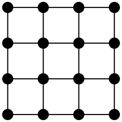
$\gamma(M) \approx 1 - \frac{1}{\log n}$

1D grid



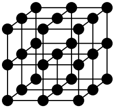
$\gamma(M) \approx 1 - \frac{1}{n^2}$

2D grid



$\gamma(M) \approx 1 - \frac{1}{n}$

3D grid



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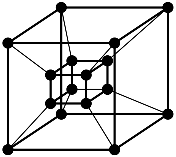
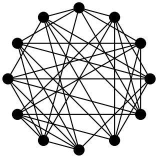
Hypercube



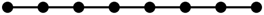
$\gamma(M) \approx 1 - \frac{1}{\log n}$

Random Graph



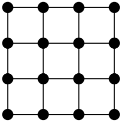
$\underline{\gamma(M) < 1}$

1D grid



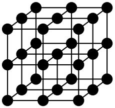
$\gamma(M) \approx 1 - \frac{1}{n^2}$

2D grid



$\gamma(M) \approx 1 - \frac{1}{n}$

3D grid



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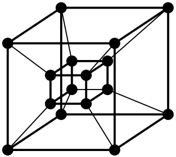
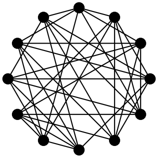
Hypercube



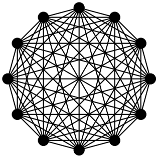
$\gamma(M) \approx 1 - \frac{1}{\log n}$

Random Graph



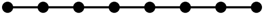
$\gamma(M) < 1$

Complete Graph



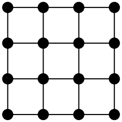
$\gamma(M) \approx 0$

1D grid



$\gamma(M) \approx 1 - \frac{1}{n^2}$

2D grid



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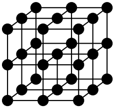

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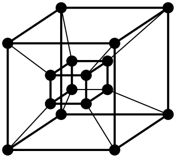

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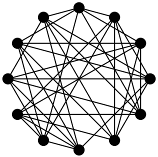
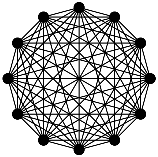
Random Graph



$\gamma(M) < 1$

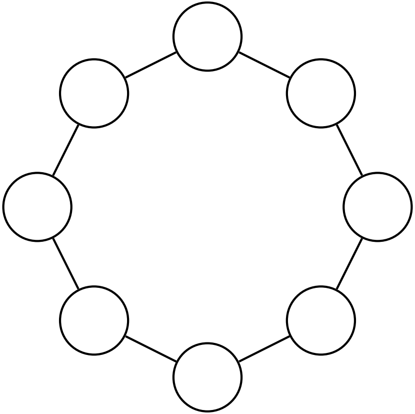
Complete Graph



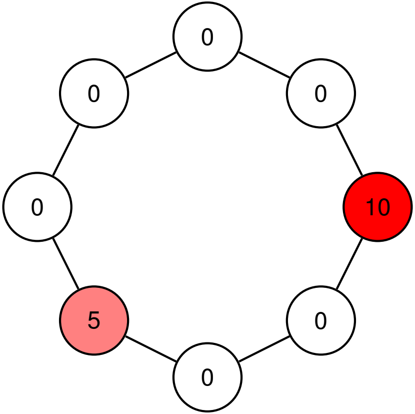
$\gamma(M) \approx 0$

$\underline{\gamma(M) \in (0,1]}$ measures connectivity of G

Diffusion on a Ring

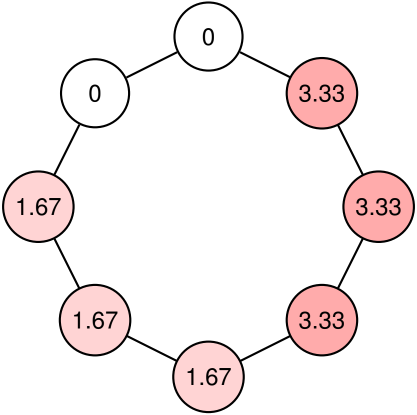


Diffusion on a Ring



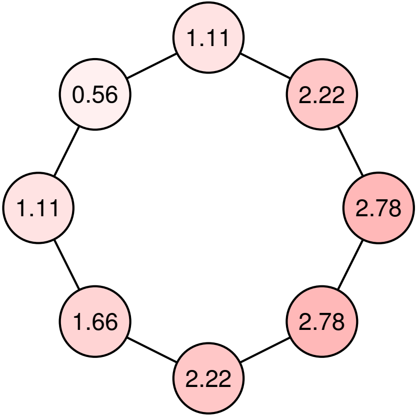
Diffusion on a Ring

after iteration 1:



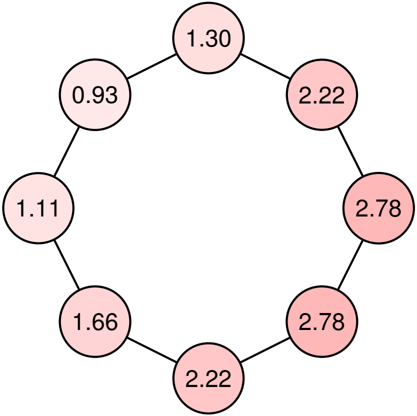
Diffusion on a Ring

after iteration 2:



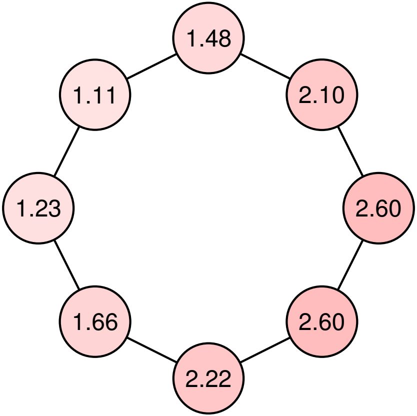
Diffusion on a Ring

after iteration 3:



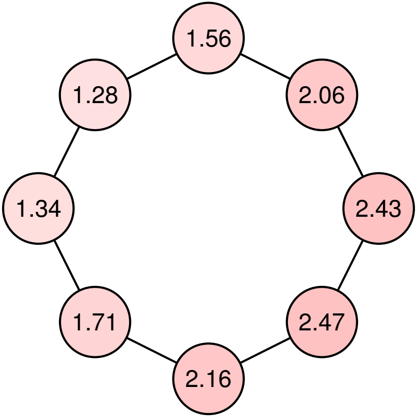
Diffusion on a Ring

after iteration 4:

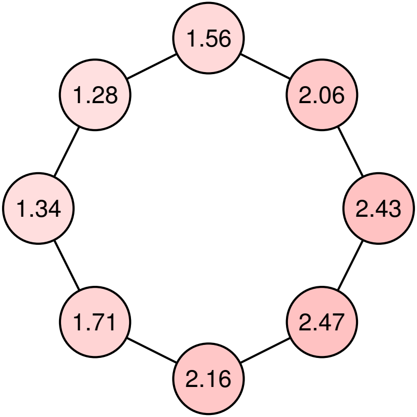


Diffusion on a Ring

after iteration 5:

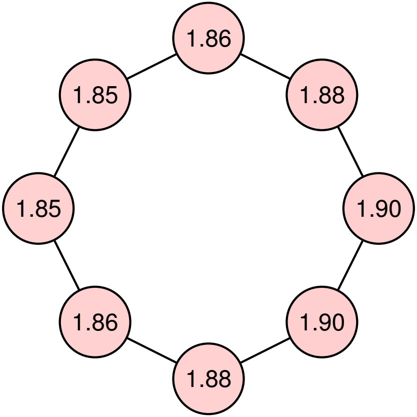


after iteration 20:



Diffusion on a Ring

after iteration 20:



Convergence of the Quadratic Error (Upper Bound)

Lemma

Let $\underline{\gamma(M)} := \max_{\mu_i \neq 1} |\mu_i|$, where $\mu_1 = 1 > \mu_2 \geq \cdots \geq \mu_n \geq -1$ are the eigenvalues of M. Then for any iteration t,

$$\underline{\Phi^t} \leq \underline{\gamma(M)^{2t}} \cdot \underline{\Phi^0}.$$

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$$e^t = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n$$

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e^t is orthogonal to v_1

$v_1 = \overline{x}$

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- For the diffusion scheme,

$$e^{t+1} = Me^t$$

$$e^{t+1} = x^{t+1} - \overline{x} = M \cdot x^t - M \cdot \overline{x}$$
$$= M \cdot (x^t - \overline{x}) = M \cdot e^t$$

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□

Convergence of the Quadratic Error (Lower Bound)

(skipped)

Lemma

For any eigenvalue μ_i, $1 \leq i \leq n$, there is an initial load vector x^0 so that

$$\Phi^t = \mu_i^{2t} \cdot \Phi^0.$$

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Outlook: Idealised versus Discrete Case

Idealised Case

Outlook: Idealised versus Discrete Case

Idealised Case

$$
\begin{aligned}
x^t &= M \cdot x^{t-1} \\
&= M^t \cdot x^0
\end{aligned}
$$

Outlook: Idealised versus Discrete Case

Idealised Case

$$\begin{aligned} x^t &= M \cdot x^{t-1} \\ &= M^t \cdot x^0 \end{aligned}$$

Linear System

- corresponds to Markov chain
- well-understood

Outlook: Idealised versus Discrete Case

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- corresponds to Markov chain
- well-understood

Given any load vector x^0, the number of iterations until x^t satisfies $\Phi^t \leq \epsilon$ is at most $\frac{\log(\Phi^0/\epsilon)}{\underline{1-}\gamma(M)}$.

Outlook: Idealised versus Discrete Case

Here load consists of integers that cannot be divided further.

Idealised Case

Discrete Case

$$\begin{aligned} x^t &= M \cdot x^{t-1} \\ &= M^t \cdot x^0 \end{aligned}$$

Linear System

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$$y^t = \underline{M \cdot y^{t-1}} + \boxed{\Delta^t}$$

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$$x^t = M \cdot x^{t-1}$$
$$= \underline{M^t} \cdot x^0$$

Discrete Case

Rounding Error

$$y^t = M \cdot y^{t-1} + \Delta^t$$
$$= \underline{M^t \cdot y^0} + \underline{\sum_{s=1}^{t} \underline{M^{t-s}} \cdot \Delta^s}$$

Linear System

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- well-understood

Given any load vector x^0, the number of iterations until x^t satisfies $\Phi^t \leq \epsilon$ is at most $\frac{\log(\Phi^0/\epsilon)}{1-\gamma(M)}$.

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Non-Linear System

- rounding of a Markov chain
- harder to analyze

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Discrete Case

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$$\begin{aligned} y^t &= M \cdot y^{t-1} + \underline{\Delta^t} \\ &= M^t \cdot y^0 + \sum_{s=1}^{t} M^{t-s} \cdot \Delta^s \end{aligned}$$

Non-Linear System

- rounding of a Markov chain
- harder to analyze

How close can it be made to the idealised case?

II. Matrix Multiplication

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

Introduction to Sorting Networks

Batcher's Sorting Network

Counting Networks

Load Balancing on Graphs

Introduction to Matrix Multiplication

Serial Matrix Multiplication

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \underline{\sum_{k=1}^{n} a_{ik} \cdot b_{kj}} \qquad \forall i, j = 1, 2, \ldots, n.$$

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```
SQUARE-MATRIX-MULTIPLY$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
for $i = 1$ to $n$
    for $j = 1$ to $n$
        $c_{ij} = 0$
        for $k = 1$ to $n$
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return $C$
```

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

```
SQUARE-MATRIX-MULTIPLY($A$, $B$)
$n = A.rows$
let $C$ be a new $n \times n$ matrix
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    for $j = 1$ to $n$
        $c_{ij} = 0$
        for $k = 1$ to $n$
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```

SQUARE-MATRIX-MULTIPLY(A, B) takes time $\Theta(n^3)$.

Matrix Multiplication

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

This definition suggests that $n \cdot n^2 = n^3$ arithmetic operations are necessary.

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        $c_{ij} = 0$
        for k = 1 to n
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C
```

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Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

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Divide & Conquer:
Partition A, B, and C into four $n/2 \times n/2$ matrices:

Divide & Conquer: First Approach

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Divide & Conquer:
Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Divide & Conquer: First Approach

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Hence the equation $C = A \cdot B$ becomes:

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} \underline{C_{11}} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ \underline{C_{12}} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ \underline{C_{21}} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ \underline{C_{22}} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

Divide & Conquer:
Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Each equation specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their products.

Divide & Conquer: First Approach (Pseudocode)

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
$$C_{11} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
**if** $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
**else** partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{11})$
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{21})$
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{12})$
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{22})$
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{11})$
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
        $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
**return** $C$
```

$$\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
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\end{aligned}$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A$, $B$)
1   $n = A.rows$
2   let $C$ be a new $n \times n$ matrix
3   if $n == 1$
4       $c_{11} = a_{11} \cdot b_{11}$
5   else partition $A$, $B$, and $C$ as in equations (4.9)
6       $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{11}$)
                 + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{21}$)
7       $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{12}$)
                 + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{22}$)
8       $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{11}$)
                 + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{21}$)
9       $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{12}$)
                 + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{22}$)
10  return $C$
```

Line 5: Handle submatrices implicitly through index calculations instead of creating them.

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
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Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
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    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{11})$
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    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{12})$
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    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure.

Divide & Conquer: First Approach (Pseudocode)

```
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          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

Divide & Conquer: First Approach (Pseudocode)

```
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8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A$, $B$)
$n = A.rows$
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    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{11}$)
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    $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{12}$)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{22}$)
    $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
    $n = A.rows$
    let $C$ be a new $n \times n$ matrix
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        $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{12}$)
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{22}$)
        $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
        $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
    return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{11})$
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{21})$
    $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{12})$
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{22})$
    $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{11})$
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$\frac{8^{\log_2 n} \cdot \Theta(1) = \Theta(n^3)}{c \cdot n^2 + c \cdot 8 \cdot \left(\frac{n}{2}\right)^2 + c \cdot 8^2 \cdot \left(\frac{n}{4}\right)^2 + \dots = \Theta(n^3)}$$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ \boxed{8} \cdot T(\boxed{n/2}) + \overset{c \cdot n^2}{\boxed{\Theta(n^2)}} & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
$n = A.rows$
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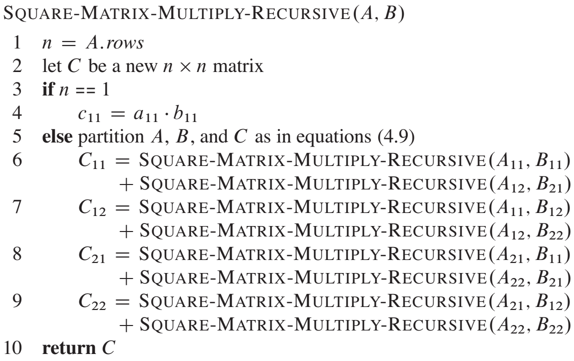
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) =$

Divide & Conquer: First Approach (Pseudocode)



```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
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          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n})$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
$n = A.rows$
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          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{22}$)
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
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```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

No improvement over the naive algorithm!

II. Matrix Multiplication

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

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```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new $n \times n$ matrix
for $i = 1$ to $n$
    for $j = 1$ to $n$
        $c_{ij} = 0$
        for $k = 1$ to $n$
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C
```

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        $c_{ij} = 0$
        for k = 1 to n
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
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```

SQUARE-MATRIX-MULTIPLY(A, B) takes time $\Theta(n^3)$.

Matrix Multiplication

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

This definition suggests that $n \cdot n^2 = n^3$ arithmetic operations are necessary.

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        $c_{ij} = 0$
        for k = 1 to n
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```

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Serial Matrix Multiplication

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Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

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Divide & Conquer: First Approach

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Divide & Conquer:
Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

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Hence the equation $C = A \cdot B$ becomes:

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

Divide & Conquer:
Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Each equation specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their products.

Divide & Conquer: First Approach (Pseudocode)

$$
\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
C_{11} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}
$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
$n = A.rows$
let $C$ be a new $n \times n$ matrix
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    $c_{11} = a_{11} \cdot b_{11}$
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return $C$
```

$$
\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
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C_{11} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}
$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A$, $B$)
1   $n = A.rows$
2   let $C$ be a new $n \times n$ matrix
3   if $n == 1$
4       $c_{11} = a_{11} \cdot b_{11}$
5   else partition $A$, $B$, and $C$ as in equations (4.9)
6       $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{11}$)
              + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{21}$)
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8       $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{11}$)
              + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{21}$)
9       $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{12}$)
              + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{22}$)
10  return $C$
```

Line 5: Handle submatrices implicitly through index calculations instead of creating them.

$$
\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
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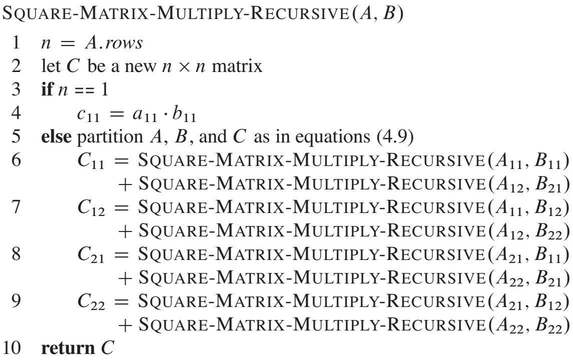
\end{aligned}
$$

Divide & Conquer: First Approach (Pseudocode)



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**return** $C$
```

Let $T(n)$ be the runtime of this procedure.

Divide & Conquer: First Approach (Pseudocode)

```
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return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

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            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{22})$
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{11})$
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
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```

Let $T(n)$ be the runtime of this procedure. Then:

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8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
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    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{11}$)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{21}$)
    $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{12}$)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{22}$)
    $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{11}$)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{21}$)
    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{12}$)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
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          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
**return** $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
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            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{21})$
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{12})$
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{22})$
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{11})$
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
            $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

4 Additions and Partitioning

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
$n = A.rows$
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    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{11}$)
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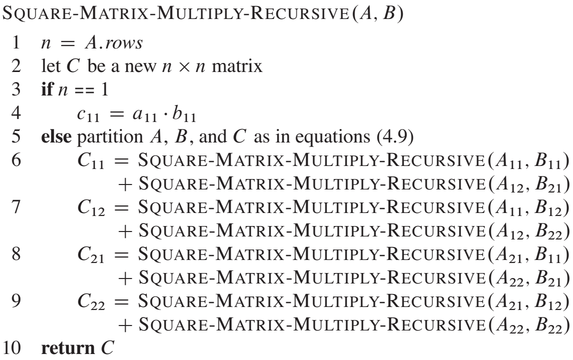
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) =$

Divide & Conquer: First Approach (Pseudocode)



```
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else partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{21}$)
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{12}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{22}$)
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n})$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A, B)$
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{11})$
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{21})$
    $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{11}, B_{12})$
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{12}, B_{22})$
    $C_{21}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{11})$
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{21})$
    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{21}, B_{12})$
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE$(A_{22}, B_{22})$
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

No improvement over the naive algorithm!

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A, B$)
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{21}$)
    $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}, B_{12}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}, B_{22}$)
    $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{11}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{21}$)
    $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}, B_{12}$)
          $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}, B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE($A$, $B$)
$n = A.rows$
let $C$ be a new $n \times n$ matrix
if $n == 1$
    $c_{11} = a_{11} \cdot b_{11}$
else partition $A$, $B$, and $C$ as in equations (4.9)
    $C_{11}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{11}$)
             + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{12}$, $B_{21}$)
    $C_{12}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{11}$, $B_{12}$)
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    $C_{22}$ = SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{21}$, $B_{12}$)
             + SQUARE-MATRIX-MULTIPLY-RECURSIVE($A_{22}$, $B_{22}$)
return $C$
```

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ \boxed{8} \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(\boxed{8}^{\log_2 n}) = \Theta(n^{\boxed{3}})$

Goal: Reduce the number of multiplications

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.
3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

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4. Compute $n/2 \times n/2$ submatrices of C by adding and subtracting various combinations of the P_i.

Time for steps 1,2,4: $\Theta(n^2)$, hence $\underline{T(n) = 7 \cdot T(n/2) + \Theta(n^2)} \boxed{\Rightarrow} \boxed{T(n) = \Theta(n^{\log 7})}$.

Solving the Recursion

$$
\begin{aligned}
T(n) &= \boxed{7} \cdot T(n/2) + c \cdot n^2 \\
&= 7 \cdot \left(7 \cdot T(n/4) + c \cdot (n/2)^2\right) + c \cdot n^2 \\
&= 7^2 \cdot T(n/4) + 7c \cdot (n/2)^2 + c \cdot n^2 \\
&= 7^2 \cdot \left(7 \cdot T(n/8) + c \cdot (n/4)^2\right) + 7c \cdot (n/2)^2 + cn^2 \\
&= 7^3 \cdot T(n/8) + \underbrace{7^2 c \cdot (n/4)^2 + 7c \cdot (n/2)^2 + cn^2} \\
&= \dots \\
&= 7^{\log_2 n} \cdot T(1) + \sum_{i=0}^{\log_2 n - 1} 7^i \cdot c \cdot (n/2^i)^2 \\
&= 7^{\log_2 n} \cdot \Theta(1) + \sum_{i=0}^{\log_2 n - 1} \underbrace{\left(\frac{7}{4}\right)}_{>1}^{i} \cdot c \cdot n^2 \\
&= 7^{\log_2 n} \cdot \Theta(1) + \Theta\left(\left(\frac{7}{4}\right)^{\log_2 n - 1} \cdot n^2\right) \\
&= 7^{\log_2 n} \cdot \Theta(1) + \Theta\left(7^{\log_2 n - 1}\right) = \Theta\left(2^{\log_2 7 \cdot \log_2 n}\right) \\
&= \Theta\left(n^{\log_2 7}\right)
\end{aligned}
$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
P_2 &= S_2 \cdot B_{22} = (A_{11} + A_{12}) \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} = (A_{21} + A_{22}) \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}
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$$\begin{aligned}
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P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}$$

Claim

$$C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \boxed{\begin{pmatrix} \underline{P_5 + P_4 - P_2 + P_6} & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}}$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Proof:

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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\end{aligned}$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Proof:

$$P_5 + P_4 - P_2 + P_6 =$$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}
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Claim

$$
\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}
$$

Proof:

$$
\begin{aligned}
P_5 + P_4 - P_2 + P_6 = & \underbrace{A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}}_{} + \underbrace{A_{22}B_{21} - A_{22}B_{11}}_{} \\
& \underbrace{- A_{11}B_{22} - A_{12}B_{22}}_{} + \underbrace{A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}}_{}
\end{aligned}
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Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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P_4 &= A_{22} \cdot S_4 = A_{22} \cdot (B_{21} - B_{11}) \\
P_5 &= S_5 \cdot S_6 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
P_6 &= S_7 \cdot S_8 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
P_7 &= S_9 \cdot S_{10} = (A_{11} - A_{21}) \cdot (B_{11} + B_{12})
\end{aligned}$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Proof:

$$\begin{aligned}
P_5 + P_4 - P_2 + P_6 = {} & \underline{A_{11}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} \\
& - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + \underline{A_{12}B_{21}} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}}
\end{aligned}$$

Details of Strassen's Algorithm

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\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}
$$

Claim

$$
\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}
$$

Proof:

$$
\begin{aligned}
P_5 + P_4 - P_2 + P_6 &= A_{11}B_{11} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} \\
&\quad - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}} \\
&= A_{11}B_{11} + A_{12}B_{21} = C_{11}
\end{aligned}
$$

Details of Strassen’s Algorithm

The 10 Submatrices and 7 Products

$$\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Other three blocks can be verified similarly.

Proof:

$$\begin{aligned}
P_5 + P_4 - P_2 + P_6 &= A_{11}B_{11} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} \\
&\quad - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}} \\
&= A_{11}B_{11} + A_{12}B_{21}
\end{aligned}$$

Details of Strassen's Algorithm

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\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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\end{aligned}
$$

$\alpha_{ij} \in \{-1, 0, 1\}$
β_{ij}

$$P_i = (\alpha_{i1} \cdot A_{11} + \alpha_{i2} \cdot A_{12} + \alpha_{i3} \cdot A_{21} + \alpha_{i4} \cdot A_{22}) \cdot (\beta_{i1} \cdot B_{11} + \beta_{i2} \cdot B_{12} + \beta_{i3} \cdot B_{21} + \beta_{i4} \cdot B_{22})$$

Claim

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_5 + P_1 - P_3 - P_7 \end{pmatrix}$$

Other three blocks can be verified similarly.

Proof:

$$
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P_5 + P_4 - P_2 + P_6 &= A_{11}B_{11} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} \\
&\quad - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}} \\
&= A_{11}B_{11} + A_{12}B_{21} \qquad \square
\end{aligned}
$$

Current State-of-the-Art

Conjecture: Does a quadratic-time algorithm exist?

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Asymptotic Complexities:

- $O(n^3)$, naive approach

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- $O(n^3)$, naive approach
- $O(n^{2.808})$, Strassen (1969)
- $O(n^{2.796})$, Pan (1978)
- $O(n^{2.522})$, Schönhage (1981)
- $O(n^{2.517})$, Romani (1982)
- $O(n^{2.496})$, Coppersmith and Winograd (1982)
- $O(n^{2.479})$, Strassen (1986)
- $O(n^{\underline{2.376}})$, Coppersmith and Winograd (1989)

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- $O(n^{2.479})$, Strassen (1986)
- $O(n^{2.376})$, Coppersmith and Winograd (1989)
- $O(n^{2.374})$, Stothers (2010)
- $O(n^{2.37\underline{28642}})$, V. Williams (2011)
- $O(n^{2\underline{.3728639}})$, Le Gall (2014)
- ...

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Memory Models

Distributed Memory

- Each processor has its private memory
- Access to memory of another processor via messages

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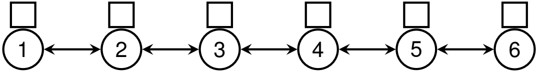

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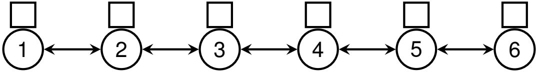

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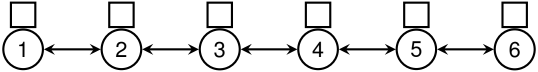

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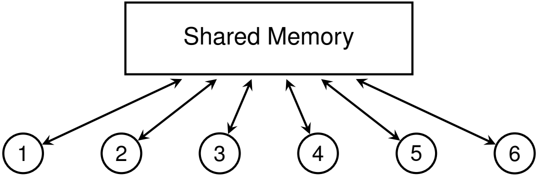

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- Programming shared-memory parallel computer difficult

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Scheduling jobs, communication protocols, load balancing etc.

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Functionalities:

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Functionalities:

- **`spawn`**

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  - (optional) prefix to a procedure call statement
  - procedure is executed in a separate thread
- **sync**

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Only logical parallelism, but not actual!
Need a scheduler to map threads to processors.

Computing Fibonacci Numbers Recursively (Fig. 27.1)

```
FIB(n)
  if n<=1 return n
  else x=FIB(n-1)
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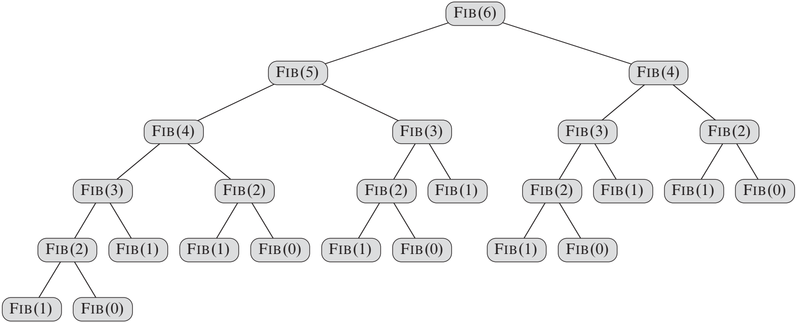
       return x+y
```

Computing Fibonacci Numbers Recursively (Fig. 27.1)



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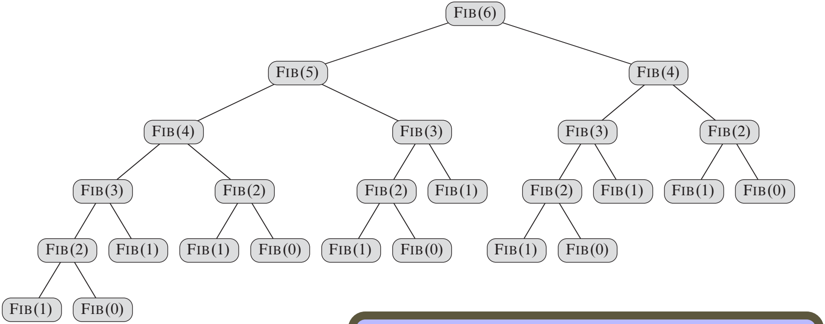

Very inefficient – exponential time!

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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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P-FIB(n)
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

- Without **spawn** and **sync** same pseudocode as before
- **spawn** does not imply parallel execution (depends on scheduler)

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Computation Dag $G = (V, E)$

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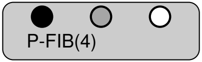

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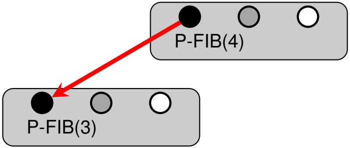

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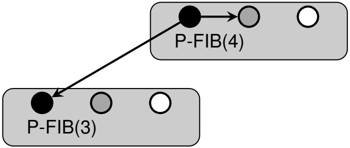

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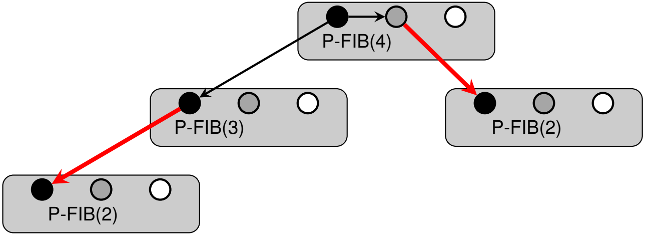

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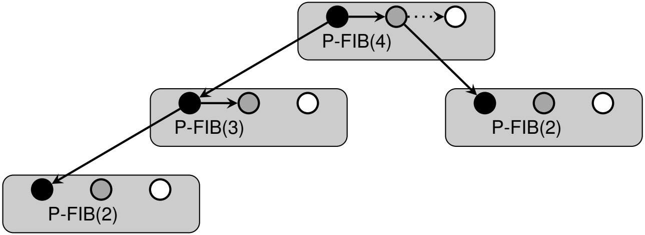

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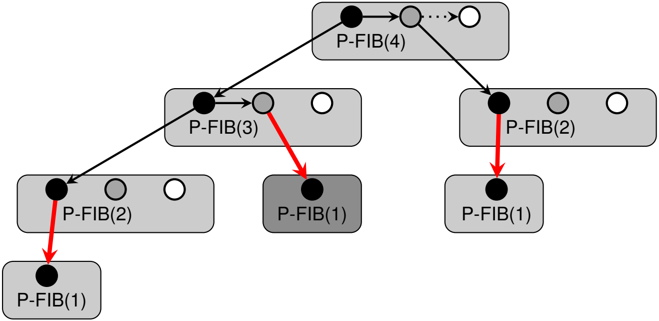

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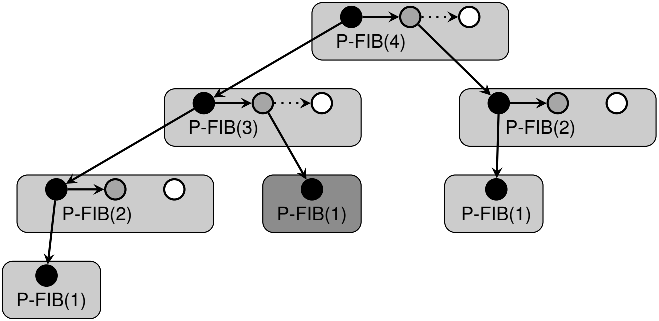

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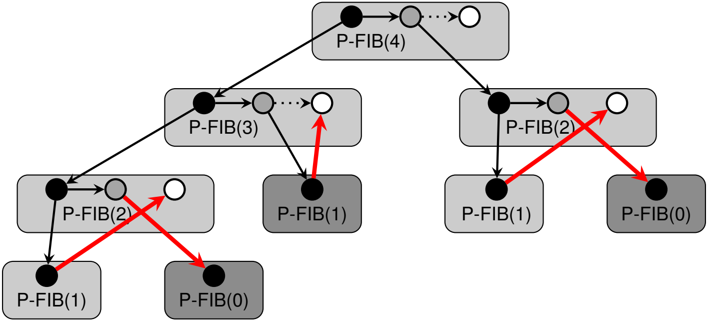

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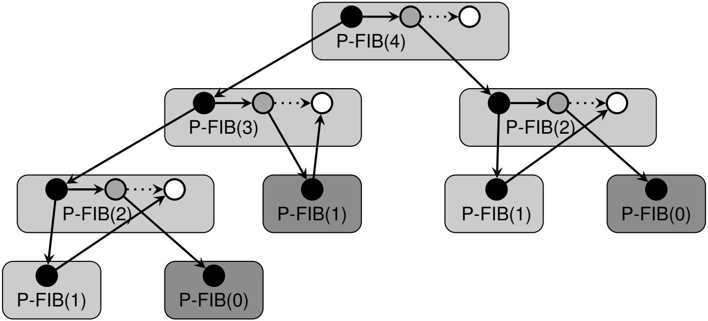

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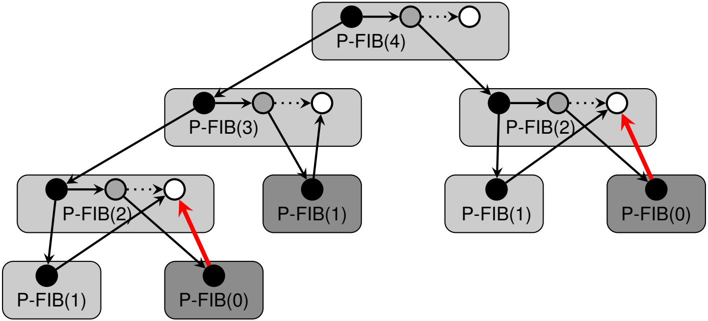

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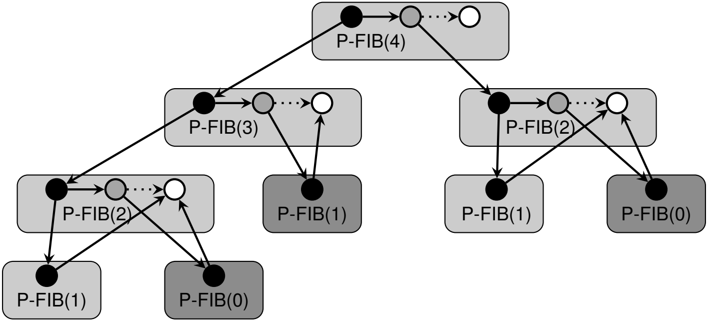

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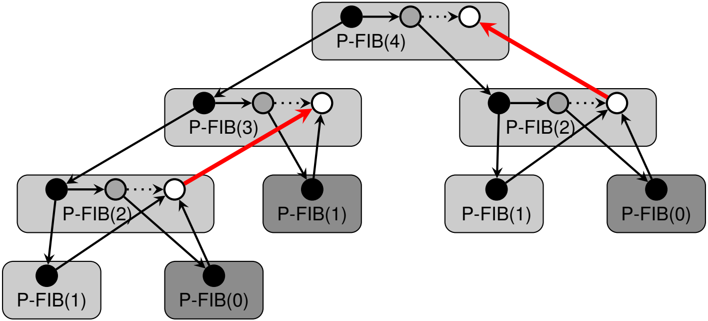

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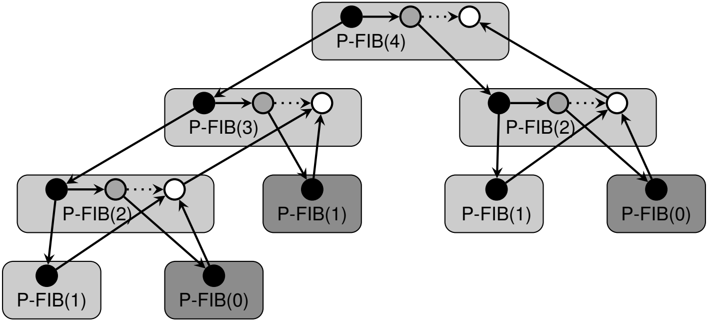

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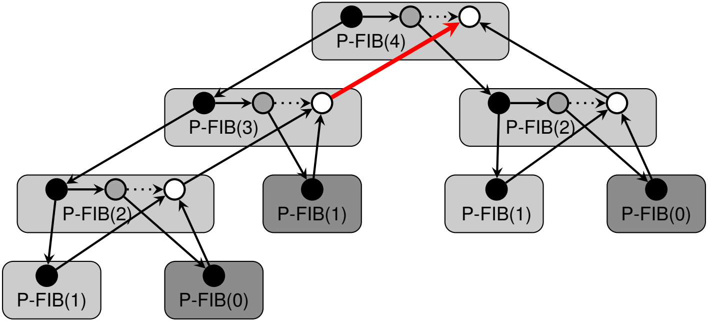

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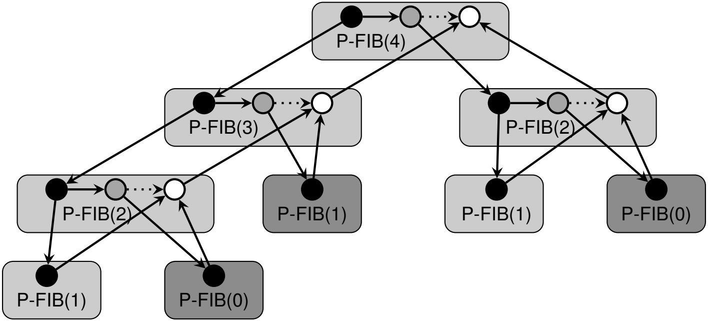

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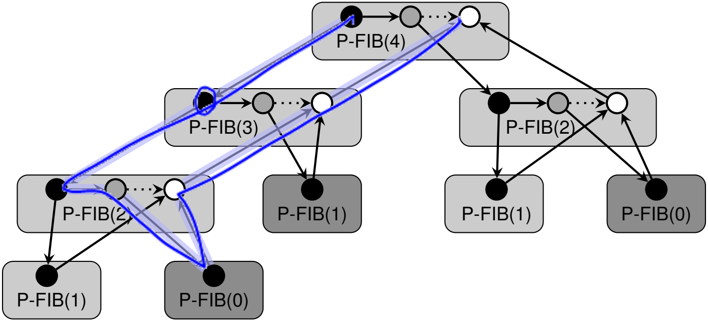

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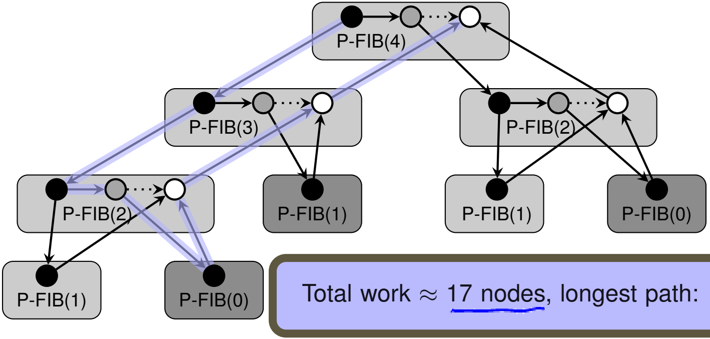

Total work ≈ 17 nodes, longest path: 8 nodes

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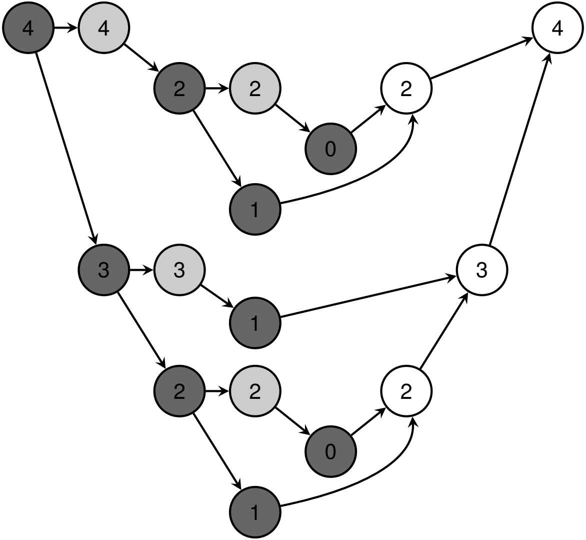

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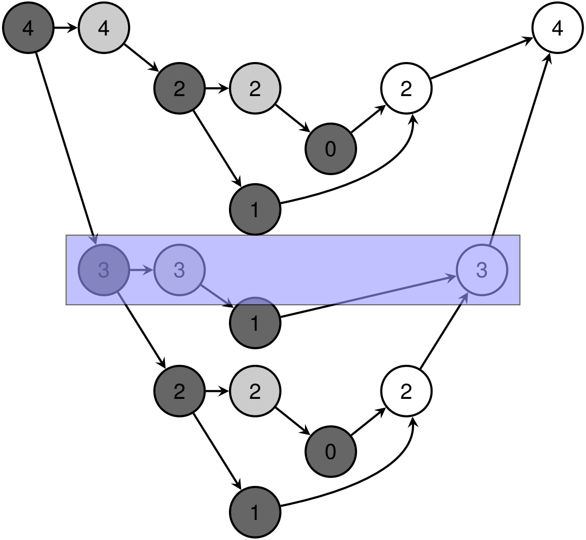

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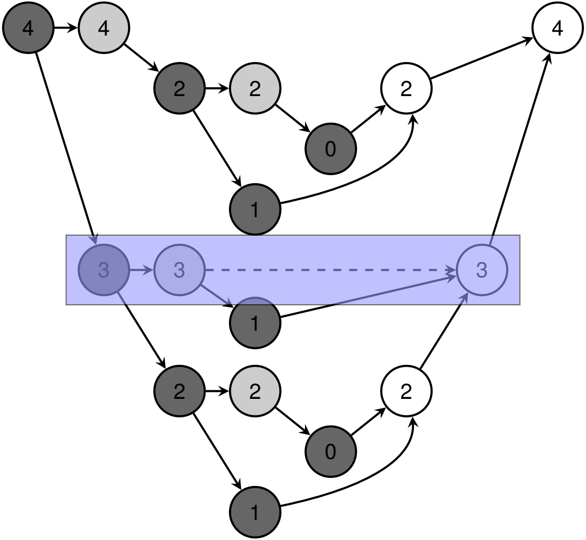

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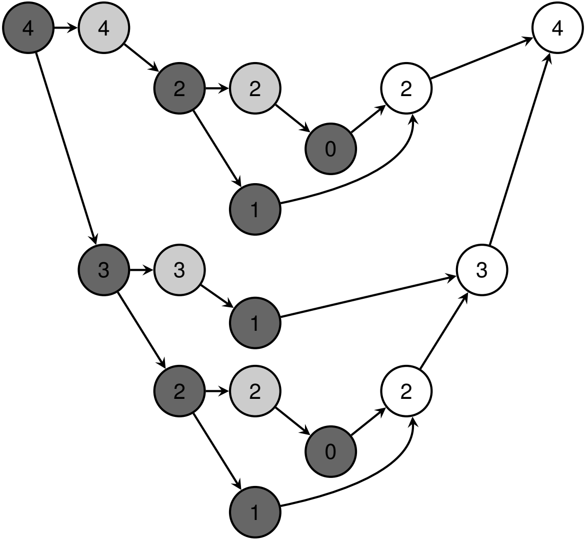

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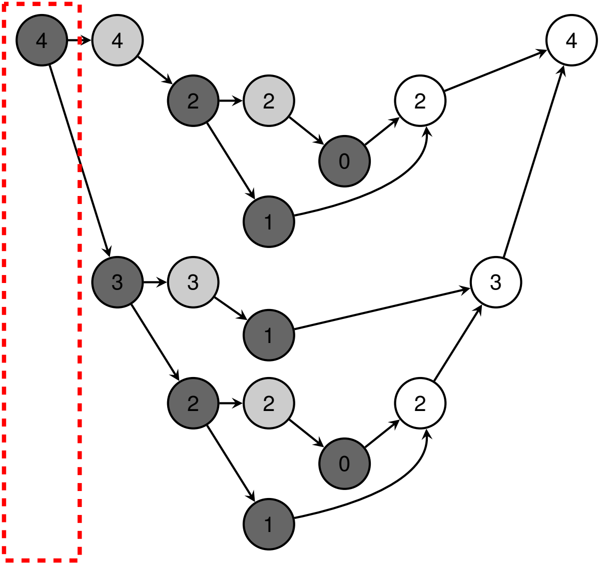

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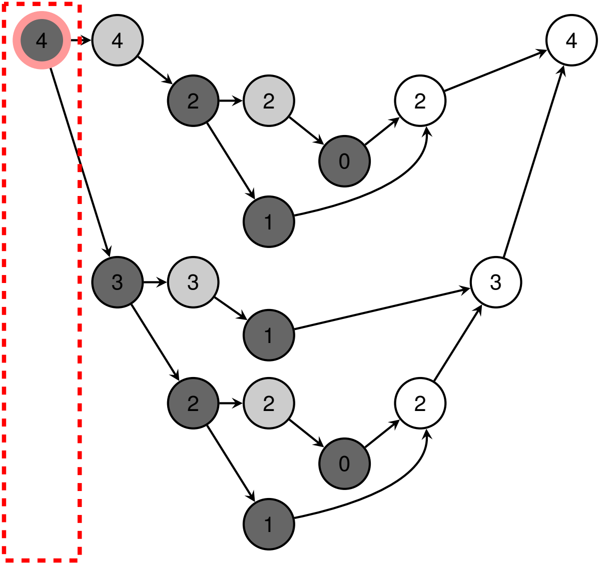

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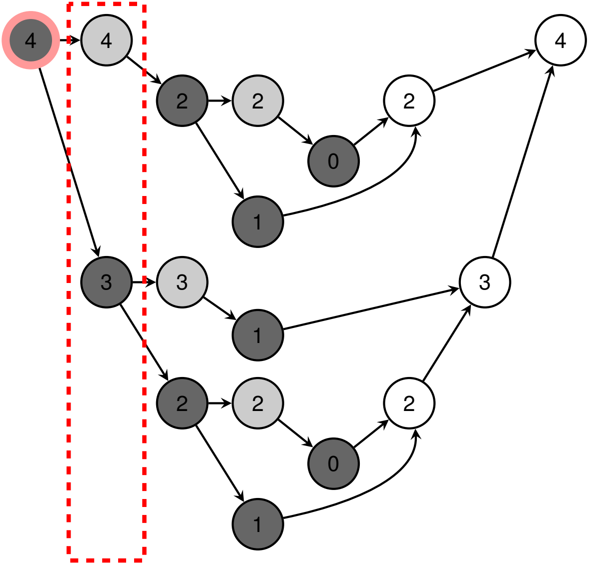

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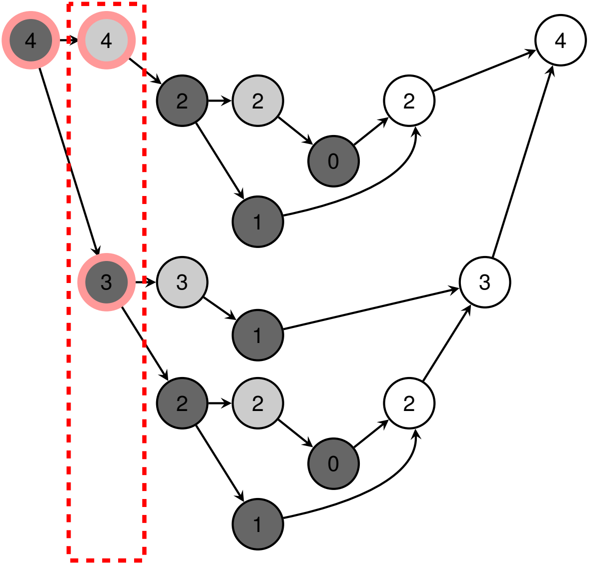

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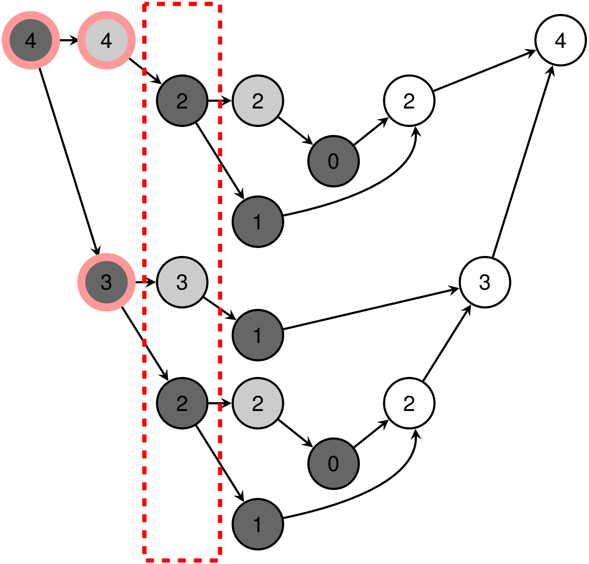

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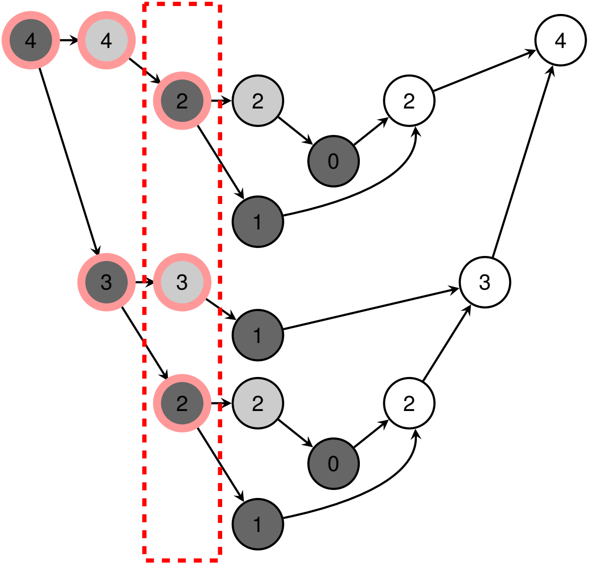

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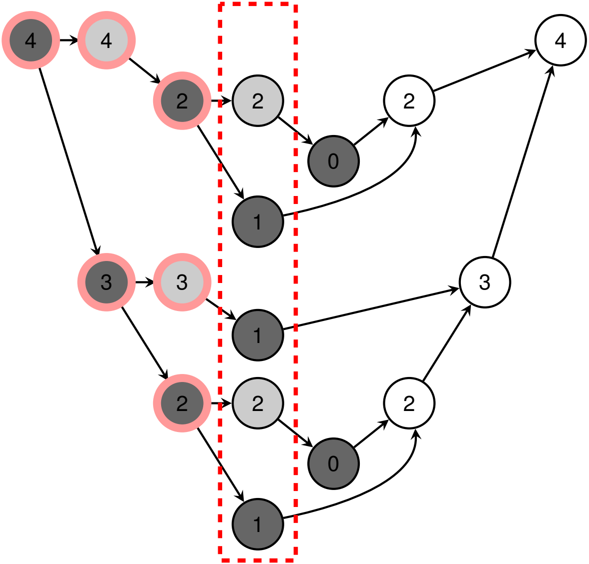

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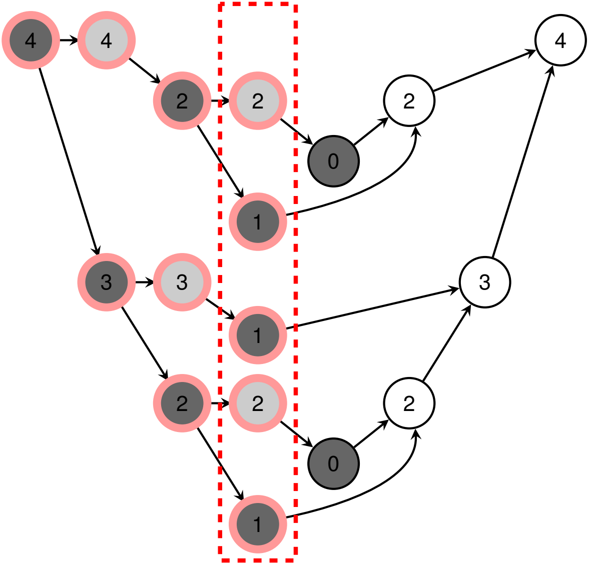

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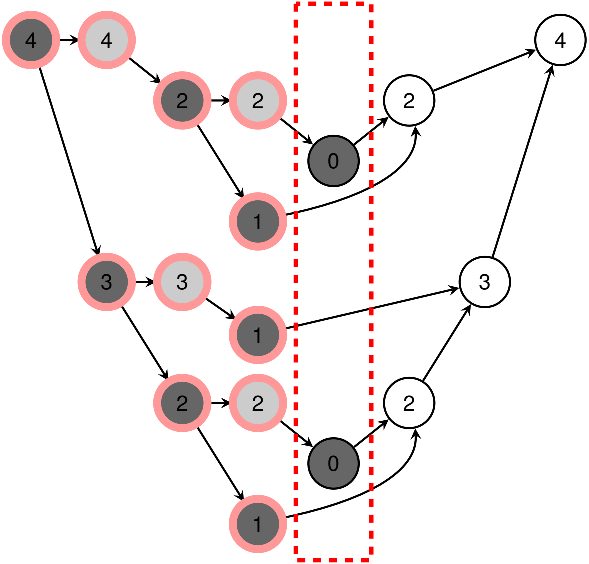

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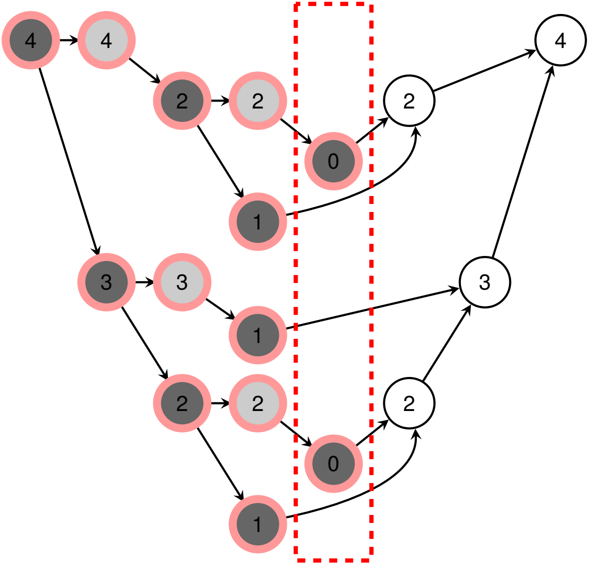

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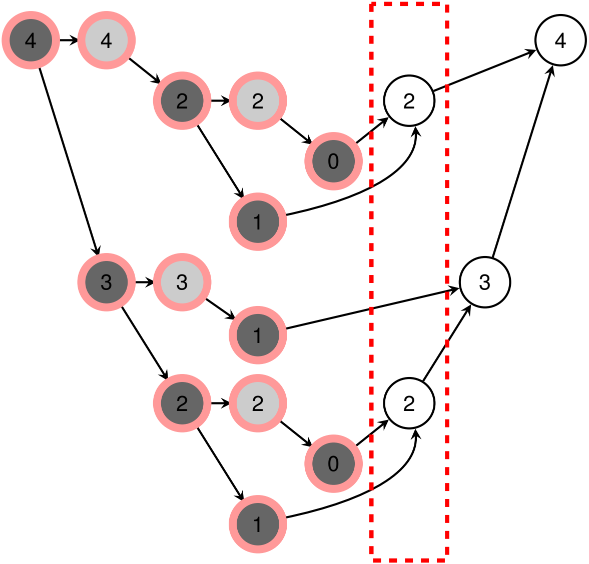

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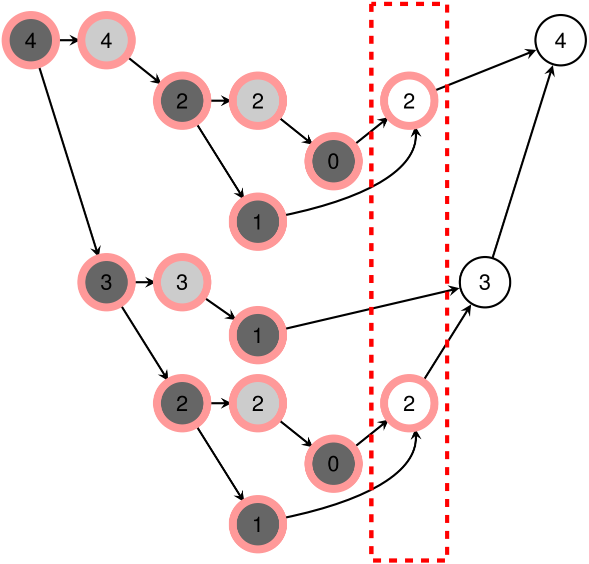

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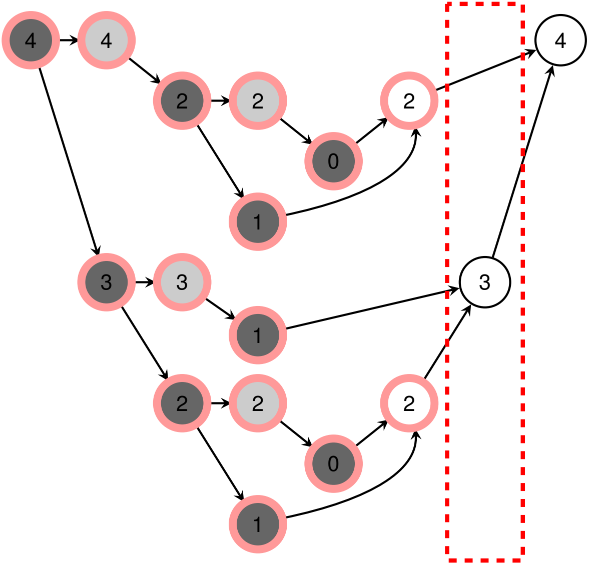

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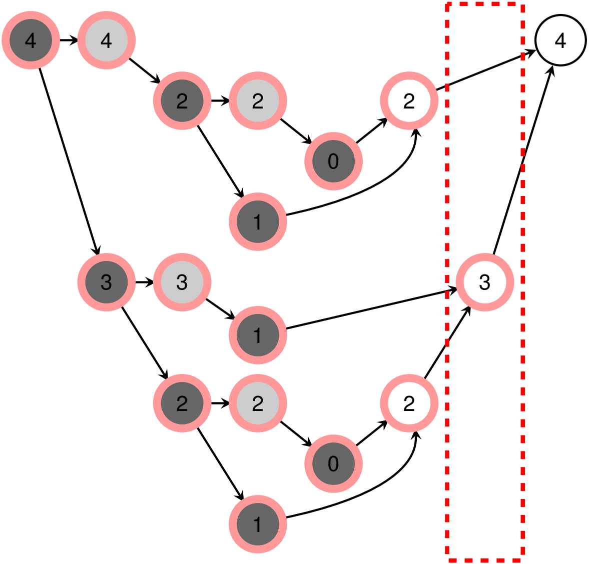

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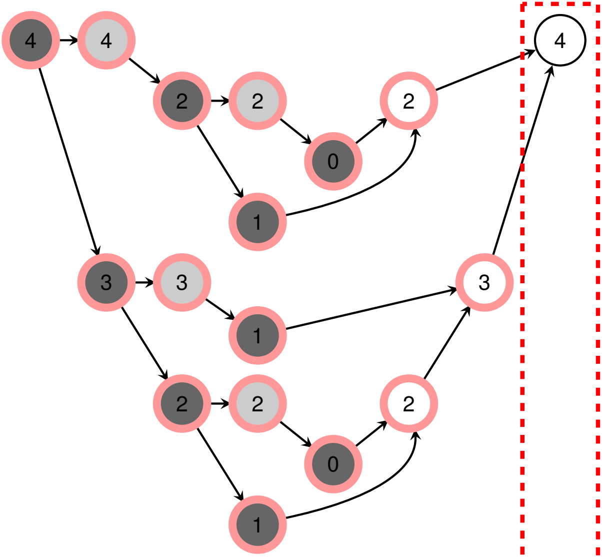

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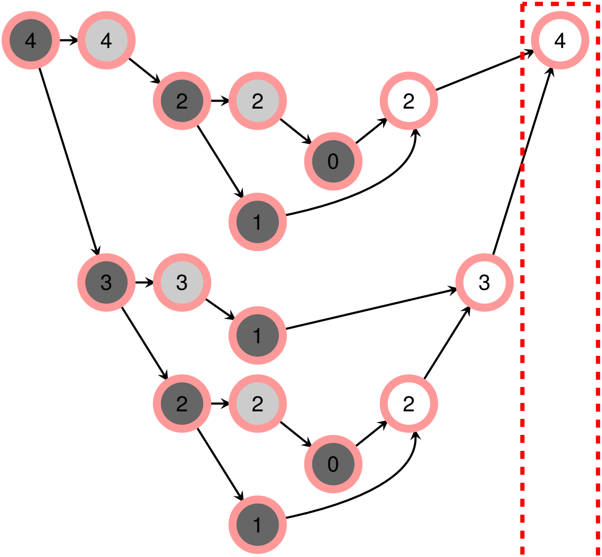
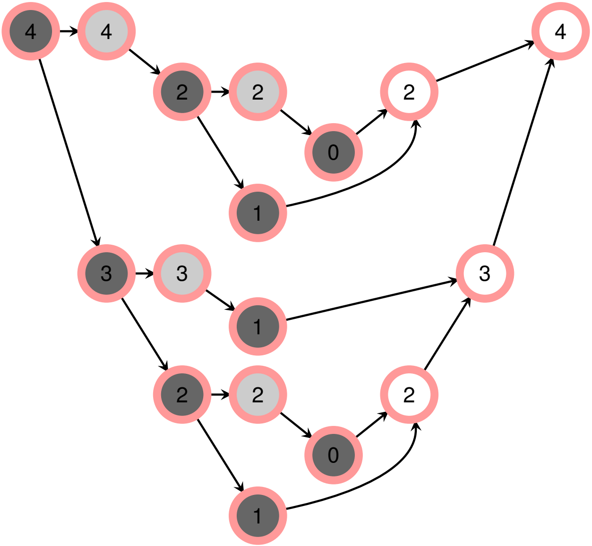


Computing Fibonacci Numbers in Parallel (DAG Perspective)



Performance Measures

Work

Total time to execute everything on single processor.

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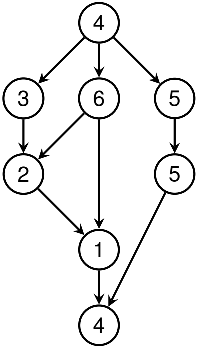
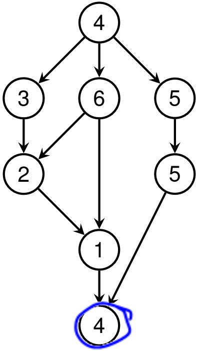


Performance Measures

Work

Total time to execute everything on single processor.

$\sum = 30$



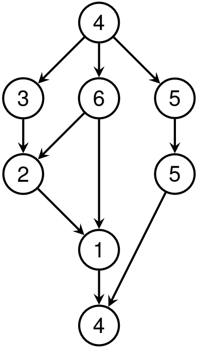
Performance Measures

Work

Total time to execute everything on single processor.

Span

Longest time to execute the threads along any path.



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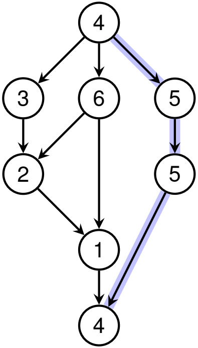


Performance Measures

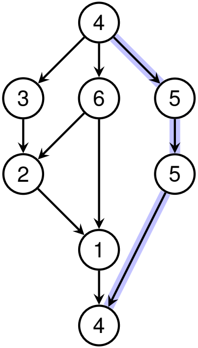
Work

Total time to execute everything on single processor.

Span

Longest time to execute the threads along any path.

$\sum = 18$



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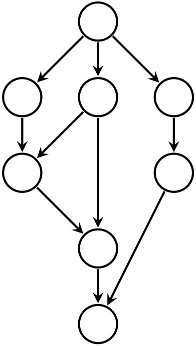

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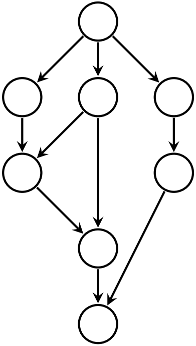
If each thread takes unit time, span is the length of the critical path.



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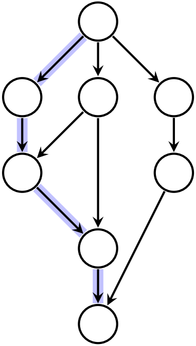

Performance Measures

Work

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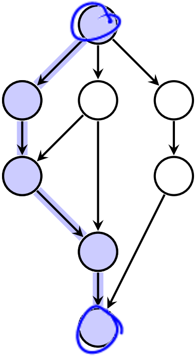
Span

Longest time to execute the threads along any path.

If each thread takes unit time, span is the length of the critical path.

$\#\text{nodes} = 5$



Work Law and Span Law

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span

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- $T_1 =$ work, $T_\infty =$ span
- $P =$ number of (identical) processors
- $T_P =$ running time on P processors

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

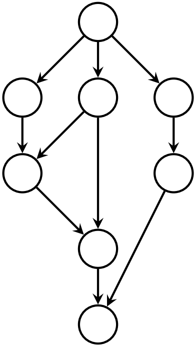
Running time actually also depends on scheduler etc.!

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

Work Law

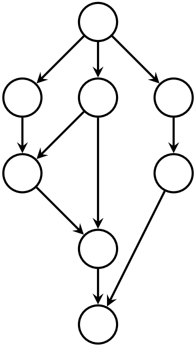
$$T_P \geq \frac{T_1}{P}$$



Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

$T_1 = 8$, $P = 2$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

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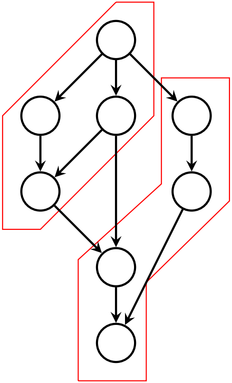

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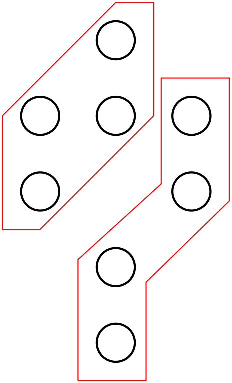

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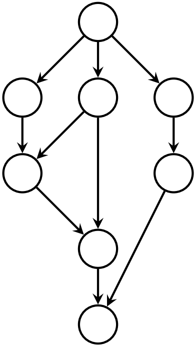

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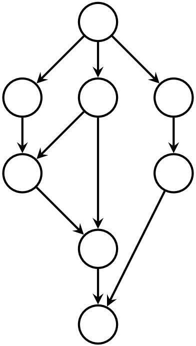
Span Law

$$T_P \geq T_\infty$$



Work Law and Span Law

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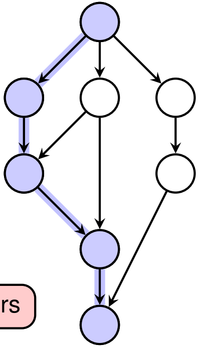
$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

Time on P processors can't be shorter than time on ∞ processors

Work Law and Span Law

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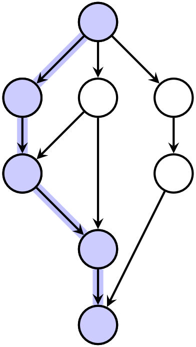

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$

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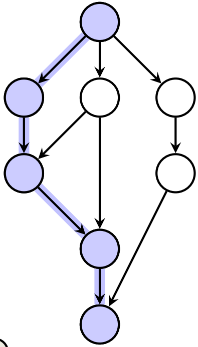

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$

Maximum Speed-Up bounded by P!

Work Law and Span Law

- T_1 = work, T_∞ = span
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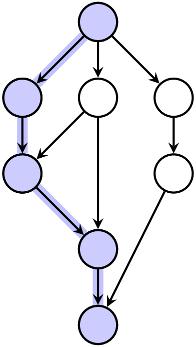
$T_\infty = 5$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Work Law and Span Law

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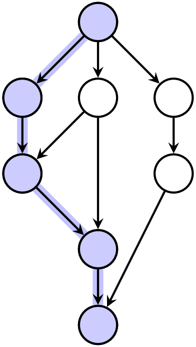

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Span Law

$$T_P \geq T_\infty$$

- Speed-Up: $\frac{T_1}{T_P}$
- Parallelism: $\frac{T_1}{T_\infty}$

Maximum Speed-Up for ∞ processors!

Outline

Introduction

Serial Matrix Multiplication

Reminder: Multithreading

Multithreaded Matrix Multiplication

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $\underline{n \times n \text{ matrix } A = (a_{ij})}$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

$$\boxed{y_i = \sum_{j=1}^{n} a_{ij} x_j} \quad \text{for } i = 1, 2, \ldots, n.$$

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MAT-VEC($A, x$)
$n = A.rows$
let $y$ be a new vector of length $n$
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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

How can a compiler implement the **parallel for**-loop?

Implementing `parallel for` based on Divide-and-Conquer

```
MAT-VEC-MAIN-LOOP($A, x, y, n, i, i'$)
if $i == i'$
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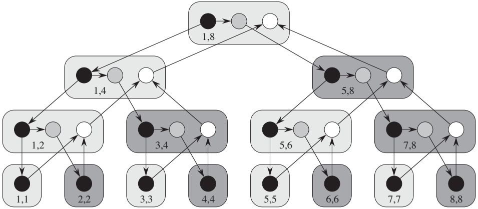

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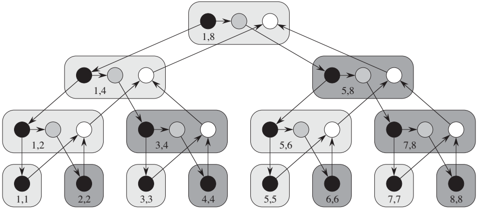

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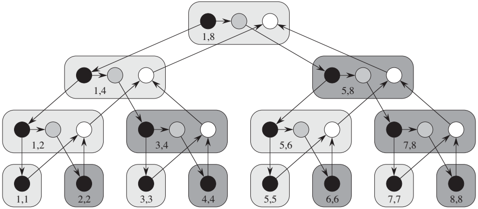

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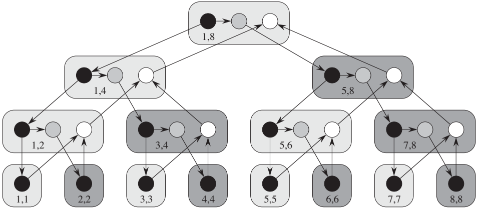
$$T_1(n) =$$

Implementing `parallel for` based on Divide-and-Conquer



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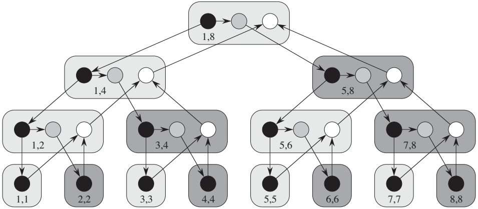
$T_1(n) =$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

Implementing `parallel for` based on Divide-and-Conquer



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$$T_1(n) = \underline{\Theta(n^2)}$$

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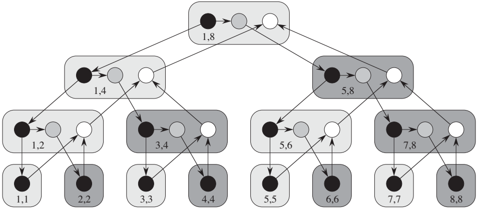
Implementing `parallel for` based on Divide-and-Conquer



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MAT-VEC-MAIN-LOOP(A, x, y, n, i, i′)
if i == i′
    for j = 1 to n
        $y_i = y_i + a_{ij} x_j$
else mid = $\lfloor (i + i')/2 \rfloor$
    spawn MAT-VEC-MAIN-LOOP(A, x, y, n, i, mid)
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```

$$T_1(n) = \Theta(n^2)$$

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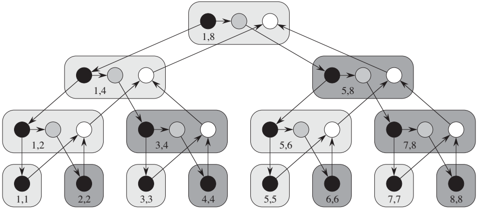
$$T_\infty(n) =$$

Implementing `parallel for` based on Divide-and-Conquer



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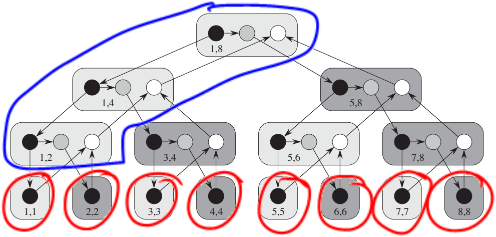
Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) =$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Implementing `parallel for` based on Divide-and-Conquer



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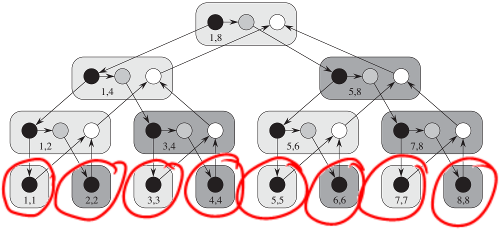
$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \boxed{\Theta(\log n)} + \boxed{\max_{1 \leq i \leq n} \text{iter}(n)}$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Implementing `parallel for` based on Divide-and-Conquer



```
MAT-VEC-MAIN-LOOP(A, x, y, n, i, i')
if $i == i'$
    for $j = 1$ to $n$
        $y_i = y_i + a_{ij} x_j$
else $mid = \lfloor (i + i')/2 \rfloor$
    spawn MAT-VEC-MAIN-LOOP(A, x, y, n, i, mid)
    MAT-VEC-MAIN-LOOP(A, x, y, n, mid + 1, i')
    sync
```

```
MAT-VEC(A, x)
$n = A.rows$
let $y$ be a new vector of length $n$
parallel for $i = 1$ to $n$
    $y_i = 0$
parallel for $i = 1$ to $n$
    for $j = 1$ to $n$
        $y_i = y_i + a_{ij} x_j$
return $y$
```

$$T_1(n) = \Theta(n^2)$$

Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \Theta(\log n) + \boxed{\max_{1 \leq i \leq n} \text{iter}(n)}$$
$$= \boxed{\Theta(n)}.$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Naive Algorithm in Parallel

```
P-SQUARE-MATRIX-MULTIPLY($A, B$)
$n = A.rows$
let $C$ be a new $n \times n$ matrix
parallel for $i = 1$ to $n$
    parallel for $j = 1$ to $n$
        $c_{ij} = 0$
        for $k = 1$ to $n$
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return $C$
```

Naive Algorithm in Parallel

```
P-SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
parallel for i = 1 to n
    parallel for j = 1 to n
        $c_{ij} = 0$
        for k = 1 to n
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C
```

P-SQUARE-MATRIX-MULTIPLY(A, B) has work $T_1(n) = \underline{\Theta(n^3)}$ and span $T_\infty(n) = \underline{\Theta(n)}$.

The first two nested for-loops parallelise perfectly.

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
    $c_{11} = a_{11}b_{11}$
else let T be a new $n \times n$ matrix
    partition A, B, C, and T into $n/2 \times n/2$ submatrices
        $A_{11}, A_{12}, A_{21}, A_{22}$; $B_{11}, B_{12}, B_{21}, B_{22}$; $C_{11}, C_{12}, C_{21}, C_{22}$;
        and $T_{11}, T_{12}, T_{21}, T_{22}$; respectively
    spawn P-MATRIX-MULTIPLY-RECURSIVE($C_{11}, A_{11}, B_{11}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($C_{12}, A_{11}, B_{12}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($C_{21}, A_{21}, B_{11}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($C_{22}, A_{21}, B_{12}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($T_{11}, A_{12}, B_{21}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($T_{12}, A_{12}, B_{22}$)
    spawn P-MATRIX-MULTIPLY-RECURSIVE($T_{21}, A_{22}, B_{21}$)
    P-MATRIX-MULTIPLY-RECURSIVE($T_{22}, A_{22}, B_{22}$)
    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            $c_{ij} = c_{ij} + t_{ij}$
```

Spawn P-M..
+ spawn P-M

} Divide-Conquer

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE($C, A, B$)
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else let $T$ be a new $n \times n$ matrix
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        $A_{11}, A_{12}, A_{21}, A_{22}$; $B_{11}, B_{12}, B_{21}, B_{22}$; $C_{11}, C_{12}, C_{21}, C_{22}$;
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    spawn P-MATRIX-MULTIPLY-RECURSIVE($C_{11}, A_{11}, B_{11}$)
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    P-MATRIX-MULTIPLY-RECURSIVE($T_{22}, A_{22}, B_{22}$)
    sync
    parallel for $i = 1$ to $n$
        parallel for $j = 1$ to $n$
            $c_{ij} = c_{ij} + t_{ij}$
```

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \boxed{\Theta(n^3)}$ and span $T_\infty(n) =$

The Simple Divide&Conquer Approach in Parallel

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    P-MATRIX-MULTIPLY-RECURSIVE($T_{22}, A_{22}, B_{22}$)
    sync
    parallel for $i = 1$ to $n$
        parallel for $j = 1$ to $n$
            $c_{ij} = c_{ij} + t_{ij}$
```

$T_\infty(1) = \Theta(1)$

$\Theta(1)$

8 multiplications in parallel

$\Theta(\log n)$

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

$$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$$

The Simple Divide&Conquer Approach in Parallel

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            $c_{ij} = c_{ij} + t_{ij}$
```

The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(\log^2 n)$.

$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

Strassen's Algorithm in Parallel

Strassen's Algorithm (parallelised)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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   This step takes $\Theta(1)$ work and span by index calculations.

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1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.

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   Can create all 10 matrices with $\Theta(n^2)$ work and $\Theta(\log n)$ span using doubly nested **`parallel for`** loops.

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3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$

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   Using doubly nested **parallel for** this takes $\Theta(n^2)$ work and $\Theta(\log n)$ span.

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   Using doubly nested **`parallel for`** this takes $\Theta(n^2)$ work and $\Theta(\log n)$ span.

$$T_1(n) = \Theta(n^{\log 7})$$

Strassen's Algorithm in Parallel

	$T_1(n)$	$T_\infty(n)$
Naive	$\Theta(n^3)$	$\Theta(n)$
Simple DC	$\Theta(n^3)$	$\Theta(\log^2 n)$
Strassen	$\Theta(n^{2.81})$	$\Theta(\log^2 n)$

Strassen's Algorithm (parallelised)

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$$T_1(n) = \underline{\Theta(n^{\log 7})}$$
$$T_\infty(n) = \underline{\Theta(\log^2 n)}$$

Matrix Multiplication and Matrix Inversion

Speedups for Matrix Inversion by an equivalence with Matrix Multiplication.

Matrix Multiplication and Matrix Inversion

Speedups for Matrix Inversion by an equivalence with Matrix Multiplication.

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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Proof:

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Proof:

- Define a $3n \times 3n$ matrix D by:

$$D = \begin{pmatrix} I_n & A & 0 \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix}$$

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$$D = \begin{pmatrix} I_n & A & 0 \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix} \qquad \Rightarrow \qquad D^{-1} = \begin{pmatrix} I_n & -A & \boxed{AB} \\ 0 & I_n & -B \\ 0 & 0 & I_n \end{pmatrix}.$$

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- Matrix D can be constructed in $\Theta(n^2) = O(I(n))$ time,

Matrix Multiplication and Matrix Inversion

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- Matrix D can be constructed in $\Theta(n^2) = O(I(n))$ time,
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Matrix Multiplication and Matrix Inversion

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- Matrix D can be constructed in $\Theta(n^2) = O(I(n))$ time,
- and we can invert D in $O(I(3n)) = O(I(n))$ time.

$\Rightarrow$ We can compute AB in $O(I(n))$ time. □

The Other Direction

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

Theorem 28.2 (Inversion is no harder than Multiplication)

Suppose we can multiply two $n \times n$ real matrices in time $M(n)$ and $M(n)$ satisfies the two regularity conditions $M(n+k) = O(M(n))$ for any $0 \leq k \leq n$ and $M(n/2) \leq c \cdot M(n)$ for some constant $c < 1/2$. Then we can compute the inverse of any real nonsingular $n \times n$ matrix in time $O(M(n))$.

The Other Direction

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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Proof of this directon much harder (CLRS) – relies on properties of SPD matrices.

The Other Direction

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

Allows us to use Strassen's Algorithm to invert a matrix!

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III. Linear Programming

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Introduction

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

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Example: Political Advertising

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Example: Political Advertising

- Imagine you are a politician trying to win an election

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- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters

Introduction

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- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters
- Aim: at least half of the registered voters in each of the three regions should vote for you

Introduction

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

Example: Political Advertising

- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters
- Aim: at least half of the registered voters in each of the three regions should vote for you
- Possible Actions: Advertise on one of the primary issues which are (i) building more roads, (ii) gun control, (iii) farm subsidies and (iv) a gasoline tax dedicated to improve public transit.

Political Advertising Continued

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

Political Advertising Continued

policy	urban	suburban	rural
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farm subsidies	0	0	10
gasoline tax	10	0	−2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

- Possible Solution:
  - $20,000 on advertising to building roads
  - $0 on advertising to gun control
  - $4,000 on advertising to farm subsidies
  - $9,000 on advertising to a gasoline tax

Political Advertising Continued

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

urban: $20{,}000 \times (-2) + 0 \times (+8) + 4{,}000 \times 0 + 9{,}000 \times 10 = 50{,}000$ ✓

- Possible Solution:
    - $20,000 on advertising to building roads
    - $0 on advertising to gun control
    - $4,000 on advertising to farm subsidies
    - $9,000 on advertising to a gasoline tax
- Total cost: $33,000

Political Advertising Continued

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

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  - $20,000 on advertising to building roads
  - $0 on advertising to gun control
  - $4,000 on advertising to farm subsidies
  - $9,000 on advertising to a gasoline tax
- Total cost: $33,000

What is the best possible strategy?

Towards a Linear Program

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

Towards a Linear Program

policy	urban	suburban	rural
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gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

- $x_1 =$ number of thousands of dollars spent on advertising on building roads
- $x_2 =$ number of thousands of dollars spent on advertising on gun control
- $x_3 =$ number of thousands of dollars spent on advertising on farm subsidies
- $x_4 =$ number of thousands of dollars spent on advertising on gasoline tax

Towards a Linear Program

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gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

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- $x_3 =$ number of thousands of dollars spent on advertising on farm subsidies
- $x_4 =$ number of thousands of dollars spent on advertising on gasoline tax

Constraints:

Towards a Linear Program

policy	urban	suburban	rural
build roads	-2	5	3
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farm subsidies	0	0	10
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- x_2 = number of thousands of dollars spent on advertising on gun control
- x_3 = number of thousands of dollars spent on advertising on farm subsidies
- x_4 = number of thousands of dollars spent on advertising on gasoline tax

Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq \underbrace{50}$

Towards a Linear Program

policy	urban	suburban	rural
build roads	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
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- $x_3 =$ number of thousands of dollars spent on advertising on farm subsidies
- $x_4 =$ number of thousands of dollars spent on advertising on gasoline tax

Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$

Towards a Linear Program

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- x_4 = number of thousands of dollars spent on advertising on gasoline tax

Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Towards a Linear Program

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- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Objective: Minimize $x_1 + x_2 + x_3 + x_4$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & \boxed{x_1, x_2, x_3, x_4 \qquad\qquad\qquad\qquad \geq \quad 0} & & & & & &
\end{array}$$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & x_1, x_2, x_3, x_4 & & & & & & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{7}{c}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr} \text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & & & & & & & & & \\ & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\ & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\ & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\ & & & x_1, x_2, x_3, x_4 & & & & & \geq & 0 \end{array}$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n. = \langle a, x \rangle$$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & \multicolumn{3}{c}{x_1, x_2, x_3, x_4} & & & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

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$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr} \text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & & & & & & & & & \\ & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\ & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\ & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\ & & & x_1, x_2, x_3, x_4 & & & & & \geq & 0 \end{array}$$

The solution of this linear program yields the optimal advertising strategy.

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- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \overset{\geq}{\leq} b$

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{7}{c}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
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Linear Constraints

The Linear Program

Linear Program for the Advertising Problem

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & \multicolumn{7}{l}{\underline{x_1 + x_2 + x_3 + x_4}} & & \\
\text{subject to} & & & & & & & & & \\
& -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
& 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
& 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
& \multicolumn{7}{c}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

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- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \overset{\geq}{\leq} b$

Linear Constraints

- Linear-Progamming Problem: either minimize or maximize a linear function subject to a set of linear constraints

A Small(er) Example

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

A Small(er) Example

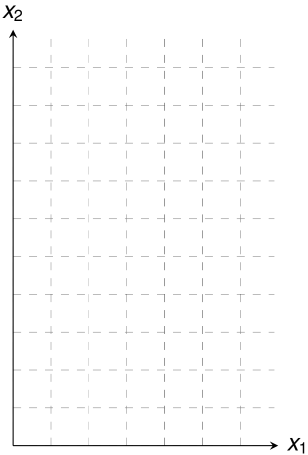
$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1\,,\,x_2 & & & \geq & 0
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

A Small(er) Example

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution



A Small(er) Example

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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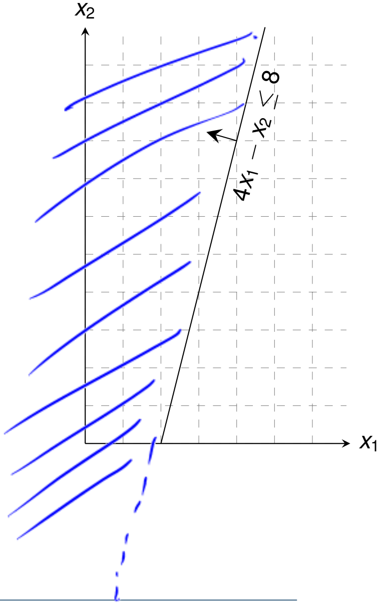
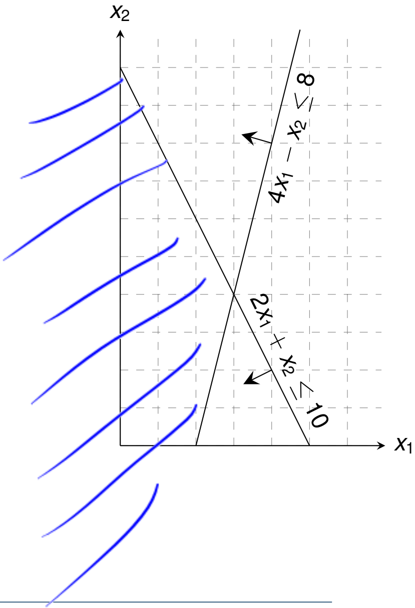


A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

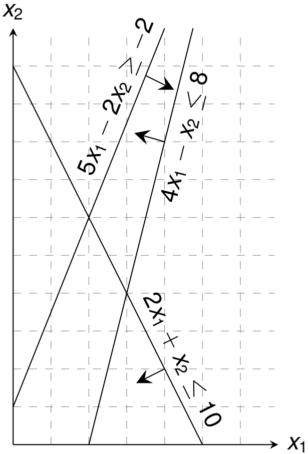
Any setting of x_1 and x_2 satisfying all constraints is a feasible solution



A Small(er) Example

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}
$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution



A Small(er) Example

$$
\begin{array}{llcccl}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}
$$

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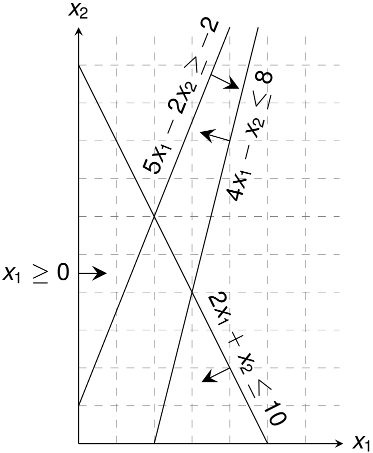
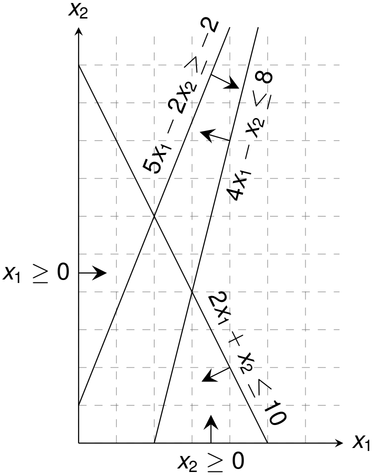


A Small(er) Example

$$\begin{array}{llrcrcr} \text{maximize} & & x_1 & + & x_2 & & \\ \text{subject to} & & & & & & \\ & & 4x_1 & - & x_2 & \leq & 8 \\ & & 2x_1 & + & x_2 & \leq & 10 \\ & & 5x_1 & - & 2x_2 & \geq & -2 \\ & & x_1, x_2 & & & \geq & 0 \end{array}$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution



A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
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 & x_1, x_2 & & & \geq & 0
\end{array}
$$

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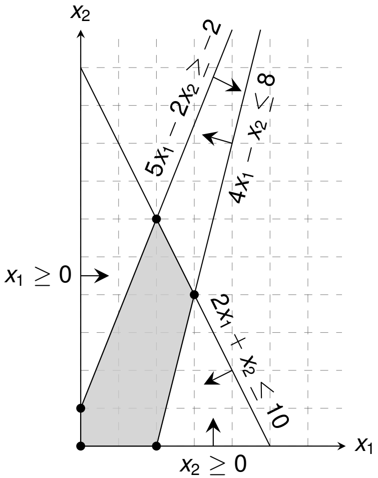
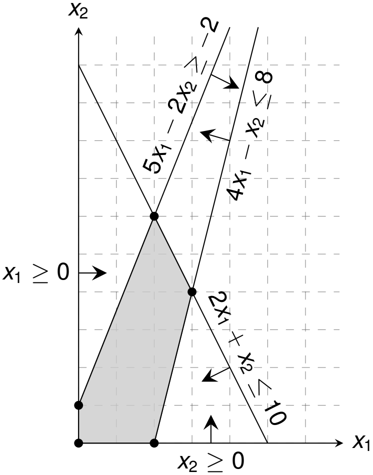


A Small(er) Example

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}$$

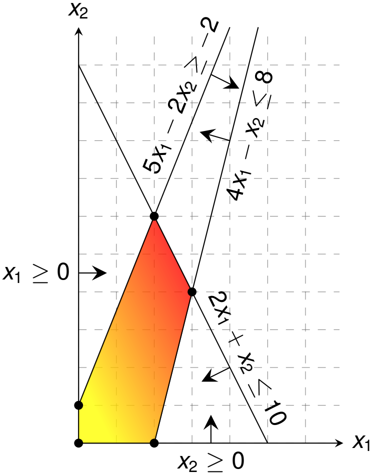
Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& x_1, x_2 & & & \geq & 0
\end{array}$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
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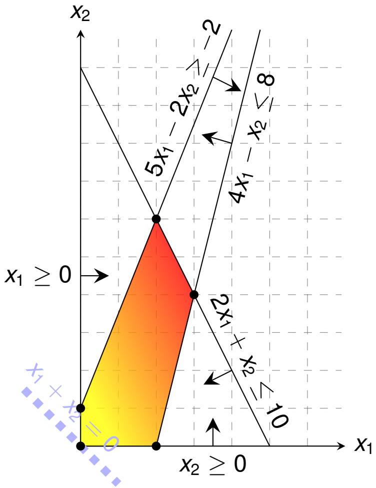
\end{array}
$$

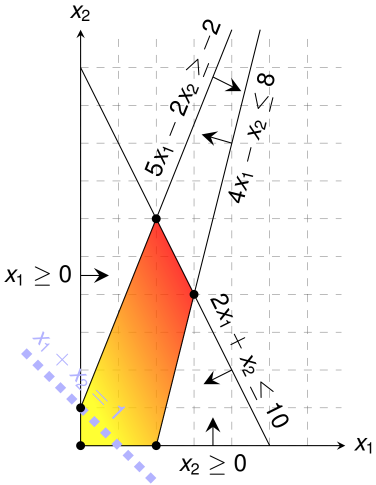
Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}$$

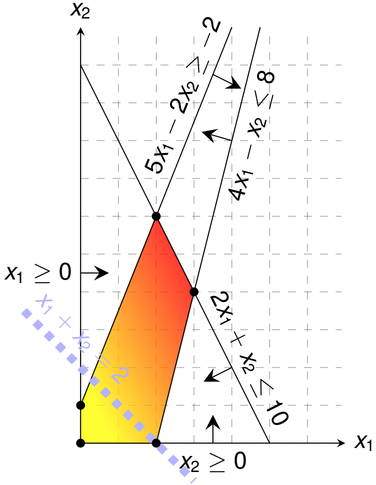
Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



A Small(er) Example

$$\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
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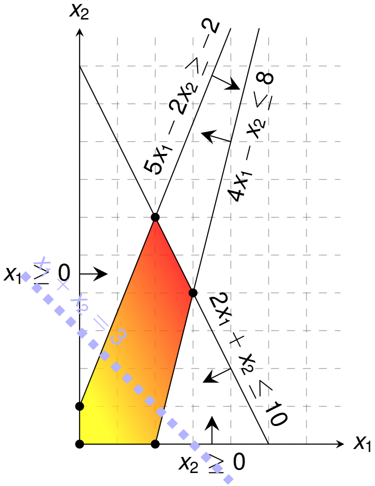

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$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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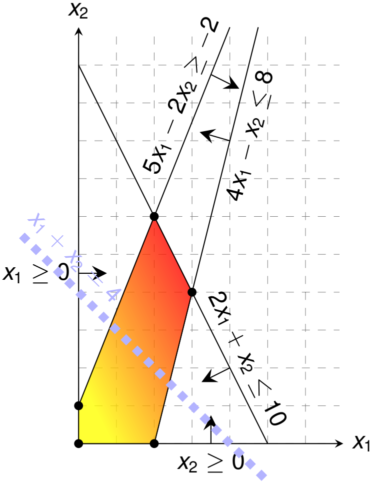

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$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
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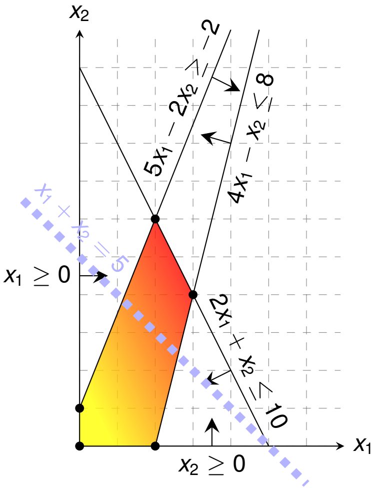

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$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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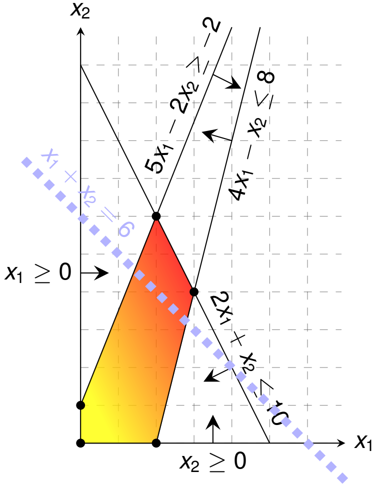

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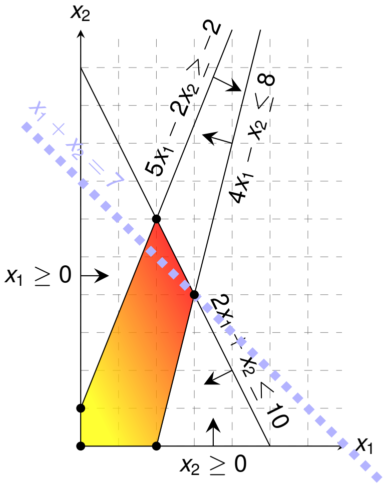

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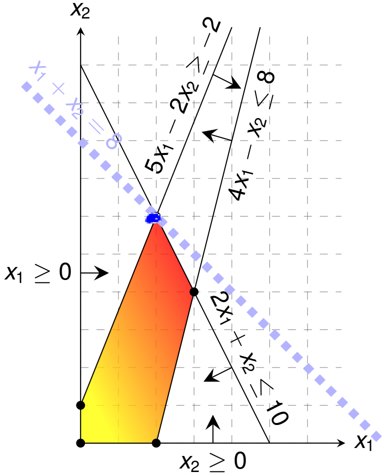

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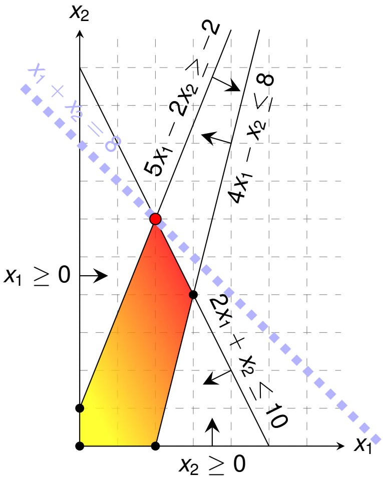

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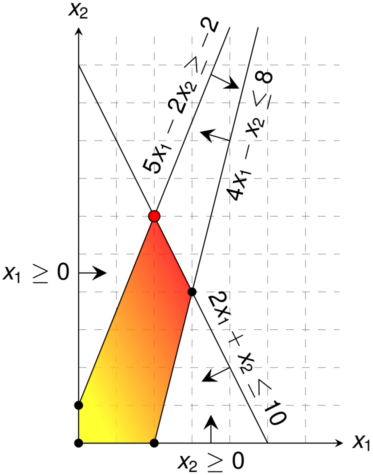

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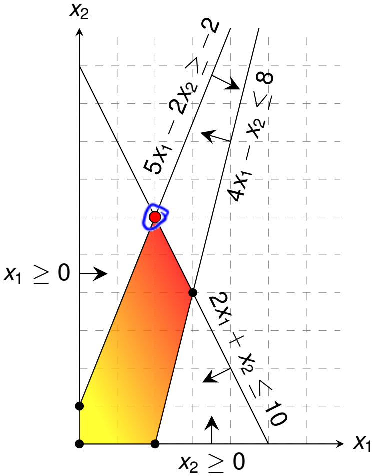

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Standard and Slack Forms

Standard Form

maximize $$\sum_{j=1}^{n} \underline{c_j} x_j$$

subject to

$$\sum_{j=1}^{n} \underline{a_{ij}} x_j \underline{\leq b_i} \qquad \text{for } i = 1, 2, \ldots, m$$

$$\underline{x_j \geq 0} \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximize $\sum_{j=1}^{n} c_j x_j$ Objective Function

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$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

$n + m$ Constraints

$$\left\{ \begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \qquad \text{for } i = 1, 2, \ldots, m \\ x_j &\geq 0 \qquad \text{for } j = 1, 2, \ldots, n \end{aligned} \right.$$

Standard and Slack Forms

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$n + m$ Constraints

Non-Negativity Constraints

Standard and Slack Forms

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subject to

$$n+m \text{ Constraints} \quad \left\{ \begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{array} \right.$$

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

maximize $c^T x$ Inner product of two vectors

subject to

$Ax \leq b$ Matrix-vector product

$x \geq 0$

similar to Gaussian Elimination $(A \cdot x = b)$

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimization rather than maximization.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

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Equivalence: a correspondence (not necessarily a bijection) between solutions so that their objective values are identical.

When switching from maximization to minimization, sign of objective value changes.

Converting into Standard Form (1/5)

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$$\begin{array}{lrcrcr}
\boxed{\text{minimize} \quad -2x_1 + 3x_2} & & & & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}$$

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subject to

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$\Big\downarrow$ Negate objective function

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\end{array}
$$

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$$
\begin{array}{llrcrcr}
\boxed{\text{maximize} & & 2x_1 & - & 3x_2} & & \\
\text{subject to} & & & & & & \\
 & & x_1 & + & x_2 & = & 7 \\
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$$\boxed{x_1 \geq 0}$$

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\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & \boxed{x_1} & & & \boxed{\geq} & \boxed{0}
\end{array}
$$

Replace x_2 by two non-negative variables x_2' and x_2''

$x_2 = x_2' - x_2''$

$x_2' \geq 0, x_2'' \geq 0$

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$$\begin{array}{llcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & \boxed{x_1 \geq 0} & & & &
\end{array}$$

Replace x_2 by two non-negative variables x_2' and x_2''

$$\begin{array}{llcccl}
\text{maximize} & 2x_1 & - & \boxed{3x_2' + 3x_2''} & & \\
\text{subject to} & & & & & \\
 & x_1 & + & \boxed{x_2' - x_2''} & = & 7 \\
 & x_1 & - & \boxed{2x_2' + 2x_2''} & \leq & 4 \\
 & \boxed{x_1, x_2', x_2'' \geq 0} & & & &
\end{array}$$

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$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & \\ & \boxed{x_1 + x_2' - x_2'' = 7} \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$$

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\begin{array}{ll}
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\end{array}
$$

Replace each equality by two inequalities.

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$$
\begin{array}{llcrcrcrl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& \boxed{x_1 \;+\; x_2' \;-\; x_2'' \;=\; 7} & & & & & & \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

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$$
\begin{array}{llcrcrcrl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
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$$

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$$
\begin{array}{lrcrcrcr}
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\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

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$$
\begin{array}{ll}
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\text{subject to} & \\
& x_1 + x_2' - x_2'' \leq 7 \\
& \boxed{x_1 + x_2' - x_2'' \geq 7} \\
& x_1 - 2x_2' + 2x_2'' \leq 4 \\
& x_1, x_2', x_2'' \geq 0
\end{array}
$$

Negate respective inequalities.

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$$
\begin{array}{lrcrcrcr}
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\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
\hline
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
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$$

$\Downarrow$ Negate respective inequalities.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
\hline
 & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
\hline
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

$$
\begin{array}{llrcrcrcr}
\boxed{\text{maximize}} & & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & & \\
& & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{llrcrcrcr}
\text{maximize} & & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
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It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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For the simplex algorithm, it is more convenient to work with equality constraints.

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- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
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$$\boxed{s = b_i - \sum_{j=1}^{n} a_{ij} \underline{x_j}}$$

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$$\boxed{s \geq 0.}$$

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- Introduce a slack variable s by

$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

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For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the ith inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& \underline{x_1} & \underline{+} & \underline{x_2} & \underline{-} & \underline{x_3} & \underline{\leq} & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

$\big\downarrow$ Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & \boxed{x_4} & = & 7 & - & x_1 & - & x_2 & + & x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

$\downarrow$ Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Introduce slack variables

$$\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

Introduce slack variables

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\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
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 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

Converting Standard Form into Slack Form (2/3)

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\begin{array}{lrcrcrcr}
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 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3} & \geq & 0
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$$

Introduce slack variables

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 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{llrcrcrcr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{lrrrrrrrr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 7, \underset{\uparrow}{-7}, 4)$

not feasible!

This is called slack form.

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z &=& & & 2x_1 &-& 3x_2 &+& 3x_3 \\
x_4 &=& 7 &-& x_1 &-& x_2 &+& x_3 \\
x_5 &=& -7 &+& x_1 &+& x_2 &-& x_3 \\
x_6 &=& 4 &-& x_1 &+& 2x_2 &-& 2x_3
\end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z &=& 0 &+& 2x_1 &-& 3x_2 &+& 3x_3\\ x_4 &=& 7 &-& x_1 &-& x_2 &+& x_3\\ x_5 &=& -7 &+& x_1 &+& x_2 &-& x_3\\ x_6 &=& 4 &-& x_1 &+& 2x_2 &-& 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple $(\underline{N}, \underline{B}, \underline{A}, \underline{b}, \underline{c}, \underline{v})$ so that

$$z = \underline{v} + \sum_{j \in N} \underline{c_j} x_j$$

$$x_i = \underline{b_i} - \sum_{j \in N} \underline{a_{ij}} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Variables on the right hand side are indexed by the entries of N.

Slack Form (Example)

$$\begin{array}{rc|r|crcrcr}
\cline{3-3}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \\
\cline{3-3}
\end{array}$$

$(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 4, 0, 18, 0, 0)$
is a feasible solution with objective value 28

Slack Form (Example)

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}$$

Slack Form Notation

Slack Form (Example)

$$\begin{array}{rcrcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

Slack Form (Example)

$$\begin{array}{rcrcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \underline{\{3, 5, 6\}}$
- $$A = \begin{pmatrix} \underline{a_{13}} & \underline{a_{15}} & \underline{a_{16}} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Slack Form (Example)

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

Slack Form (Example)

$$
\begin{array}{rcrcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Slack Form Notation

- $B = \{1,2,4\}$, $N = \{3,5,6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$

Slack Form (Example)

$$\begin{array}{rcrcrcrcr}
z & = & \boxed{28} & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$

The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a $\boxed{\text{strict}}$ convex combination of two other points in the feasible set.

$$x = \lambda \cdot y + (1-\lambda) \cdot z$$

$$\lambda \in (0,1)$$

The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

The set of feasible solutions is a convex set.

= finite intersection of halfspaces

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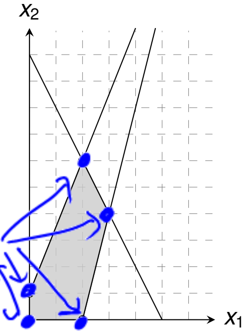
The set of feasible solutions is a convex set.

Theorem

If there exists an optimal solution, ~~it~~ occurs at a vertex of the polygon.

one of them

suffices to check a finite number of points



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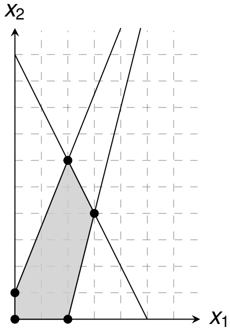
Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof: (non-examinable)

- Let x be an optimal solution which is not a vertex



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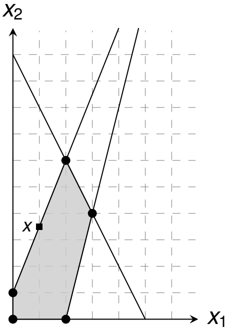

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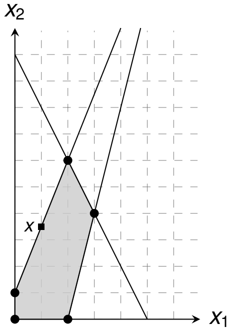
The set of feasible solutions is a convex set.

Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof:

- Let x be an optimal solution which is not a vertex
  $\Rightarrow \exists$ vector d so that $x - d$ and $x + d$ are feasible



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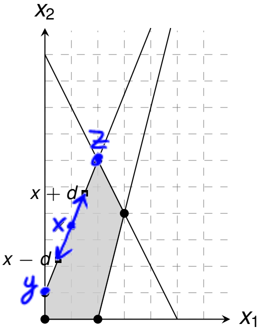

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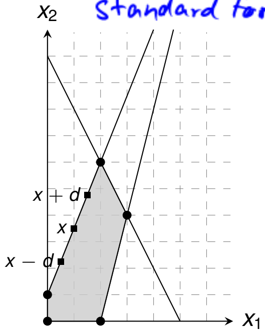
- Let x be an optimal solution which is not a vertex
  $\Rightarrow \exists$ vector d so that $x - d$ and $x + d$ are feasible

  slack form
- Since $\boxed{A(x+d) = b}$ and $\boxed{Ax = b} \Rightarrow \boxed{Ad = 0}$

this illustration is in standard form



The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

The set of feasible solutions is a convex set.

Theorem

If there exists an optimal solution, it occurs at a vertex of the polygon.

Proof:

- Let x be an optimal solution which is not a vertex
  $\Rightarrow \exists$ vector d so that $x - d$ and $x + d$ are feasible
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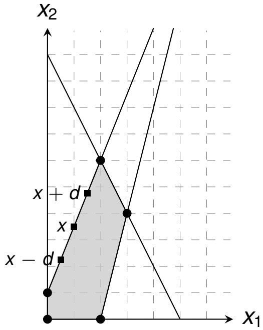

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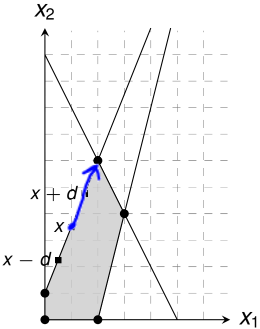

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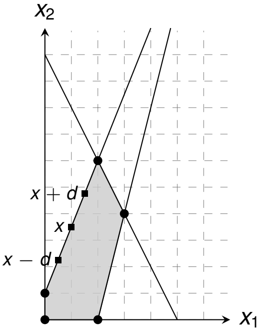

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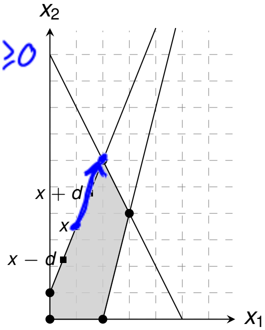
$x_j + d_j \geq 0$

$\boxed{x_j > 0}$

- Case 1: There exists j with $d_j < 0$
  - Increase $\underline{\lambda}$ from 0 to λ' until a new entry of $\boxed{x + \lambda d}$ becomes zero



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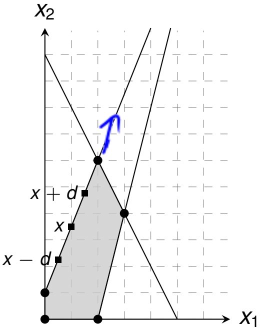
  - $\boxed{x + \lambda' d \text{ feasible}}$ since $\boxed{A(x + \lambda' d) = Ax = b}$ and $\boxed{x + \lambda' d \geq 0}$



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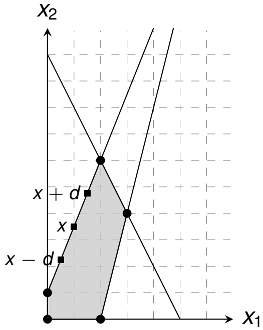
- $\boxed{\text{Case 1}}$ There exists j with $d_j < 0$
  - Increase λ from 0 to λ' until a new entry of $x + \lambda d$ becomes zero !
  - $x + \lambda' d$ feasible, since $A(x + \lambda' d) = Ax = b$ and $x + \lambda' d \geq 0$
  - $c^T(x + \lambda' d) = c^T x + c^T \lambda' d \geq c^T x$



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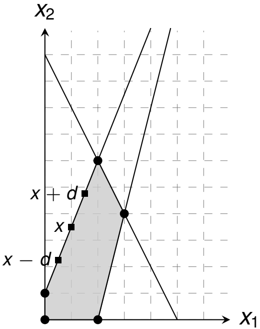

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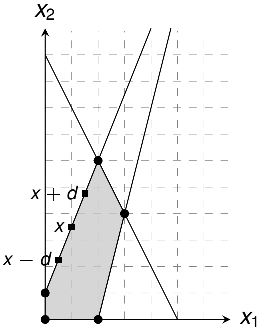

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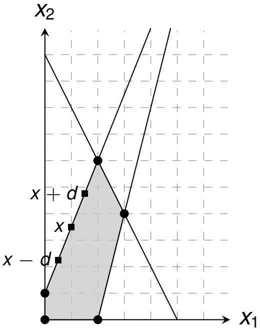
  - $x + \lambda d$ is feasible for all $\lambda \geq 0$: $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
  - If $\lambda \to \infty$, then $c^T(x + \lambda d) \to \infty$



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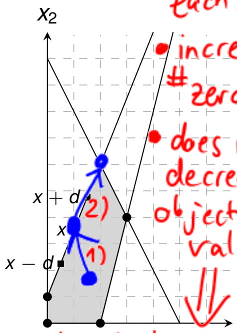
  $\Rightarrow$ This contradicts the assumption that there exists an optimal solution.



each step:
• increases # zero's
• does not decrease objective value
⇓
has to hit vertex (eventually)

Outline

Introduction

Standard and Slack Forms

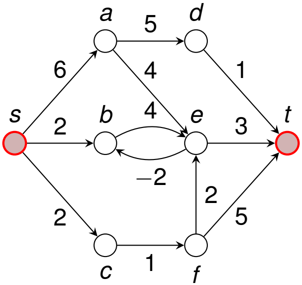
Formulating Problems as Linear Programs

Simplex Algorithm

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$



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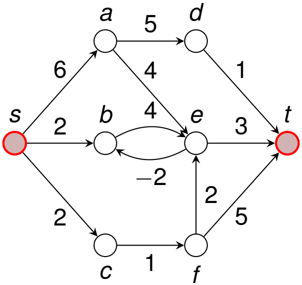

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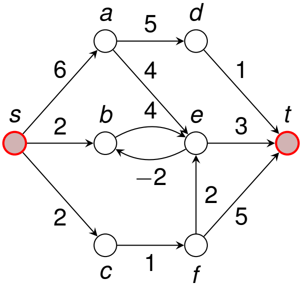
$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimized.



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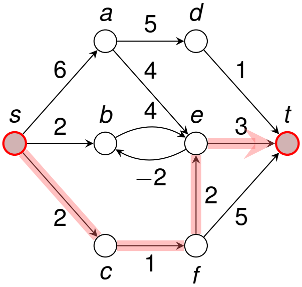

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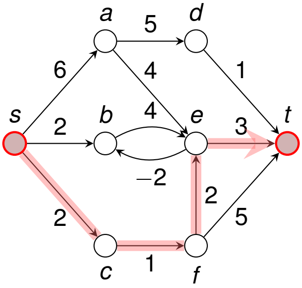

Shortest Paths as LP

subject to

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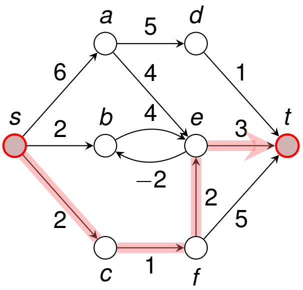

Shortest Paths as LP

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$$\begin{array}{llll} \underline{d_v \leq d_u + w(u,v)} & \text{for each edge } (u,v) \in E, \\ \underline{d_s = 0.} & \end{array}$$

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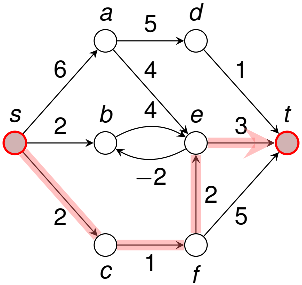

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$$
\begin{array}{llllll}
\text{maximize} & d_t & & & & \\
\text{subject to} & & & & & \\
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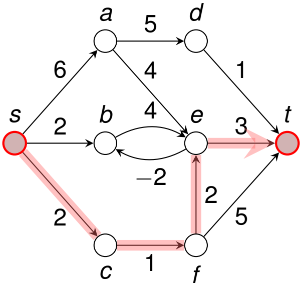

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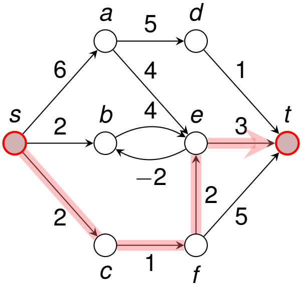

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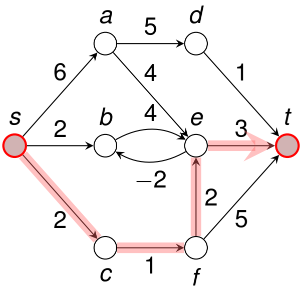

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Solution $\overline{d}$ satisfies $\overline{d}_v \boxed{=} \min_{u:\,(u,v)\in E} \left\{ \overline{d}_u + w(u,v) \right\}$

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Maximum Flow Problem

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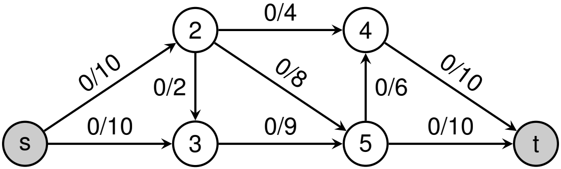

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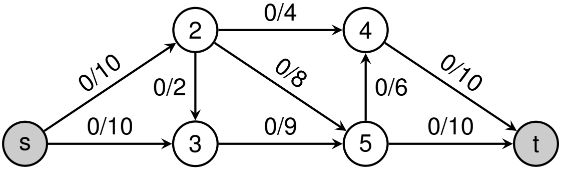

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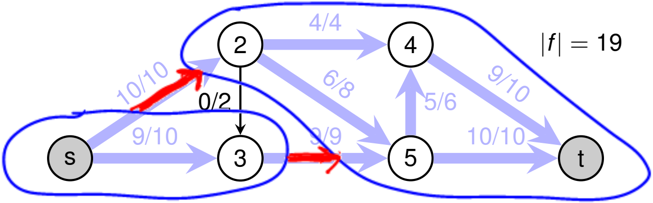

$|f| = 19$

cut ({s, 3}, {2, 4, 5, t}) has capacity

10 + 9 = 19 ⇒ f is max-flow

↑

Max-Flow Min-Cut Theorem

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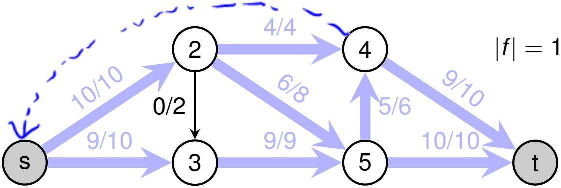

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Maximum Flow as LP

$$
\begin{array}{lrcll}
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\text{subject to} & & & & \\
 & f_{uv} & \leq & c(u, v) & \text{for each } u, v \in V, \\
 & \sum_{v \in V} f_{vu} & = & \sum_{v \in V} f_{uv} & \text{for each } u \in V \setminus \{s, t\}, \\
 & f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow

Generalization of the Maximum Flow Problem

Minimum-Cost-Flow Problem

Minimum-Cost Flow

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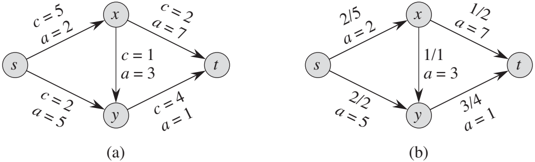

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

Minimum-Cost Flow

Generalization of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units
- Goal: Find a flow $f : V \times V \to \mathbb{R}$ from s to t with $|f| = d$ while minimising the total cost $\sum_{(u,v)\in E} a(u, v) f_{uv}$ incurrred by the flow.

Optimal Solution with total cost:
$\sum_{(u,v)\in E} a(u, v) f_{uv} = (2{\cdot}2)+(5{\cdot}2)+(3{\cdot}1)+(7{\cdot}1)+(1{\cdot}3) = 27$

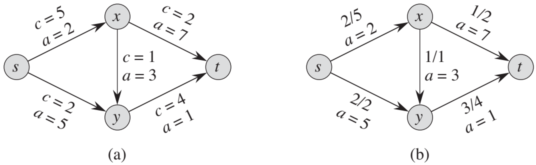

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

minimize $\sum_{(u,v)\in E} a(u,v)f_{uv}$

subject to

$$\begin{array}{rcll} f_{uv} & \leq & c(u,v) & \text{for each } u,v \in V,\\ \sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} & = & 0 & \text{for each } u \in V \setminus \{s,t\},\\ \multicolumn{3}{r}{\boxed{\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} \;=\; d\,,}} & \\ f_{uv} & \geq & 0 & \text{for each } u,v \in V.\end{array}$$

only new constraint

Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

minimize $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$\begin{aligned} f_{uv} &\leq c(u,v) && \text{for each } u,v \in V, \\ \sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} &= 0 && \text{for each } u \in V \setminus \{s,t\}, \\ \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\,, && \\ f_{uv} &\geq 0 && \text{for each } u,v \in V. \end{aligned}$$

Real power of Linear Programming comes from the ability to solve **new problems**!

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Simplex Algorithm: Introduction

Simplex Algorithm

- classical method for solving linear programs (Dantzig, 1947)
- usually fast in practice although worst-case runtime not polynomial
- iterative procedure somewhat similar to Gaussian elimination

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- iterative procedure somewhat similar to Gaussian elimination

Basic Idea:

features of slack form

- Each iteration corresponds to a "basic solution" of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints

new ideas

- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion ("pivoting") is achieved by switching the roles of one basic and one non-basic variable

Simplex Algorithm: Introduction

Simplex Algorithm

- classical method for solving linear programs (Dantzig, 1947)
- usually fast in practice although worst-case runtime not polynomial
- iterative procedure somewhat similar to Gaussian elimination

Basic Idea:

- Each iteration corresponds to a "basic solution" of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion ("pivoting") is achieved by switching the roles of one basic and one non-basic variable

In that sense, it is a greedy algorithm.

Extended Example: Conversion into Slack Form

$$
\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Extended Example: Conversion into Slack Form

$$
\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & \multicolumn{3}{c}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

$\Big\downarrow$ Conversion into slack form

Extended Example: Conversion into Slack Form

$$\begin{array}{llllllll} \text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\ & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\ & & & x_1, x_2, x_3 & & & \geq & 0 \end{array}$$

Conversion into slack form

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

Extended Example: Iteration 1

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Extended Example: Iteration 1

$$
\begin{array}{rcrrrrrrr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\boxed{0, 0, 0}, \boxed{30, 24, 36})$

Extended Example: Iteration 1

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

This basic solution is **feasible**

Extended Example: Iteration 1

$$\begin{array}{rclcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

This basic solution is **feasible**

Objective value is 0.

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$
\begin{array}{rcccccccc}
z & = & & & \boxed{3x_1} & + & x_2 & + & 2x_3 \\
x_4 & = & \boxed{30} & - & \boxed{x_1} & - & x_2 & - & 3x_3 \\
x_5 & = & \boxed{24} & - & \boxed{2x_1} & - & 2x_2 & - & 5x_3 \\
x_6 & = & \boxed{36} & - & \boxed{4x_1} & - & x_2 & - & 2x_3
\end{array}
$$

$x_1 \leq \frac{30}{1} = 30$

$x_1 \leq \frac{24}{2} = 12$

$x_1 \leq \frac{36}{4} = 9$

Basic solution: $(\overline{x_1}, \overline{x_2}, \dots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

This basic solution is **feasible**

Objective value is 0.

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$\begin{array}{rcrcrcrcrl}
z &=& & & 3x_1 &+& x_2 &+& 2x_3 & \\
x_4 &=& 30 &-& x_1 &-& x_2 &-& 3x_3 & x_1 \leq \frac{30}{1} = 30 \\
x_5 &=& 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 & x_1 \leq \frac{24}{2} = 12 \\
x_6 &=& 36 &-& 4x_1 &-& x_2 &-& 2x_3 & x_1 \leq \frac{36}{4} = 9
\end{array}$$

The third constraint is the tightest and limits how much we can increase x_1.

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
\boxed{x_6} & = & 36 & - & \boxed{4x_1} & - & x_2 & - & 2x_3
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

- Solving for x_1 yields:

$$\boxed{x_1 = 9} - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

↑ new value of x_1 in the next iteration

Extended Example: Iteration 1

Increasing the value of x_1 would increase the objective value.

$$\begin{array}{rclcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

substitution equivalent to elementary row operations in Gaussian Elimination

⇓

new slack form will be equivalent

The third constraint is the tightest and limits how much we can increase x_1.

Switch roles of x_1 and x_6:

- Solving for x_1 yields:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

- Substitute this into x_1 in the other three equations

Extended Example: Iteration 2

$$
\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Extended Example: Iteration 2

$$\begin{array}{rclcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\boxed{9}, \boxed{0, 0}, \boxed{21}, \boxed{6}, \boxed{0})$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (9, 0, 0, 21, 6, 0)$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}$$

The third constraint is the tightest and limits how much we can increase x_3.

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{rclcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4}
\end{array}
$$

$$
\boxed{x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4}
\end{array}$$

$$\boxed{x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}}$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{array}{ccccccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4}
\end{array}$$

$$\boxed{x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}}$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

- Substitute this into x_3 in the other three equations

Extended Example: Iteration 3

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Extended Example: Iteration 3

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$\boxed{x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{array}{lcccccccc}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
\hline
\multicolumn{1}{|l}{x_3} & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \multicolumn{1}{c|}{\frac{x_6}{8}} \\
\hline
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_{\boxed{3}}}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$

$$\boxed{x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}}$$

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

- Substitute this into x_2 in the other three equations

Extended Example: Iteration 4 (=after Iteration 3)

$$\begin{array}{rcrcrcrcr} z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\ x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\ x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\ x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & & \end{array}$$

Extended Example: Iteration 4

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

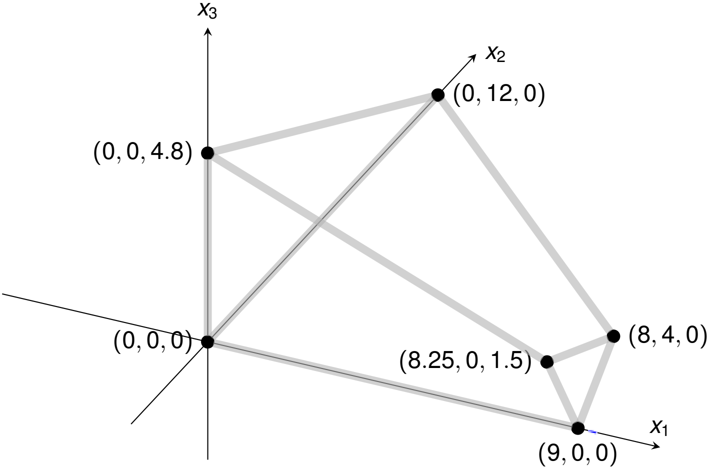
Extended Example: Iteration 4

All coefficients are negative, and hence this basic solution is **optimal**!

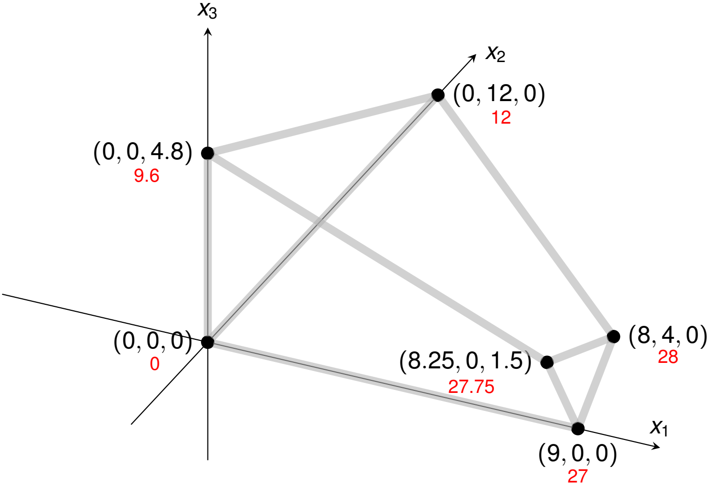
$$
\begin{aligned}
z &= 28 \boxed{- \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

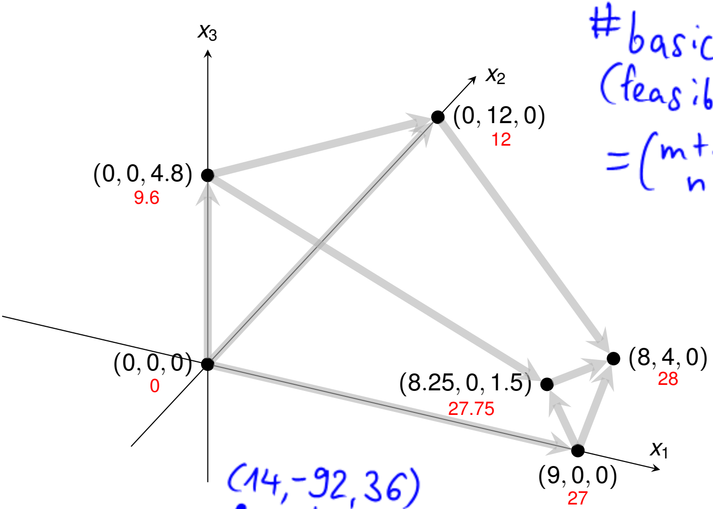
Extended Example: Visualization of SIMPLEX



Extended Example: Visualization of SIMPLEX



Extended Example: Visualization of SIMPLEX



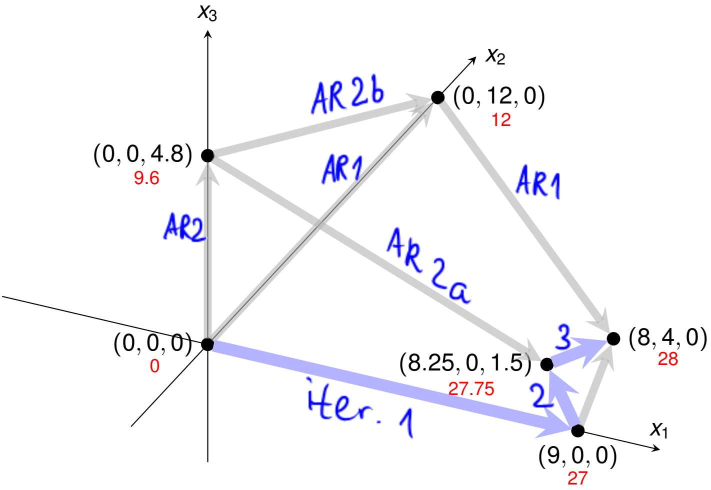
basic solutions (feasible/infeas.)

$= \binom{m+n}{n} = \binom{6}{3}$

$= 20$!

infeasible basic solution

Extended Example: Visualization of SIMPLEX



Extended Example: Alternative Runs (1/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_2 and x_5

Extended Example: Alternative Runs (1/2) (AR 1 in the illustration)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

$\Big\downarrow$ Switch roles of x_2 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}$$

Extended Example: Alternative Runs (1/2)

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

$\downarrow$ Switch roles of x_2 and x_5

$$\begin{array}{rcrcrcrcr} z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\ x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\ x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\ x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2} \end{array}$$

$\downarrow$ Switch roles of x_1 and x_6

Extended Example: Alternative Runs (1/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_2 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}$$

Switch roles of x_1 and x_6

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Extended Example: Alternative Runs (2/2) (AR 2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

$\Big\downarrow$ Switch roles of x_3 and x_5

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rclclclcl}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

$$\begin{array}{rclclclcl}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

$\downarrow$ Switch roles of x_3 and x_5

$$\begin{array}{rcrcrcrcr} z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\ x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\ x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\ x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5} \end{array}$$

Switch roles of x_1 and x_6 $\swarrow$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rclclclcl}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rclclclcl}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

$$
\begin{array}{rclclclcl}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

Switch roles of x_2 and x_3

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}$$

2a)

Switch roles of x_1 and x_6

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

2 b)

Switch roles of x_2 and x_3

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

The Pivot Step Formally

Precondition: $a_{le} \neq 0$

(Simplex ensures $a_{le} > 0$!)

```
PIVOT$(N, B, A, b, c, v, l, e)$
// Compute the coefficients of the equation for new basic variable $x_e$.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1/a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e\widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
$\widehat{c}_l = -c_e\widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

$$x_l = b_l - a_{l1}\cdot x_1 - a_{l2}\cdot x_2 - \dots - a_{le}\cdot x_e$$

$$x_e = \frac{b_l}{a_{le}} - \frac{a_{l1}}{a_{le}} x_1 - \frac{a_{l2}}{a_{le}}\cdot x_2 - \dots - \frac{1}{a_{le}}\cdot x_l$$

$$x_3 = b_3 - a_{31}\cdot x_1 - a_{32}\cdot x_2 - \dots - a_{3e}\cdot x_e$$

$$= \left(b_3 - a_{3l}\cdot \frac{b_l}{a_{le}}\right) - \left(a_{31} - a_{3e}\cdot \frac{a_{l1}}{a_{le}}\right) x_1$$

$$- \left(a_{32} - a_{3l}\cdot \frac{a_{l2}}{a_{le}}\right)\cdot x_2 - \dots$$

$$- \left(-\frac{a_{3e}}{a_{le}}\right)\cdot x_l$$

The Pivot Step Formally

```
PIVOT$(N, B, A, b, c, v, l, e)$
    // Compute the coefficients of the equation for new basic variable $x_e$.
    let $\widehat{A}$ be a new $m \times n$ matrix
    $\widehat{b}_e = b_l / a_{le}$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ej} = a_{lj} / a_{le}$
    $\widehat{a}_{el} = 1 / a_{le}$
    // Compute the coefficients of the remaining constraints.
    for each $i \in B - \{l\}$
        $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
        for each $j \in N - \{e\}$
            $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
        $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
    // Compute the objective function.
    $\widehat{v} = v + c_e\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
    $\widehat{c}_l = -c_e\widehat{a}_{el}$
    // Compute new sets of basic and nonbasic variables.
    $\widehat{N} = N - \{e\} \cup \{l\}$
    $\widehat{B} = B - \{l\} \cup \{e\}$
    return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

Rewrite “tight” equation for enterring variable x_e.

The Pivot Step Formally

```
PIVOT$(N, B, A, b, c, v, l, e)$
    // Compute the coefficients of the equation for new basic variable $x_e$.
    let $\widehat{A}$ be a new $m \times n$ matrix
    $\widehat{b}_e = b_l / a_{le}$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ej} = a_{lj} / a_{le}$
    $\widehat{a}_{el} = 1/a_{le}$
    // Compute the coefficients of the remaining constraints.
    for each $i \in B - \{l\}$
        $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
        for each $j \in N - \{e\}$
            $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
        $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
    // Compute the objective function.
    $\widehat{v} = v + c_e\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
    $\widehat{c}_l = -c_e\widehat{a}_{el}$
    // Compute new sets of basic and nonbasic variables.
    $\widehat{N} = N - \{e\} \cup \{l\}$
    $\widehat{B} = B - \{l\} \cup \{e\}$
    return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

The Pivot Step Formally

```
PIVOT($N, B, A, b, c, v, l, e$)
// Compute the coefficients of the equation for new basic variable $x_e$.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie} \widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

The Pivot Step Formally

```
PIVOT($N, B, A, b, c, v, l, e$)
// Compute the coefficients of the equation for new basic variable $x_e$.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e\widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
$\widehat{c}_l = -c_e\widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

Update non-basic and basic variables

The Pivot Step Formally

```
PIVOT$(N, B, A, b, c, v, l, e)$
// Compute the coefficients of the equation for new basic variable $x_e$.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
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        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
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// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$
```

Need that $a_{le} \neq 0$!

Rewrite "tight" equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

Update non-basic and basic variables

Effect of the Pivot Step

Lemma 29.1

Consider a call to $\textsc{Pivot}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

Effect of the Pivot Step

→ just summarizing previous pseudocode

Lemma 29.1

Consider a call to PIVOT(N, B, A, b, c, v, l, e) in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

1. $\underline{\overline{x}_j = 0}$ for each $j \in \widehat{N}$.
2. $\underline{\overline{x}_e = b_l/a_{le}}$.
3. $\underline{\overline{x}_i = b_i - a_{ie}\widehat{b}_e}$ for each $i \in \widehat{B} \setminus \{e\}$.

Effect of the Pivot Step

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Consider a call to $\textsc{Pivot}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

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Proof:

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Proof:

1. holds since the basic solution always sets all non-basic variables to zero.

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Proof:

1. holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint

$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j,$$

Effect of the Pivot Step

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$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j,$$

   we have $\overline{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Hence $\overline{x}_e = \widehat{b}_e = b_l / a_{le}$.
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Proof:

1. holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint

$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j,$$

we have $\overline{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Hence $\overline{x}_e = \widehat{b}_e = b_l / a_{le}$.

3. After the substituting in the other constraints, we have

$$\overline{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e. \qquad \square$$

Formalizing the Simplex Algorithm: Questions

Questions:

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Formalizing the Simplex Algorithm: Questions

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- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Example before was a particularly nice one!

The formal procedure SIMPLEX

```
SIMPLEX$(A, b, c)$
(N, B, A, b, c, ν) = INITIALIZE-SIMPLEX$(A, b, c)$
let $\Delta$ be a new vector of length $n$
while some index $j \in N$ has $c_j > 0$
    choose an index $e \in N$ for which $c_e > 0$
    for each index $i \in B$
        if $a_{ie} > 0$
            $\Delta_i = b_i / a_{ie}$
        else $\Delta_i = \infty$
    choose an index $l \in B$ that minimizes $\Delta_i$
    if $\Delta_l == \infty$
        return "unbounded"
    else $(N, B, A, b, c, \nu)$ = PIVOT$(N, B, A, b, c, \nu, l, e)$
for $i = 1$ to $n$
    if $i \in B$
        $\bar{x}_i = b_i$
    else $\bar{x}_i = 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

The formal procedure SIMPLEX

→ "Black Box" (for now)

```
SIMPLEX$(A, b, c)$
$(N, B, A, b, c, \nu) = \boxed{\textsc{Initialize-Simplex}(A, b, c)}$
let $\Delta$ be a new vector of length $n$
while some index $j \in N$ has $c_j > 0$
    choose an index $e \in N$ for which $c_e > 0$
    for each index $i \in B$
        if $a_{ie} > 0$
            $\Delta_i = b_i / a_{ie}$
        else $\Delta_i = \infty$
    choose an index $l \in B$ that minimizes $\Delta_i$
    if $\Delta_l == \infty$
        return "unbounded"
    else $(N, B, A, b, c, \nu) = \textsc{Pivot}(N, B, A, b, c, \nu, l, e)$
for $i = 1$ to $n$
    if $i \in B$
        $\bar{x}_i = b_i$
    else $\bar{x}_i = 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

The formal procedure SIMPLEX

```
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for $i = 1$ to $n$
    if $i \in B$
        $\bar{x}_i = b_i$
    else $\bar{x}_i = 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

→ potentially many choices!

The formal procedure SIMPLEX

```
SIMPLEX$(A, b, c)$
$(N, B, A, b, c, v)$ = INITIALIZE-SIMPLEX$(A, b, c)$
let $\Delta$ be a new vector of length $n$
**while** some index $j \in N$ has $c_j > 0$
    choose an index $e \in N$ for which $c_e > 0$
    **for** each index $i \in B$
        **if** $a_{ie} > 0$
            $\Delta_i = b_i / a_{ie}$
        **else** $\Delta_i = \infty$
    choose an index $l \in B$ that minimizes $\Delta_i$
    **if** $\Delta_l == \infty$
        **return** "unbounded"
    **else** $(N, B, A, b, c, v)$ = PIVOT$(N, B, A, b, c, v, l, e)$
**for** $i = 1$ **to** $n$
    **if** $i \in B$
        $\bar{x}_i = b_i$
    **else** $\bar{x}_i = 0$
**return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

The formal procedure SIMPLEX

```
SIMPLEX$(A, b, c)$
 1  $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX$(A, b, c)$
 2  let $\Delta$ be a new vector of length $n$
 3  while some index $j \in N$ has $c_j > 0$
 4      choose an index $e \in N$ for which $c_e > 0$
 5      for each index $i \in B$
 6          if $a_{ie} > 0$
 7              $\Delta_i = b_i / a_{ie}$
 8          else $\Delta_i = \infty$
 9      choose an index $l \in B$ that minimizes $\Delta_i$
10      if $\Delta_l == \infty$
11          return "unbounded"
12      else $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
13  for $i = 1$ to $n$
14      if $i \in B$
15          $\bar{x}_i = b_i$
16      else $\bar{x}_i = 0$
17  return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
- Lines 6 – 9 pick the tightest constraint, associated with x_l
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

The formal procedure SIMPLEX

```
SIMPLEX$(A, b, c)$
 1  $(N, B, A, b, c, v) = $ INITIALIZE-SIMPLEX$(A, b, c)$
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 4      choose an index $e \in N$ for which $c_e > 0$
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10      if $\Delta_l == \infty$
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12      else $(N, B, A, b, c, v) = $ PIVOT$(N, B, A, b, c, v, l, e)$
13  for $i = 1$ to $n$
14      if $i \in B$
15          $\bar{x}_i = b_i$
16      else $\bar{x}_i = 0$
17  return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
- Lines 6 – 9 pick the tightest constraint, associated with x_l
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

Return corresponding solution.

The formal procedure SIMPLEX

```
SIMPLEX$(A, b, c)$
1  $(N, B, A, b, c, v)$ = INITIALIZE-SIMPLEX$(A, b, c)$
2  let $\Delta$ be a new vector of length $n$
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13 for $i = 1$ to $n$
14     if $i \in B$
15         $\bar{x}_i = b_i$
16     else $\bar{x}_i = 0$
17 return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$
```

Returns a slack form with a feasible basic solution (if it exists)

we will see Later that it is optimal!

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

The formal procedure SIMPLEX

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SIMPLEX$(A, b, c)$
1  $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX$(A, b, c)$
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```

Returns a slack form with a feasible basic solution (if it exists)

Proof is based on the following three-part loop invariant:

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

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8          else $\Delta_i = \infty$
9      choose an index $l \in B$ that minimizes $\Delta_i$
10     if $\Delta_l == \infty$
11         return "unbounded"
```

Returns a slack form with a feasible basic solution (if it exists)

elementary row operations (as in Gaussian Elimination)

Proof is based on the following three-part loop invariant:

1. the slack form is always equivalent to the one returned by INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$,
3. the basic solution associated with the (current) slack form is feasible.

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

Termination

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$$\begin{array}{rclllllll} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{array}{rcrrrrrrr}
z & = & & & x_1 & + & x_2 & + & x_3 \\
\boxed{x_4} & = & 8 & \boxed{- \quad x_1} & & - & x_2 & & \\
x_5 & = & & & & & x_2 & - & x_3
\end{array}$$

$\downarrow$ Pivot with x_1 entering and x_4 leaving

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{array}{rcrcrcrcr}
z & = & & & x_1 & + & x_2 & + & x_3 \\
x_4 & = & 8 & - & x_1 & - & x_2 & & \\
x_5 & = & & & & & x_2 & - & x_3
\end{array}$$

$\Big\downarrow$ Pivot with x_1 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr}
z & = & 8 & & & \boxed{+ \quad x_3} & & - & x_4 \\
x_1 & = & 8 & - & x_2 & & & - & x_4 \\
\boxed{x_5} & = & \boxed{} & & x_2 & \boxed{- \quad x_3} & & &
\end{array}$$

since $b_5 = 0$ next basic solution will be identical (in particular, objective value remains the same)

Termination

Degeneracy: One iteration of Simplex leaves the objective value unchanged.

$$\begin{array}{rcrcrcrcr} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

$\downarrow$ Pivot with x_1 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & 8 & & & + & x_3 & - & x_4 \\ x_1 & = & 8 & - & x_2 & & & - & x_4 \\ x_5 & = & & & x_2 & - & x_3 & & \end{array}$$

$\downarrow$ Pivot with x_3 entering and x_5 leaving

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{array}{rcrcrcrcr}
z & = & & & x_1 & + & x_2 & + & x_3 \\
x_4 & = & 8 & - & x_1 & - & x_2 & & \\
x_5 & = & & & & & x_2 & - & x_3
\end{array}$$

Pivot with x_1 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr}
z & = & 8 & & & + & x_3 & - & x_4 \\
x_1 & = & 8 & - & x_2 & & & - & x_4 \\
x_5 & = & & & x_2 & - & x_3 & &
\end{array}$$

Pivot with x_3 entering and x_5 leaving

$$\begin{array}{rcrcrcrcr}
z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\
x_1 & = & 8 & - & x_2 & - & x_4 & & \\
x_3 & = & & & x_2 & & & - & x_5
\end{array}$$

Pivot with x_2 ent. and x_1 leav.

$\dashrightarrow (x_1, x_2, x_3) = (0, 8, 8)$

$z = 16 + \dots$

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

$$\begin{aligned} z &= x_1 + x_2 + x_3 \\ x_4 &= 8 - x_1 - x_2 \\ x_5 &= x_2 - x_3 \end{aligned}$$

Pivot with x_1 entering and x_4 leaving

$$\begin{aligned} z &= 8 + x_3 - x_4 \\ x_1 &= 8 - x_2 - x_4 \\ x_5 &= x_2 - x_3 \end{aligned}$$

Pivot with x_3 entering and x_5 leaving

Cycling: Slack forms at two iterations are identical, and SIMPLEX fails to terminate!

$$\begin{aligned} z &= 8 + x_2 - x_4 - x_5 \\ x_1 &= 8 - x_2 - x_4 \\ x_3 &= x_2 - x_5 \end{aligned}$$

Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

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It is theoretically possible, but very rare in practice.

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Anti-Cycling Strategies

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Anti-Cycling Strategies

among j with $c_j > 0$

1. Bland's rule: Choose entering variable with smallest index

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Anti-Cycling Strategies

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Replace each b_i by $\widehat{b_i} = b_i + \boxed{\epsilon_i}$ where $\epsilon_i \gg \epsilon_{i+1}$ are all small.

"random" noise

Termination and Running Time

It is theoretically possible, but very rare in practice.

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Lemma 29.7

Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\boxed{\binom{n+m}{m} \text{ iterations.}}$

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Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\binom{n+m}{m}$ iterations.

Every set B of basic variables uniquely determines a slack form, and there are at most $\binom{n+m}{m}$ unique slack forms.

Outline

Introduction

Standard and Slack Forms

Formulating Problems as Linear Programs

Simplex Algorithm

Finding an Initial Solution

Finding an Initial Solution

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & \boxed{-4} \\
 & & x_1, x_2 & & \geq & 0
\end{array}$$

Finding an Initial Solution

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
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 & & x_1, x_2 & & \geq & 0
\end{array}$$

Conversion into slack form

Finding an Initial Solution

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 \\
\text{subject to} & \\
& \begin{array}{rcrcr}
2x_1 & - & x_2 & \leq & 2 \\
x_1 & - & 5x_2 & \leq & -4 \\
 & x_1, x_2 & & \geq & 0
\end{array}
\end{array}
$$

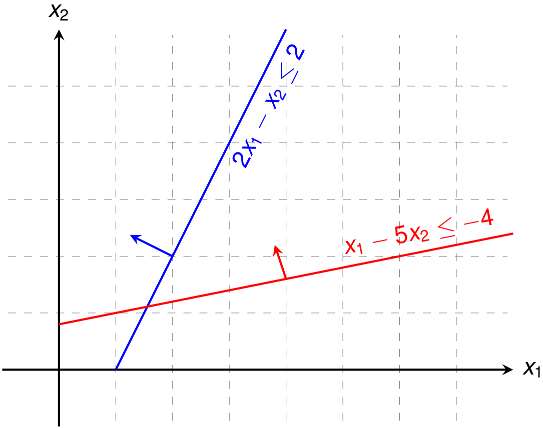
Conversion into slack form

$$
\begin{array}{rcrcrcr}
z & = & & & 2x_1 & - & x_2 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 \\
x_4 & = & -4 & - & x_1 & + & 5x_2
\end{array}
$$

Basic solution $(x_1, x_2, x_3, x_4) = (0, 0, 2, -4)$ is not feasible!

Geometric Illustration

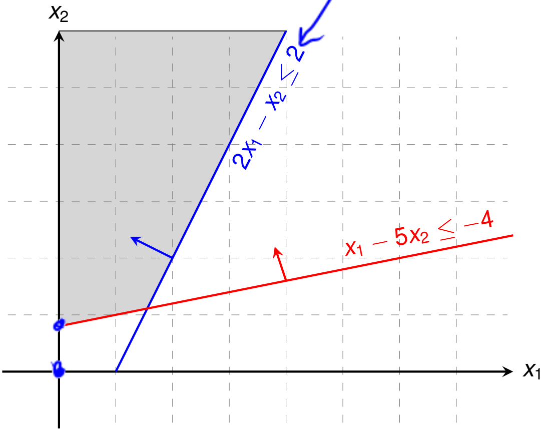
$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4 \\
 & & x_1, x_2 & & \geq & 0
\end{array}$$



Geometric Illustration

$$\begin{array}{lrcrcr}
\text{maximize} & \boxed{2x_1 & - & x_2} & & \\
\text{subject to} & & & & & \\
& \boxed{2x_1 & - & x_2} & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}$$

$\}$ all points (x_1, x_2) with $2x_1 - x_2 = 2$ are optimal



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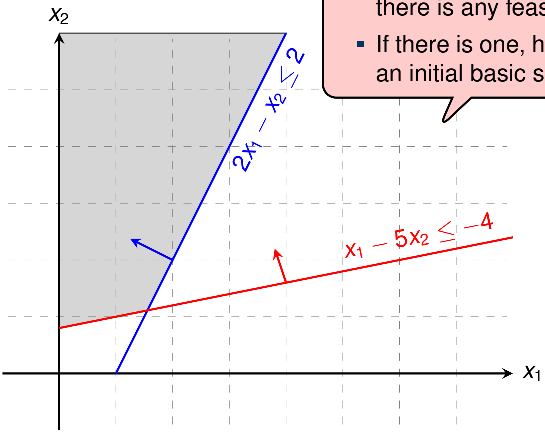

Questions:

- How to determine whether there is any feasible solution?
- If there is one, how to determine an initial basic solution?

Formulating an Auxiliary Linear Program

$$\begin{array}{lrcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & & & & \\ & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\ & x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \end{array}$$

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Formulating an Auxiliary Linear Program

"Relax" each constraint

$$\begin{array}{ll} \text{maximize} & \boxed{-x_0} \\ \text{subject to} & \end{array}$$

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j \boxed{- x_0} & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\ x_j & \geq & 0 & \text{for } j = 0, 1, \ldots, n \end{array}$$

minimize x_0, the "distance" from being feasible

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\ x_j &\geq 0 \quad \text{for } j = 1, 2, \ldots, n \end{aligned}$$

$\Big\downarrow$ Formulating an Auxiliary Linear Program

maximize $-x_0$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j - x_0 &\leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\ x_j &\geq 0 \quad \text{for } j = 0, 1, \ldots, n \end{aligned}$$

Lemma 29.11

Let L_{aux} be the auxiliary LP of a linear program L in standard form. Then L is feasible if and only if the optimal objective value of L_{aux} is 0.

Formulating an Auxiliary Linear Program

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Proof.

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\begin{array}{rcll} \boxed{\sum_{j=1}^{n} a_{ij}x_j \quad \leq \quad b_i} & & & \text{for } i = 1, 2, \ldots, m, \\ x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \end{array}$$

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x_j & \geq & 0 & \text{for } j = \underline{0,} 1, \ldots, n
\end{array}
$$

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  - $\underline{\overline{x}_0 = 0}$ combined with $\overline{x}$ is a feasible solution to L_{aux} with objective value 0.

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\ x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \end{array}$$

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- "⇐": Suppose that the optimal objective value of L_{aux} is 0
  - Then $\overline{x}_0 = 0$, and the remaining solution values $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ satisfy L.

Formulating an Auxiliary Linear Program

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INITIALIZE-SIMPLEX

```
INITIALIZE-SIMPLEX$(A, b, c)$
let $k$ be the index of the minimum $b_i$
if $b_k \geq 0$                         // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form $L_{\text{aux}}$ by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let $(N, B, A, b, c, \nu)$ be the resulting slack form for $L_{\text{aux}}$
$l = n + k$
// $L_{\text{aux}}$ has $n + 1$ nonbasic variables and $m$ basic variables.
$(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for $L_{\text{aux}}$.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to $L_{\text{aux}}$ is found
if the optimal solution to $L_{\text{aux}}$ sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of $L_{\text{aux}}$, remove $x_0$ from the constraints and
        restore the original objective function of $L$, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

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10  iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
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11  if the optimal solution to $L_{\text{aux}}$ sets $\bar{x}_0$ to 0
12      if $\bar{x}_0$ is basic
13          perform one (degenerate) pivot to make it nonbasic
14      from the final slack form of $L_{\text{aux}}$, remove $x_0$ from the constraints and
            restore the original objective function of $L$, but replace each basic
            variable in this objective function by the right-hand side of its
            associated constraint
15      return the modified final slack form
16  else return "infeasible"
```

ℓ will be the leaving variable so that x_ℓ has the most negative value.

↓

this ensures basic solution becomes feasible after line 8 (non-negative)

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

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    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of $L_{\text{aux}}$, remove $x_0$ from the constraints and
        restore the original objective function of $L$, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

Initialize-Simplex

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

```
INITIALIZE-SIMPLEX$(A, b, c)$
let $k$ be the index of the minimum $b_i$
if $b_k \geq 0$                    // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form $L_{\text{aux}}$ by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let $(N, B, A, b, c, \nu)$ be the resulting slack form for $L_{\text{aux}}$
$l = n + k$
// $L_{\text{aux}}$ has $n+1$ nonbasic variables and $m$ basic variables.
$(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for $L_{\text{aux}}$.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to $L_{\text{aux}}$ is found
if the optimal solution to $L_{\text{aux}}$ sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of $L_{\text{aux}}$, remove $x_0$ from the constraints and
        restore the original objective function of $L$, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

This pivot step does not change the value of any variable.

↓

because $\overline{x}_0 = 0$!

Example of Initialize-Simplex (1/3)

$$
\begin{array}{llrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & \boxed{-4} \longrightarrow \text{"canonical" basic solution is not feasible!} \\
 & & x_1, x_2 & & \geq & 0
\end{array}
$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4 \\
 & & x_1, x_2 & & \geq & 0
\end{array}$$

$\downarrow$ Formulating the auxiliary linear program

Example of INITIALIZE-SIMPLEX (1/3)

$$
\begin{array}{llcrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
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& & x_1, x_2 & & \geq & 0
\end{array}
$$

Formulating the auxiliary linear program

$$
\begin{array}{llcrcrcr}
\text{maximize} & & & & & -x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& x_1, x_2, x_0 & & & & & \geq & 0
\end{array}
$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{subject to} & \\ & \begin{array}{rcrcr} 2x_1 & - & x_2 & \leq & 2 \\ x_1 & - & 5x_2 & \leq & -4 \\ & x_1, x_2 & & \geq & 0 \end{array} \end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & \\ & \begin{array}{rcrcrcr} 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\ x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\ & x_1, x_2, x_0 & & & & \geq & 0 \end{array} \end{array}$$

Converting into slack form

Example of INITIALIZE-SIMPLEX (1/3)

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 \\
\text{subject to} & \\
& \begin{array}{rcrcr}
2x_1 & - & x_2 & \leq & 2 \\
x_1 & - & 5x_2 & \leq & -4 \\
 & x_1, x_2 & & \geq & 0
\end{array}
\end{array}
$$

$$\Big\downarrow$$

Formulating the auxiliary linear program

$$
\begin{array}{ll}
\text{maximize} & -x_0 \\
\text{subject to} & \\
& \begin{array}{rcrcrcr}
2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
 & x_1, x_2, x_0 & & & & \geq & 0
\end{array}
\end{array}
$$

$$\Big\downarrow$$

Converting into slack form

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{llcrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
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\end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{llcrcrcr}
\text{maximize} & & & & & -x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& & x_1, x_2, x_0 & & & & \geq & 0
\end{array}$$

Basic solution
$(0, 0, 0, 2, -4)$ not feasible!

Converting into slack form

$$\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{aligned} z &= \boxed{-\ x_0} \\ x_3 &= 2 - 2x_1 + x_2 + x_0 \\ x_4 &= \boxed{-4} - x_1 + 5x_2 + x_0 \end{aligned}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\underline{\begin{array}{rcrcrcrcr} z & = & & & & & & \boxed{-} & \boxed{x_0} \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}}$$

Pivot with x_0 entering and x_4 leaving

(this "degenerate" pivot step ensures all basic variables are non-negative)

Example of Initialize-Simplex (2/3)

$$\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}$$

$$\Big\downarrow \quad \text{Pivot with } x_0 \text{ entering and } x_4 \text{ leaving}$$

$$\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & \boxed{4} & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & \boxed{6} & - & x_1 & - & 4x_2 & + & x_4
\end{array}$$

$$\downarrow$$

$$6 = 2 + 4$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rclrcrcrcr}
z & = & & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rclrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{array}{rcrcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{aligned} z &= && && && - x_0 \\ x_3 &= 2 &&- 2x_1 &&+ x_2 &&+ x_0 \\ x_4 &= -4 &&- x_1 &&+ 5x_2 &&+ x_0 \end{aligned}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{aligned} z &= -4 &&- x_1 &&+ 5x_2 &&- x_4 \\ x_0 &= 4 &&+ x_1 &&- 5x_2 &&+ x_4 \\ x_3 &= 6 &&- x_1 &&- 4x_2 &&+ x_4 \end{aligned}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{aligned} z &= &&- x_0 && && \\ x_2 &= \frac{4}{5} &&- \frac{x_0}{5} &&+ \frac{x_1}{5} &&+ \frac{x_4}{5} \\ x_3 &= \frac{14}{5} &&+ \frac{4x_0}{5} &&- \frac{9x_1}{5} &&+ \frac{x_4}{5} \end{aligned}$$

Optimal solution has $x_0 = 0$, hence the initial problem was feasible!

Example of INITIALIZE-SIMPLEX (3/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}
$$

Example of Initialize-Simplex (3/3)

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Set $x_0 = 0$ and express objective function by non-basic variables

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{array}{rcl}
z & = & -x_0 \\
x_2 & = & \boxed{\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5}} \\
x_3 & = & \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{array}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$$2x_1 - 2x_2 = 2x_1 - \left(\frac{4}{5} - \cancel{\frac{x_0}{5}} + \frac{x_1}{5} + \frac{x_4}{5}\right)$$

$$\begin{array}{rcl}
z & = & -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\
x_2 & = & \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5}
\end{array}$$

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{aligned} z &= & & - & x_0 & & & & \\ x_2 &= & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 &= & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{aligned}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$$2x_1 - 2x_2 = 2x_1 - \left(\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5}\right)$$

$$\boxed{\begin{aligned} z &= -\frac{4}{5} & + & \frac{9x_1}{5} & - & \frac{x_4}{5} \\ x_2 &= \frac{4}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 &= \frac{14}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{aligned}}$$

Slack form returned by INITIALIZE-SIMPLEX

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Main Loop of SIMPLEX can be now executed

Example of Initialize-Simplex (3/3)

$$\begin{aligned} z &= & & - & x_0 & & & & \\ x_2 &= & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 &= & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{aligned}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$2x_1 - 2x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

$$\begin{aligned} z &= & -\frac{4}{5} & + & \frac{9x_1}{5} & - & \frac{x_4}{5} \\ x_2 &= & \frac{4}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 &= & \frac{14}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{aligned}$$

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Lemma 29.12

If a linear program L has no feasible solution, then Initialize-Simplex returns “infeasible”. Otherwise, it returns a valid slack form for which the basic solution is feasible.

Fundamental Theorem of Linear Programming

Theorem 29.13

Any linear program *L*, given in standard form, either

1. has an optimal solution with a finite objective value,
2. is infeasible, or
3. is unbounded.

If *L* is infeasible, SIMPLEX returns "infeasible". If *L* is unbounded, SIMPLEX returns "unbounded". Otherwise, SIMPLEX returns an optimal solution with a finite objective value.

proof non-trivial and requires concept of "dual Linear program".
[CLRS3, Chapter 29.4]

Linear Programming and Simplex: Summary

Linear Programming

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- extremely versatile tool for modelling problems of all kinds

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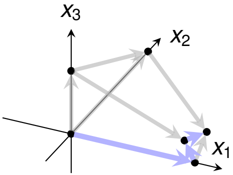
Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m+n)$



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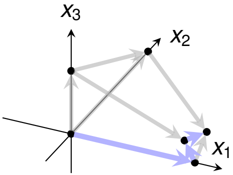
- In theory: even with anti-cycling may need exponential time



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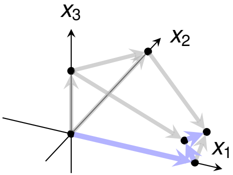

Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?

↳ different rules lead to different instantiations of SIMPLEX

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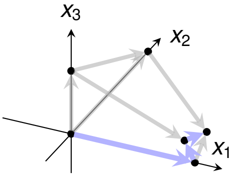

Polynomial-Time Algorithms

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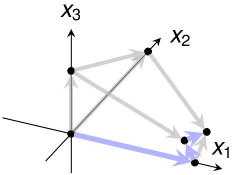

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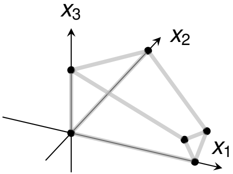
Polynomial-Time Algorithms

- Interior-Point Methods: traverses the interior of the feasible set of solutions (not just vertices!)



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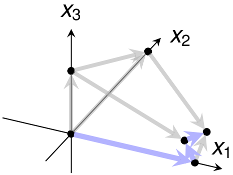
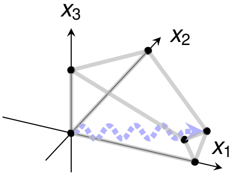


Research Problem: Is there a pivoting rule which makes SIMPLEX a polynomial-time algorithm?

Polynomial-Time Algorithms

- Interior-Point Methods: traverses the interior of the feasible set of solutions (not just vertices!)

↓ more complicated



IV. Approximation Algorithms: Covering Problems

Thomas Sauerwald

Easter 2015

Outline

Introduction

Vertex Cover

The Set-Covering Problem

Motivation

Many fundamental problems are **NP-complete**, yet they are too important to be abandoned.

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Examples: HAMILTON, 3-SAT, VERTEX-COVER, KNAPSACK,...

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Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

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3. Develop algorithms which find near-optimal solutions in polynomial-time.

Motivation

Many fundamental problems are **NP-complete**, yet they are too important to be abandoned.

Examples: Hamilton, 3-SAT, Vertex-Cover, Knapsack,. . .

Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

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This covers both maximization and minimization problems.

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- Minimization problem: $\frac{C}{C^*} \geq 1$

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- Maximization problem: $\frac{C^*}{C} \geq 1$
- Minimization problem: $\frac{C}{C^*} \geq 1$

This covers both maximization and minimization problems.

For many problems: tradeoff between runtime and approximation ratio.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

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This covers both maximization and minimization problems.

For many problems: tradeoff between runtime and approximation ratio.

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Performance Ratios for Approximation Algorithms

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Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Vertex-Cover Problem

Vertex Cover Problem

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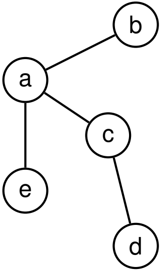
- Goal: Find a minimum-cardinality subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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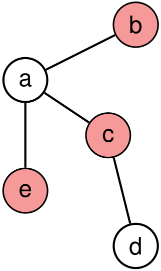

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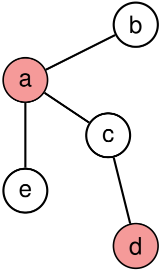

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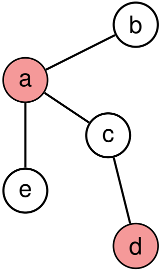

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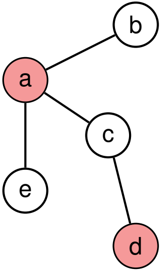

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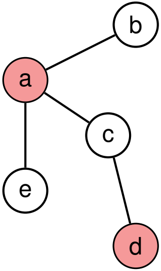

Applications:

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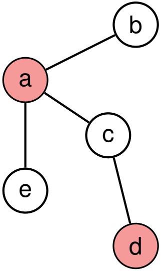

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task

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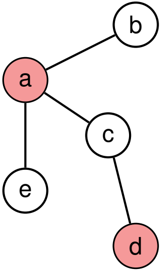

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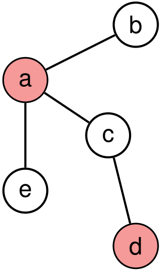

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Perform all tasks with the minimal amount of resources
- Extensions: weighted ~~edges~~ vertices or hypergraphs

An Approximation Algorithm based on Greedy

```
APPROX-VERTEX-COVER($G$)
$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
    let $(u, v)$ be an arbitrary edge of $E'$ → potentially many choices
    $C = C \cup \{u, v\}$
    remove from $E'$ every edge incident on either $u$ or $v$
return $C$
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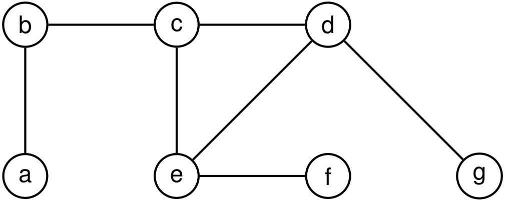

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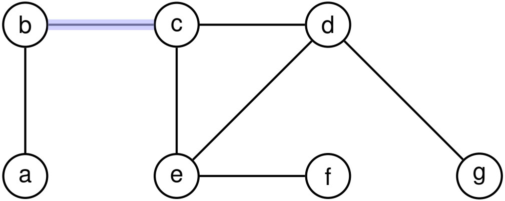

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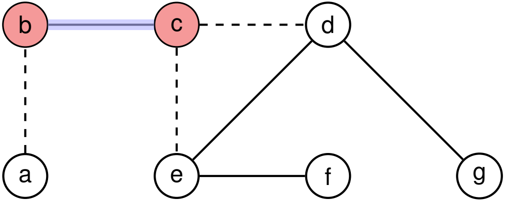

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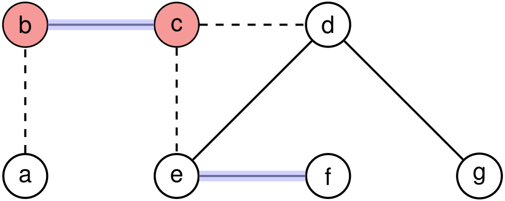

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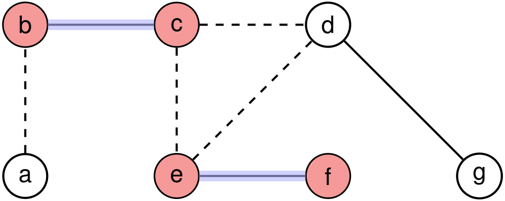

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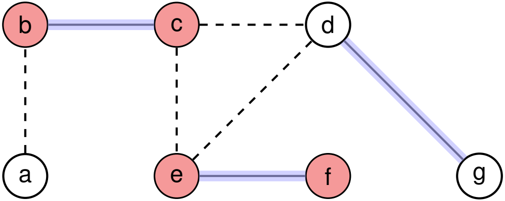

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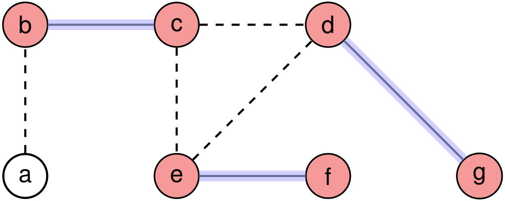

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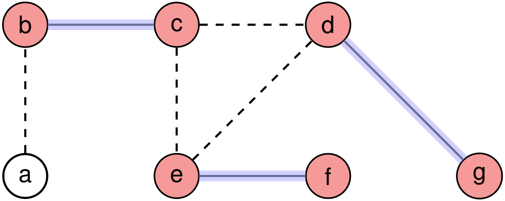

APPROX-VERTEX-COVER produces a set of size 6.

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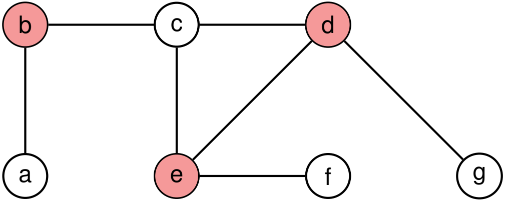

The optimal solution has size 3.

Analysis of Greedy for Vertex Cover

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APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

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We can bound the size of the returned solution without knowing the (size of an) optimal solution!

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A "vertex-based" Greedy-Alg. which adds one vertex at each iteration does not achieve approx. ratio of 2 (Exercise 18)

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Analysis of Greedy for Vertex Cover

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APPROX-VERTEX-COVER($G$)
$C = \emptyset$
$E' = G.E$
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Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V + E)$ (using adjacency lists to represent E')
- Let $A \subseteq E$ denote the set of edges picked in line 4
- Every optimal cover C^* must include at least one endpoint of edges in A, and edges in A do not share a common endpoint: $\boxed{|C^*| \geq |A|}$
- Every edge in A contributes 2 vertices to $|C|$: $\boxed{|C| = 2|A| \leq 2|C^*|.}$ □

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```

We can bound the size of the returned solution without knowing the (size of an) optimal solution!

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof: → Key idea: A is a maximal matching

- Running time is $O(V + E)$ (using adjacency lists to represent E')
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Solving Special Cases

Strategies to cope with NP-complete problems

1. If inputs are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

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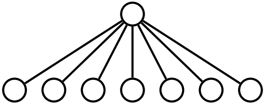

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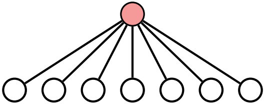

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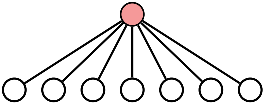

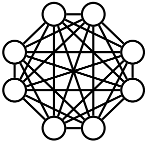

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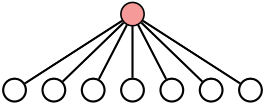

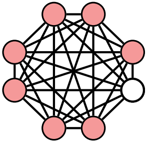

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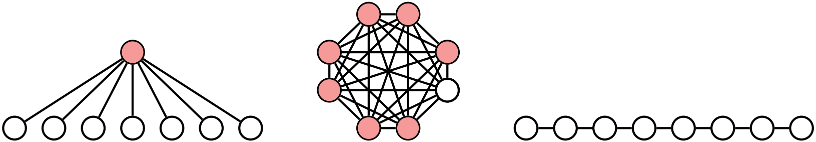

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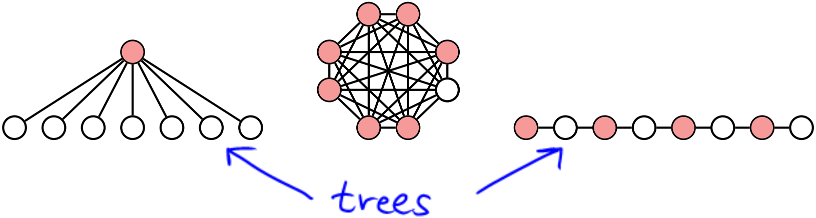
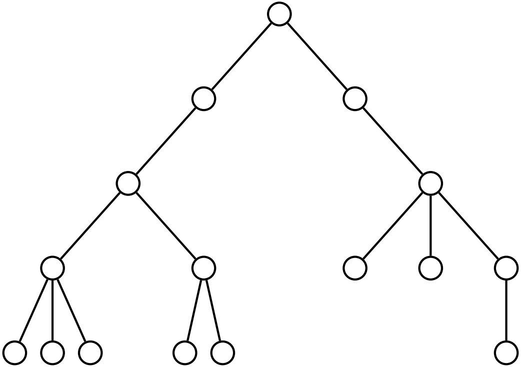


Vertex Cover on Trees



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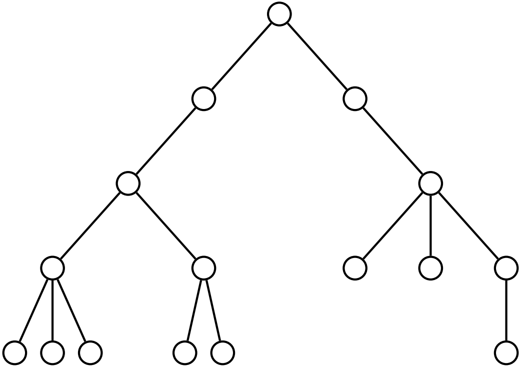
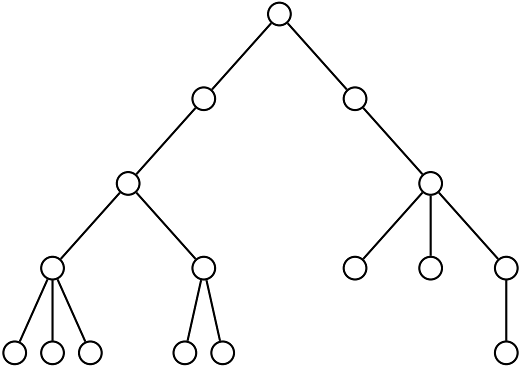


There exists an optimal vertex cover which does not include any leaves.

Vertex Cover on Trees



There exists an optimal vertex cover which does not include any leaves.

Exchange-Argument: Replace any leaf in the cover by its parent.

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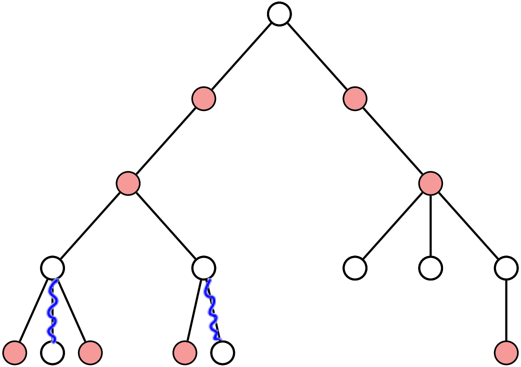

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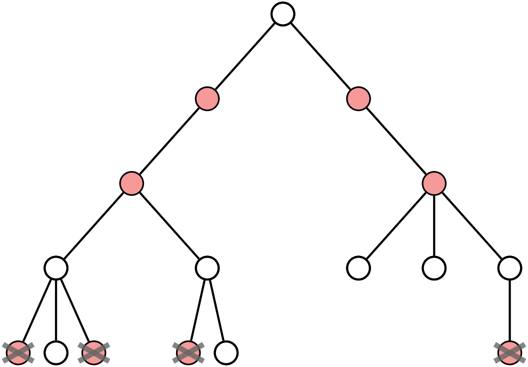

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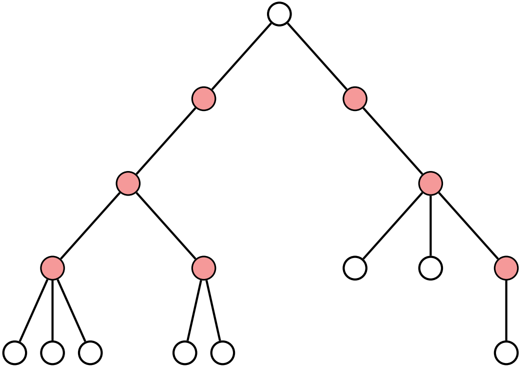

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Solving Vertex Cover on Trees

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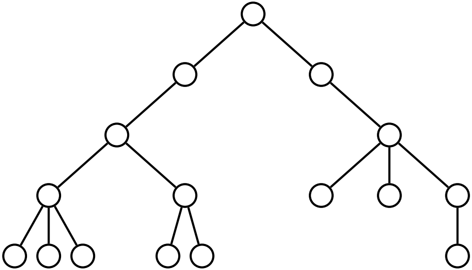
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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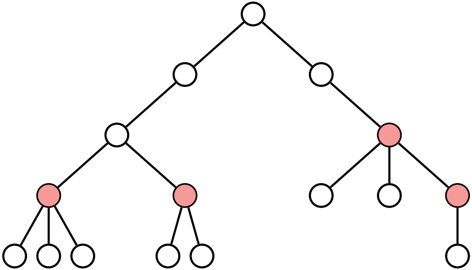

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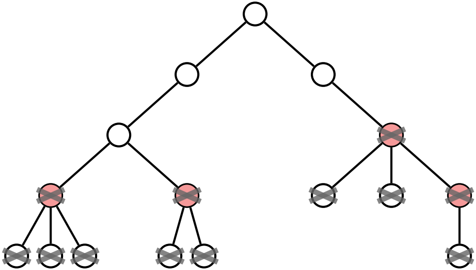

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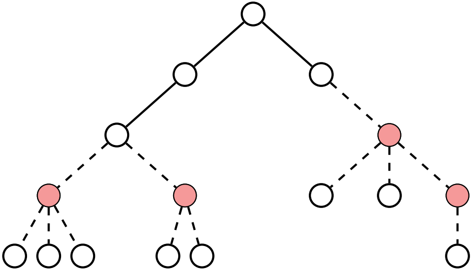

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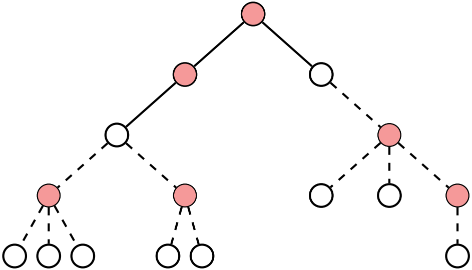

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C = ∅
while ∃ leaves in G
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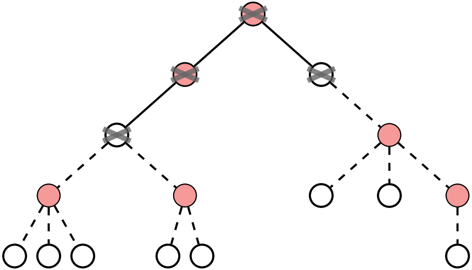

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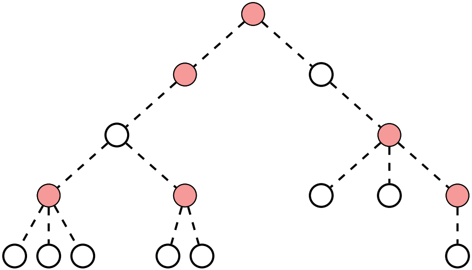

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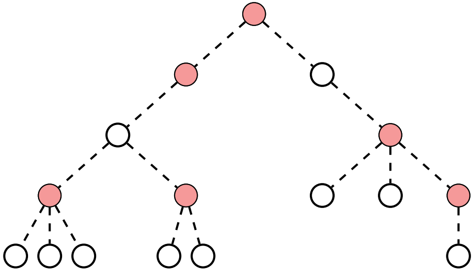

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory
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Focus on instances of where the minimum vertex cover is small, that is, smaller than some given integer k.

Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $(u, v) \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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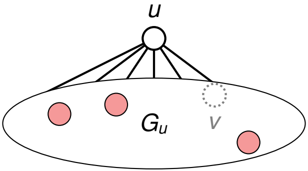

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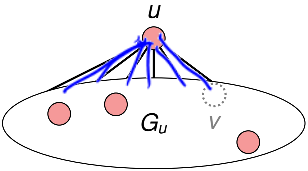
Adding u yields a vertex cover of G which is of size k



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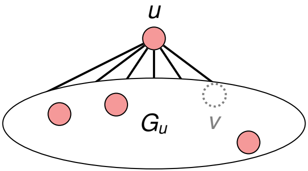
$\Rightarrow$ Assume G has a vertex cover C of size k, which contains, say u.



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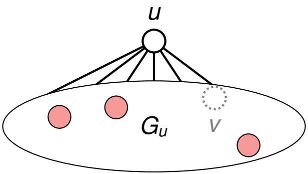
Removing u from C yields a vertex cover of G_u which is of size $k - 1$. □



A More Efficient Search Algorithm

```
VERTEX-COVER-SEARCH($G, k$)
If $E = \emptyset$ **return** $\{\perp\}$
If $k = 0$ and $E \neq \emptyset$ **return** $\emptyset$
Pick an arbitrary edge $(u, v) \in E$
$S_1 =$ VERTEX-COVER-SEARCH($G_u, k - 1$)
$S_2 =$ VERTEX-COVER-SEARCH($G_v, k - 1$)
**if** $S_1 \neq \emptyset$ **return** $S_1 \cup \{u\}$
**if** $S_2 \neq \emptyset$ **return** $S_2 \cup \{v\}$
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Correctness follows by the Substructure Lemma and induction.

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Running time:

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Running time:

- Depth k, branching factor 2

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Running time:

- Depth k, branching factor 2 ⇒ total number of calls is $O(2^k)$

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- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

A More Efficient Search Algorithm

```
Vertex-Cover-Search($G, k$)
If $E = \emptyset$ return $\{\perp\}$
If $k = 0$ and $E \neq \emptyset$ return $\emptyset$
Pick an arbitrary edge $(u, v) \in E$
$S_1$ = Vertex-Cover-Search($G_u, k - 1$)
$S_2$ = Vertex-Cover-Search($G_v, k - 1$)
if $S_1 \neq \emptyset$ return $S_1 \cup \{u\}$
if $S_2 \neq \emptyset$ return $S_2 \cup \{v\}$
return $\emptyset$
```

Running time:

- Depth k, branching factor 2 ⇒ total number of calls is $O(2^k)$
- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Set-Covering Problem

Set Cover Problem

- Given: set X of size n and family of subsets $\mathcal{F}$
- Goal: Find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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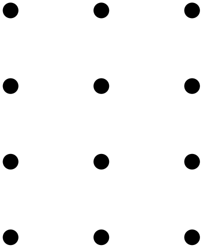

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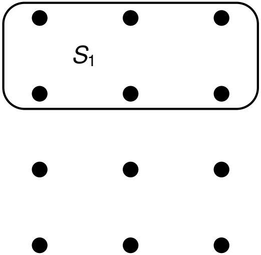

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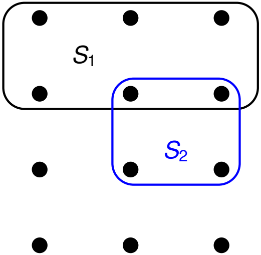

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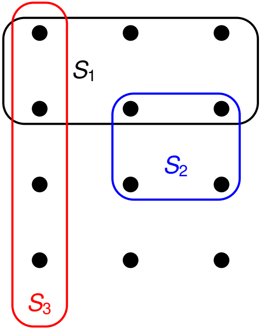

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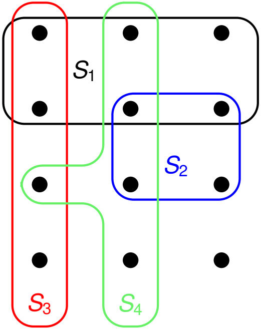

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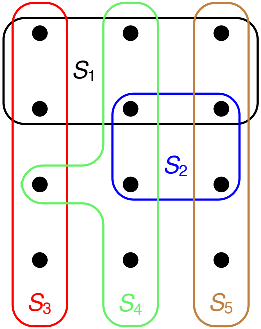

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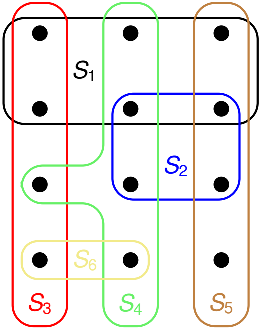

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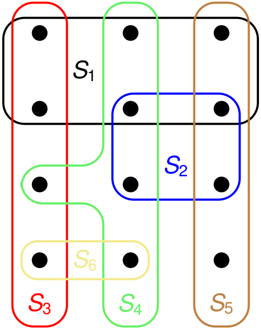

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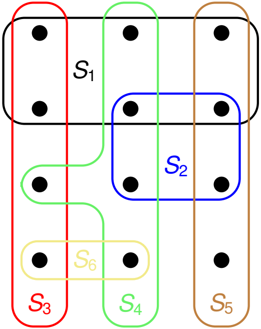

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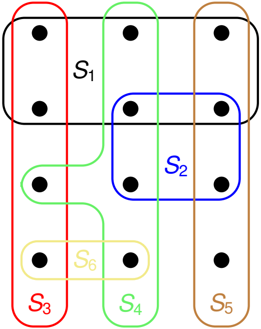

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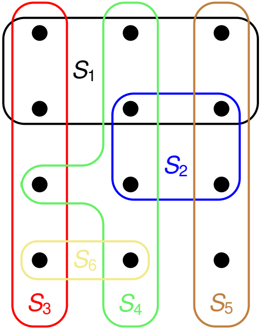

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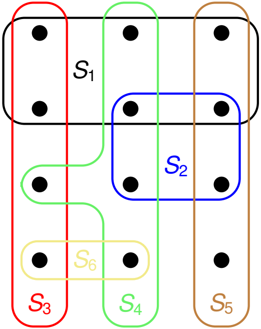

Remarks:

- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems

Greedy

Strategy: Pick the set S that covers the largest number of uncovered elements.

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$U = X$
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while $U \neq \emptyset$
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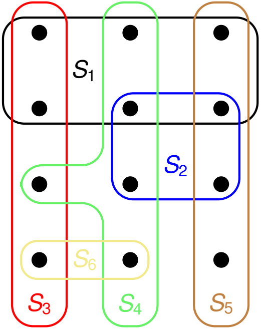

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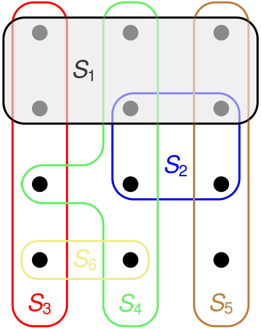

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```
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while U ≠ ∅
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    U = U − S
    C = C ∪ {S}
return C
```

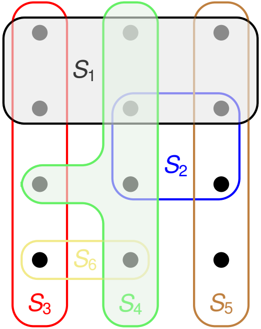

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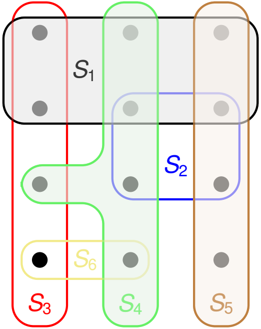

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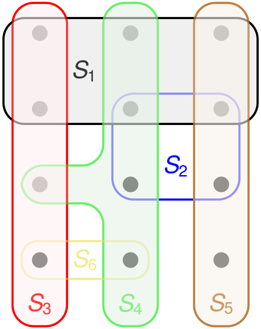

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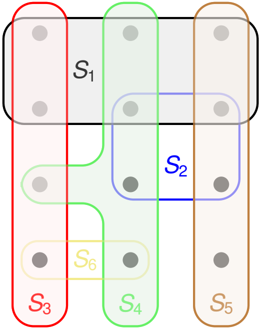

Greedy chooses S_1, S_4, S_5 and S_3 (or S_6), which is a cover of size 4.

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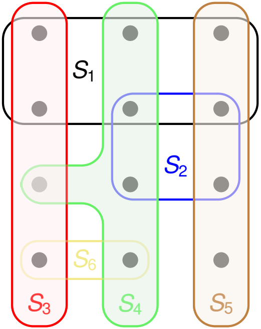

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Optimal cover is $\mathcal{C} = \{\underline{S_3, S_4, S_5}\}$

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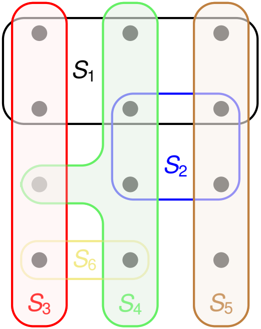
Can be easily implemented to run in time polynomial in $|X|$ and $|\mathcal{F}|$



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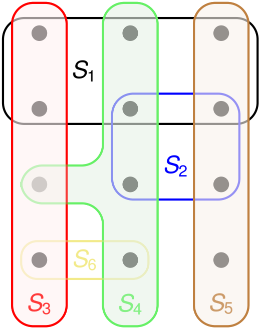

How good is the approximation ratio?

Approximation Ratio of Greedy

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\underline{\rho(n)}$-algorithm, where

$$\rho(n) = \underbrace{H(\max\{|S| : |S| \in \mathcal{F}\})}_{\text{in general, not a constant}}$$

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If an element x is covered for the first time by set S_i in iteration i, then

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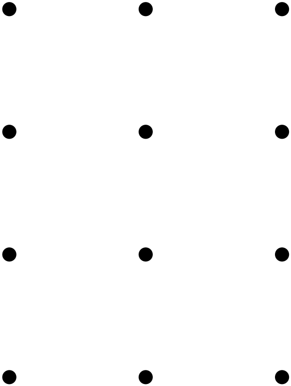
(in the mathematical analysis, S_i is the set chosen in iteration i – not to be confused with $S_1, S_2, \ldots, S_6$ in the example)

Illustration of Costs



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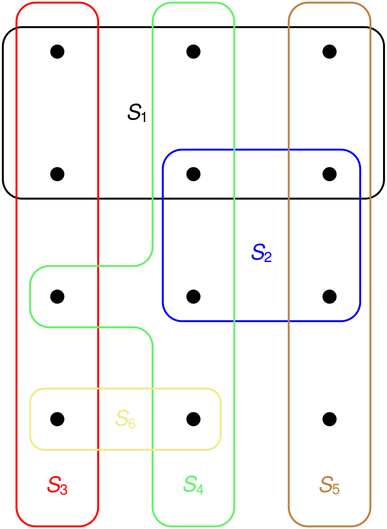

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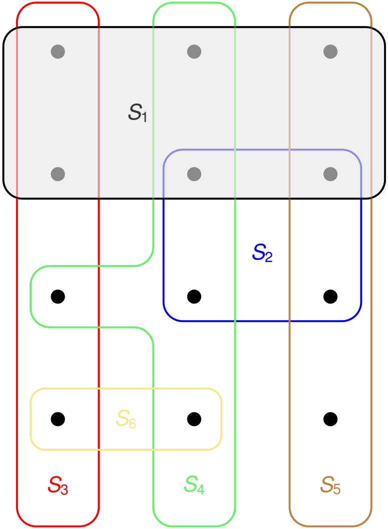

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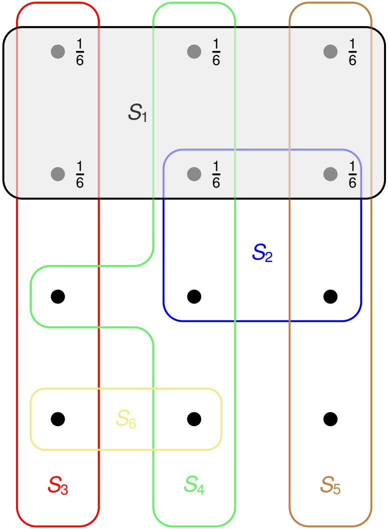

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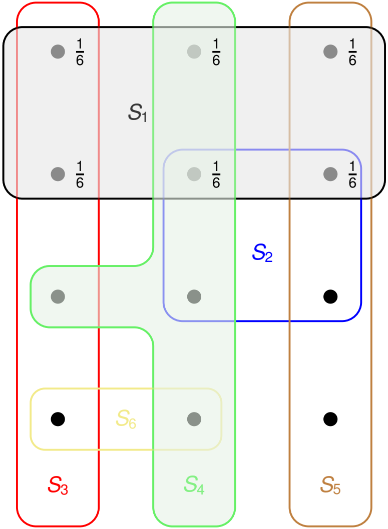

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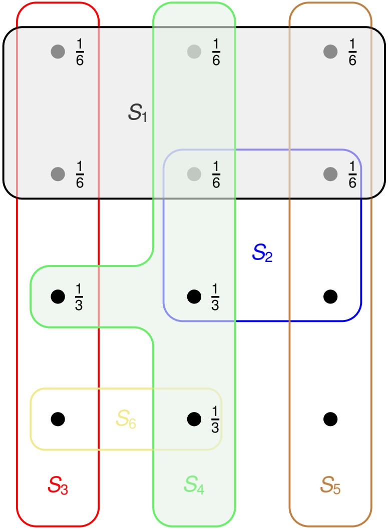

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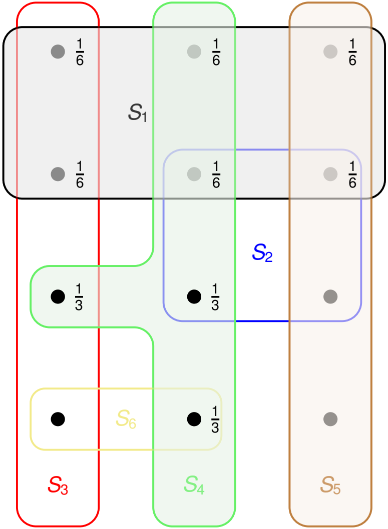

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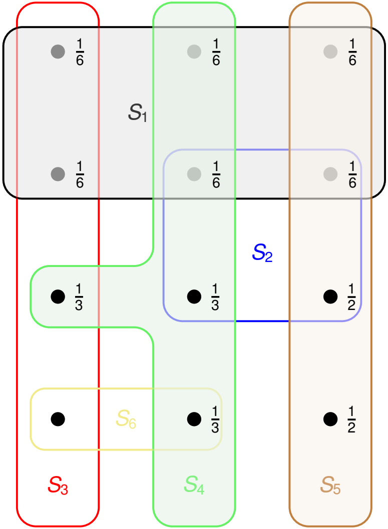

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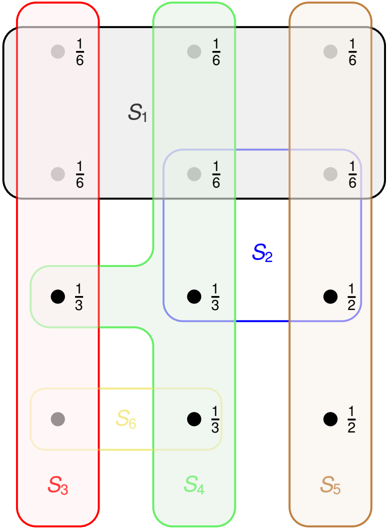

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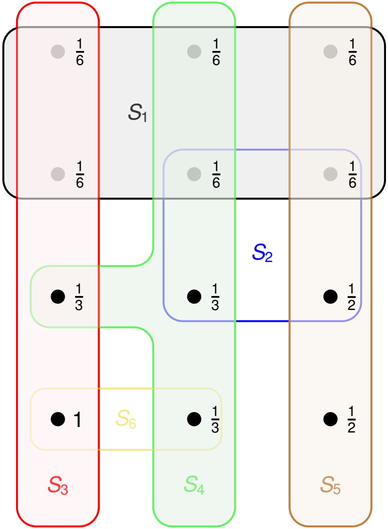

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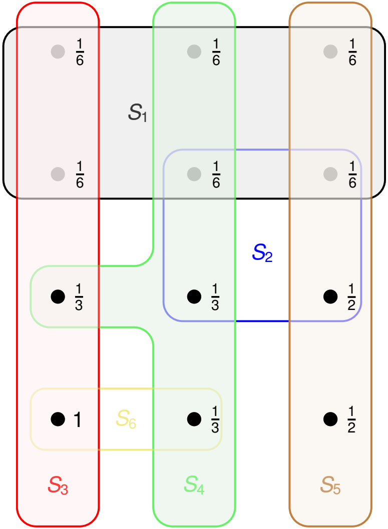

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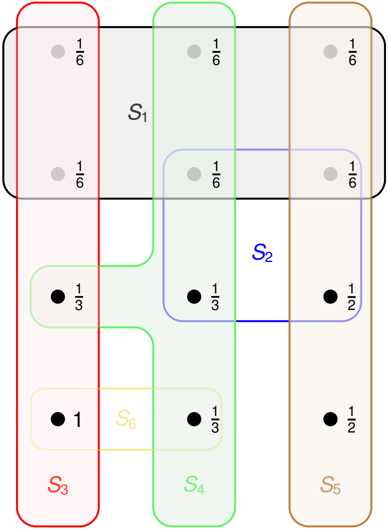

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Proof of Theorem 35.4 (1/2)

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(1)

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- Combining 1 and 2 gives

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Proof.

- Each step of the algorithm assigns one unit of cost, so

$$|\mathcal{C}| = \sum_{x \in X} c_x \tag{1}$$

- Each element $x \in X$ is in at least one set in the optimal cover $\mathcal{C}^*$, so

$$\sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \geq \sum_{x \in X} c_x \tag{2}$$

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Remaining uncovered elements in S

- For any $S \in \mathcal{F}$ and $i = 1, 2, \ldots, |\mathcal{C}| = k$ let $u_i := |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)|$

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Sets chosen by the algorithm

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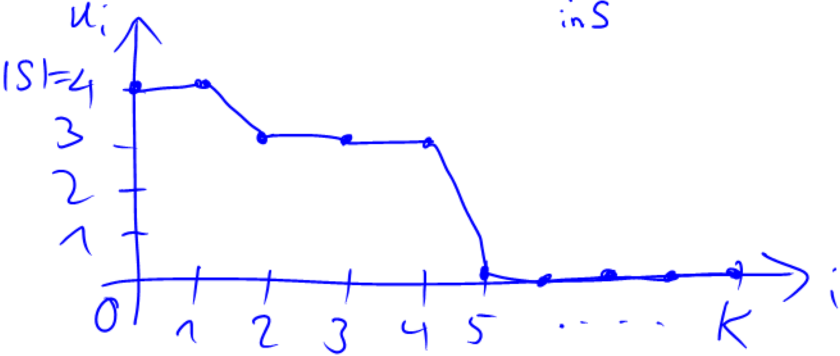

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Each factor is at most one.

$$\sum_{x \in S} c_x = \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|}$$

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u_i's are integers

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$$= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) = H(u_0) - H(u_k)$$

0

$\shortparallel$

$H(0)$

$\shortparallel$

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\end{aligned}$$

Set-Covering Problem (Summary)

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| : |S| \in \mathcal{F}\}) \leq \ln(n) + 1.$$

"Toy" Application:

Vertex Cover for Graphs with maximum degree 3

$$\begin{tikzcd}
 & G=(V,E) \arrow[dl] \arrow[dr] & \\
\mathcal{F}=\{S_1, S_2, \ldots, S_{|V|}\} & & X = E
\end{tikzcd}$$

Apply GREEDY-SET-COVER $\Rightarrow \rho(n) = H(3) = 1 + \frac{1}{2} + \frac{1}{3} < 2$

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polynomial-time

Lower Bound

Unless P=NP, there is no $c \cdot \ln(n)$ approximation algorithm for set cover for some constant $0 < c < 1$.

Set-Covering Problem (Summary)

The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : S \to \mathbb{Z}^+$

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Example where Greedy is a $(1/2) \cdot \log_2 n$ factor off

Instance

- Given any integer $k \geq 3$

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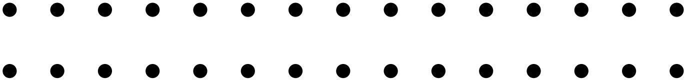
$\boxed{k = 4:}$



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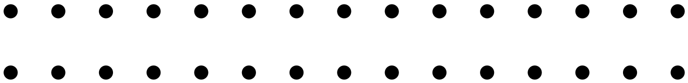

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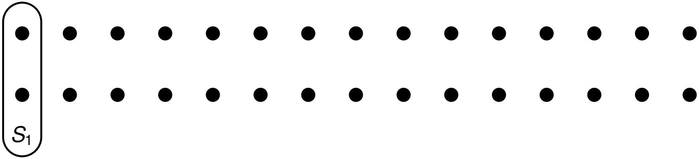

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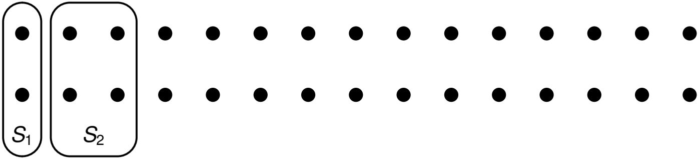

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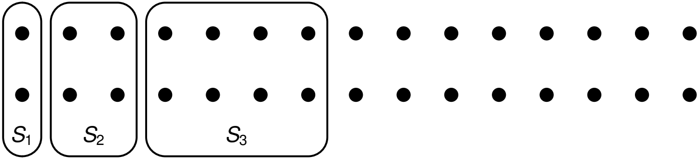

Example where Greedy is a $(1/2) \cdot \log_2 n$ factor off

Instance

- Given any integer $k \geq 3$
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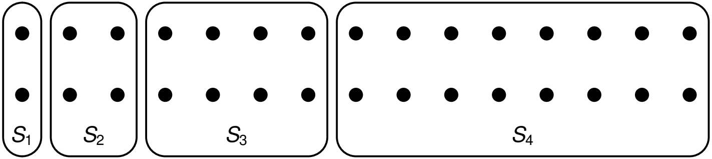

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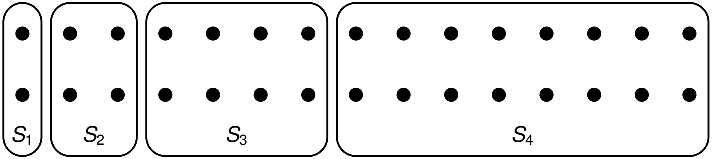

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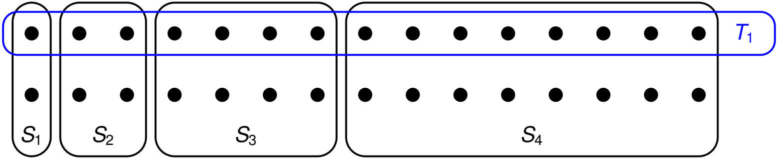

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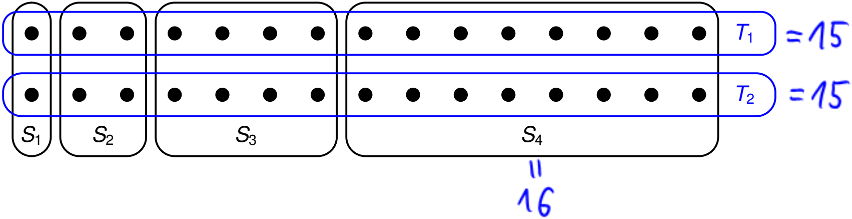
$k = 4$: $(n = 32 - 2 = 30)$



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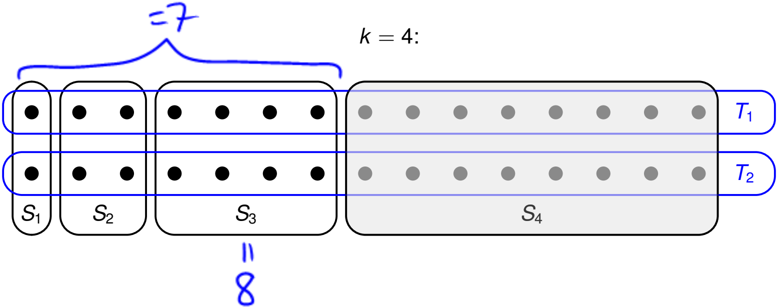

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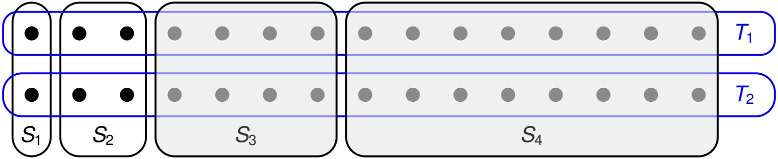

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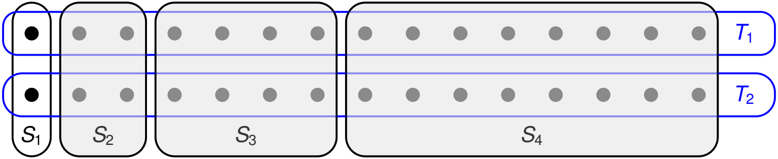

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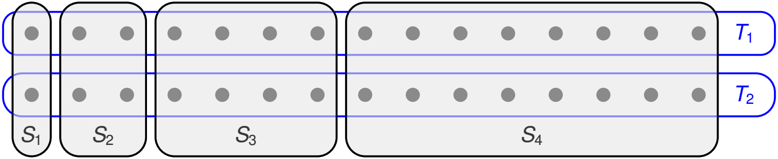

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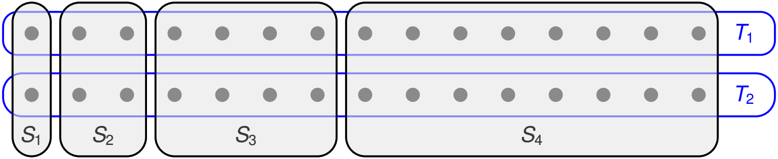

Solution of Greedy consists of k sets.

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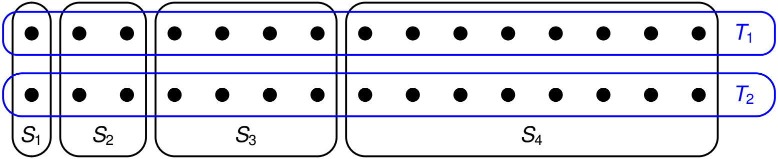

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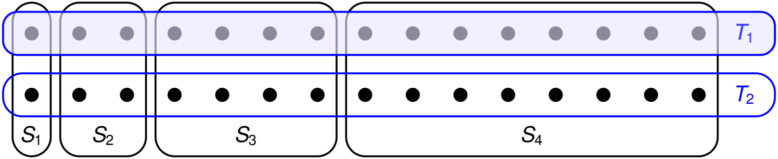

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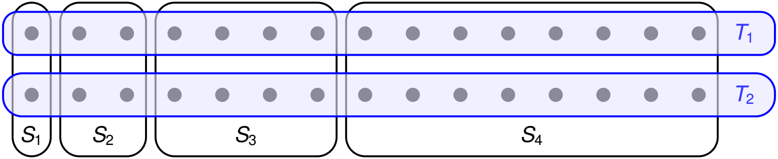

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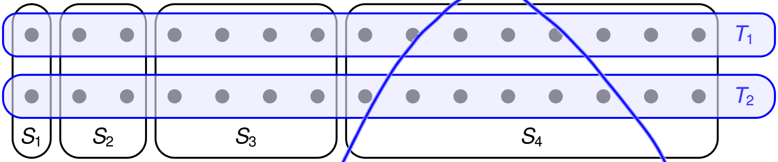
$k = 4$: $\rho(n) \geq \frac{k}{2} = \frac{\log_2(\frac{n}{2}) - 1}{2} = \frac{\log_2(n) - 2}{2}$



Solution of Greedy consists of $\underline{k}$ sets.

Optimum consists of $\underline{2}$ sets.

V. Approximation Algorithms via Exact Algorithms

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

The Subset-Sum Problem

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This problem is NP-hard

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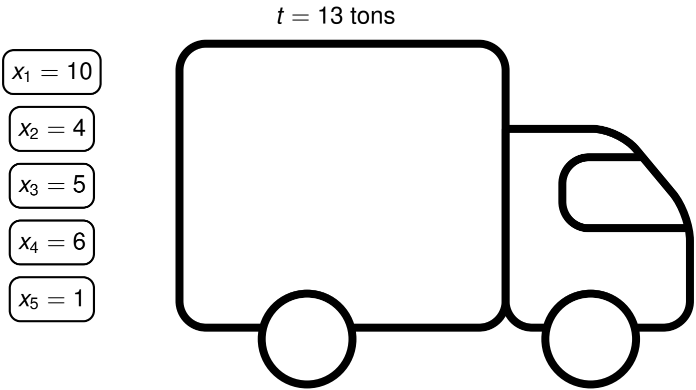

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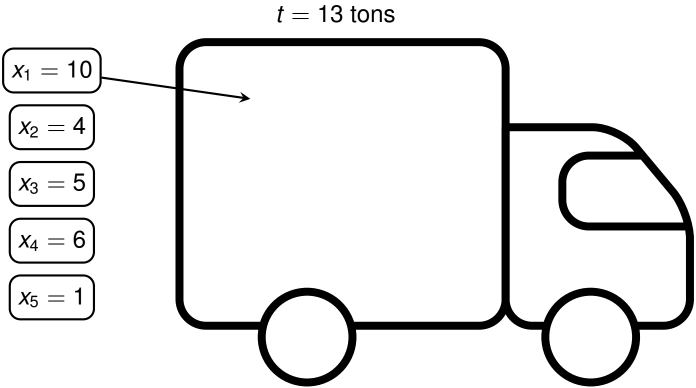

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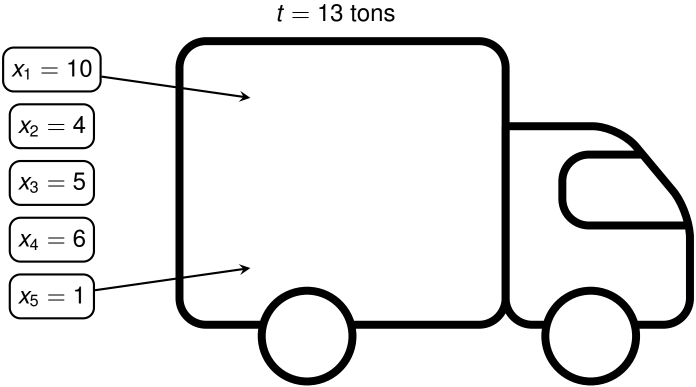

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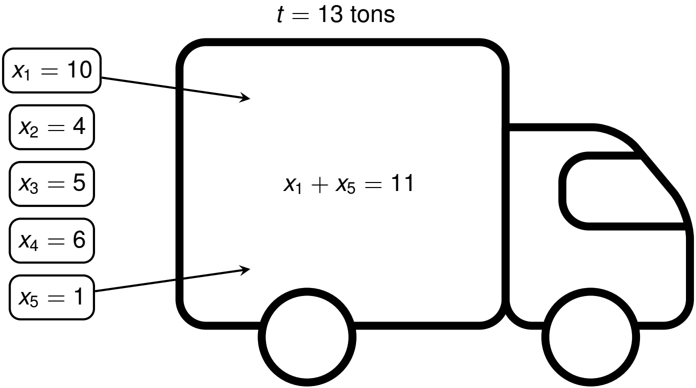

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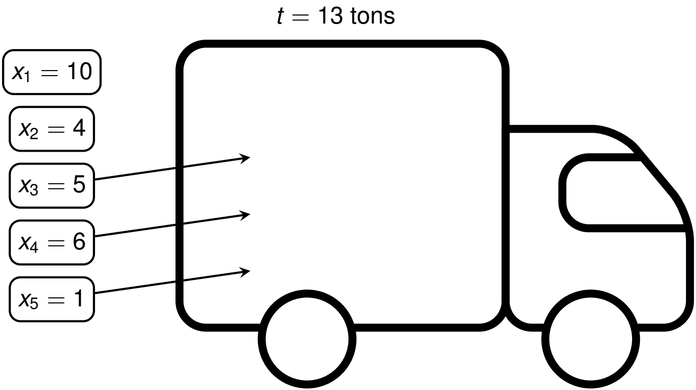

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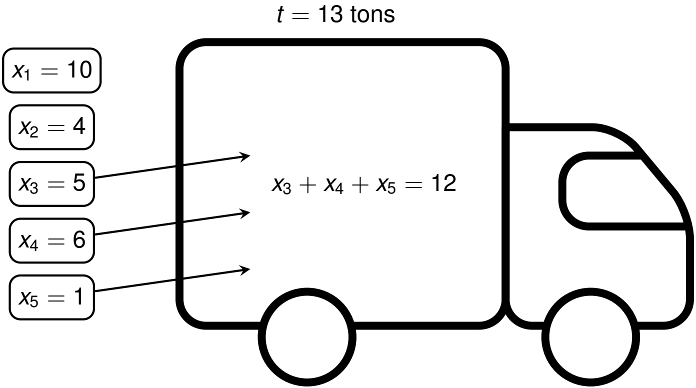

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

An Exact (Exponential-Time) Algorithm

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```
EXACT-SUBSET-SUM$(S, t)$
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$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
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Returns the merged list (in sorted order and without duplicates)

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implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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Example:

- $S = \{1, 4, 5\}$, $t=10$

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5=0+5=1+4

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- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$

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can be shown by induction on n

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- **Runtime:** $O(2^1 + 2^2 + \cdots + 2^n) = O(2^n)$

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There are 2^i subsets of $\{x_1, x_2, \ldots, x_i\}$.

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EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$
- **Runtime:** $O(2^1 + 2^2 + \cdots + 2^n) = O(2^n)$

There are 2^i subsets of $\{x_1, x_2, \ldots, x_i\}$.

Better runtime if t and/or $|L_i|$ are small.

Example:

- $S = \{1, 4, 5\}$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Towards a FPTAS

Idea: Don't need to maintain two values in *L* which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$

Towards a FPTAS

Idea: Don’t need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\underbrace{\exists z \in L'}_{\text{"approximate representative"}}$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
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- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

Towards a FPTAS

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- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$
- $\delta = 0.1$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
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$$\frac{y}{1+\delta} \leq z \leq y.$$

- $L = \langle 10, \cancel{11}, 12, 15, 20, \cancel{21}, \cancel{22}, 23, \cancel{24}, 29 \rangle$
- $\delta = 0.1$
- $L' = \langle 10, 12, 15, 20, 23, 29 \rangle$

Towards a FPTAS

Idea: Don’t need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

```
TRIM$(L, \delta)$
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1+\delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

```
TRIM(L, δ)
let $m$ be the length of $L$
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for $i = 2$ to $m$
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        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

maintained inductively

Trims list in time $\Theta(m)$, if L is given in sorted order.

Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
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for $i = 2$ to $m$
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```

Illustration of the Trim Operation

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        $last = y_i$
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```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle \rangle$$

Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
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return $L'$
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

```
TRIM($L, \delta$)
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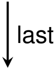
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$\delta = 0.1$



$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10 \rangle$

Illustration of the Trim Operation

```
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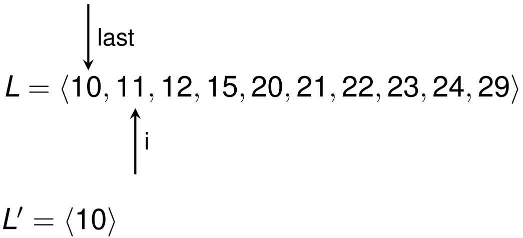
```

$\delta = 0.1$



$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

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Illustration of the Trim Operation

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        $last = y_i$
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```
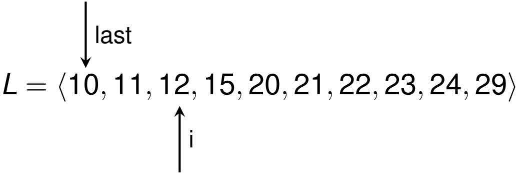

$\delta = 0.1$



$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10 \rangle$

Illustration of the Trim Operation

```
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        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$$\delta = 0.1$$

$$L = \langle \overset{\text{last}}{\overset{\downarrow}{10}}, 11, \underset{\underset{i}{\uparrow}}{12}, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12 \rangle$$

Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
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    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```
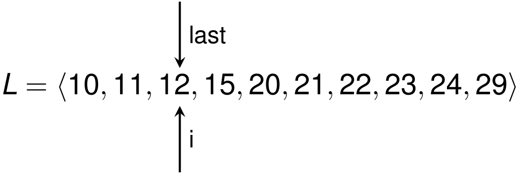

$\delta = 0.1$



last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12 \rangle$

Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
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```

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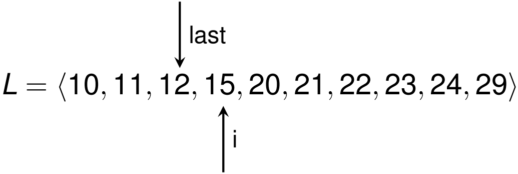

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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Illustration of the Trim Operation

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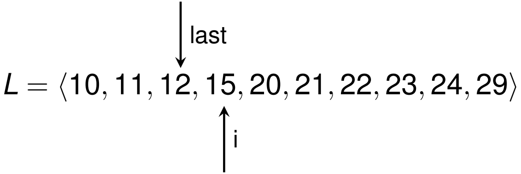
$\delta = 0.1$



last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15 \rangle$

Illustration of the Trim Operation

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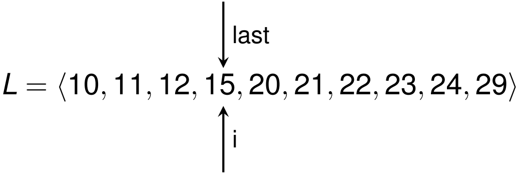

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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Illustration of the Trim Operation

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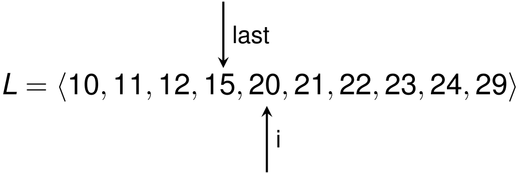

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

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Illustration of the Trim Operation

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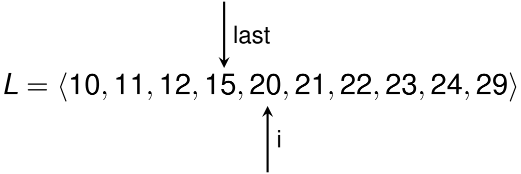

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

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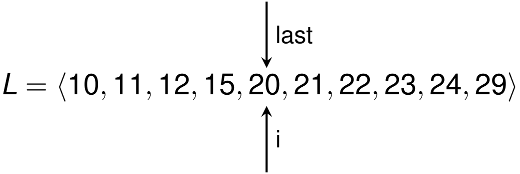

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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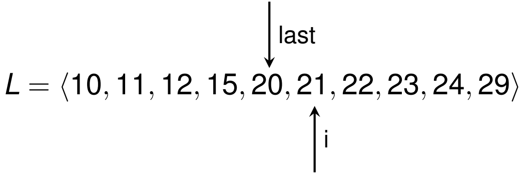
$\delta = 0.1$



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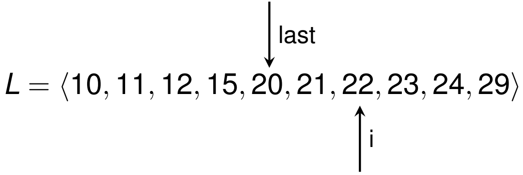
```

$$\delta = 0.1$$



$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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Illustration of the Trim Operation

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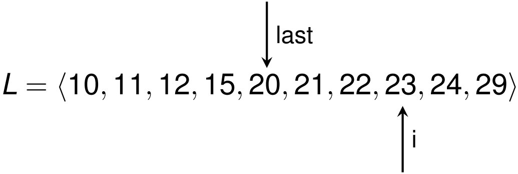

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

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Illustration of the Trim Operation

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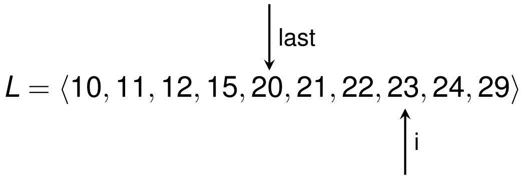

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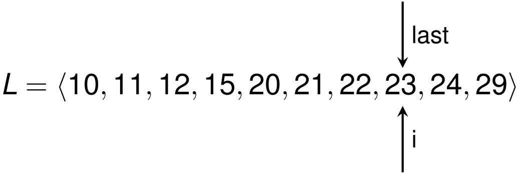

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Illustration of the Trim Operation

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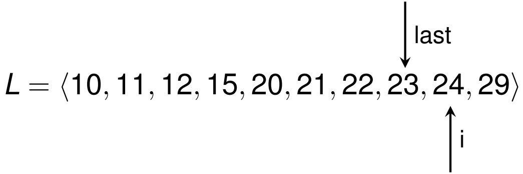

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

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Illustration of the Trim Operation

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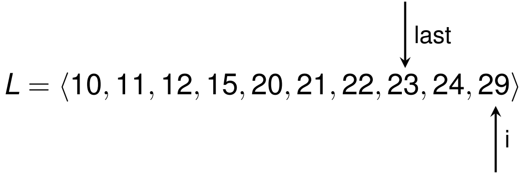

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

Illustration of the Trim Operation

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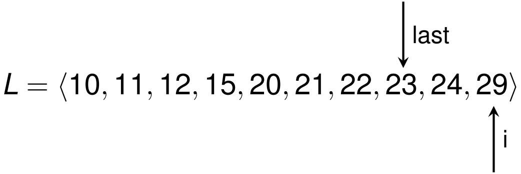
$$\delta = 0.1$$



$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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```
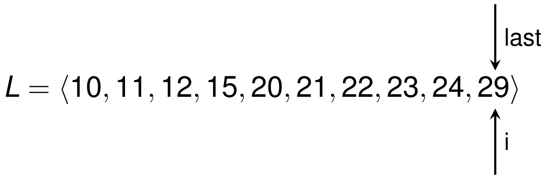

$$\delta = 0.1$$



last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
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```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

Repeated application of TRIM to make sure L_i's remain short.

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

Repeated application of TRIM to make sure L_i's remain short.

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

- We must bound the inaccuracy introduced by repeated trimming

The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \boxed{\epsilon/2n})$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

Repeated application of TRIM to make sure L_i's remain short.

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

proper choice of δ meets these conflicting goals

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

Running through an Example

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
    $L_i = $ TRIM($L_i, \epsilon/2n$)
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\boxed{\epsilon = 0.4}$

→ z^* should be at least $\frac{1}{1.4}$ times the optimum

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i$ = TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i$ = TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
5      $L_i$ = TRIM($L_i, \epsilon/2n$)
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i =$ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i =$ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
5      $L_i = $ TRIM($L_i, \epsilon/2n$)
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i =$ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, \boxed{302} \rangle$ → output

Running through an Example

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i$ = TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = \boxed{0.4}$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

much better than
$1+\varepsilon$ - approximation!

Returned solution $z^* = 302$, which is $\boxed{2\%}$ within the optimum $307 = 104 + 102 + 101$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓

all elements in the trimmed lists are solutions

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i$ s.t.:

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i$ s.t.:

$$\frac{y}{(1 + \epsilon/(2n))^i} \leq z \leq y$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

List after trimming

$$\frac{y}{\boxed{(1+\epsilon/(2n))^{i}}} \leq z \leq y$$

Can be shown by induction on i

$i=1$: clear

$i=2$: $y = x_1 + x_2$, $x_1 \in L_1 \Rightarrow \exists\, z_1 \in L_1' : z_1 \geq \frac{x_1}{(1+\delta)}$

$z_1 + x_2 \in L_2 \Rightarrow \exists\, z \in L_2' : z \geq \frac{z_1 + x_2}{(1+\delta)}$

$\Rightarrow z \geq \frac{x_1}{(1+\delta)^2} + \frac{x_2}{(1+\delta)} \geq \frac{x_1 + x_2}{(1+\delta)^2} = \frac{y}{(1+\delta)^2}$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \underset{i=n}{\overset{y=y^*}{\Rightarrow}}$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

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$$\leq 1+\epsilon/2+(\epsilon/2)^2 \leq 1+\epsilon$$

Analysis of APPROX-SUBSET-SUM

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- Strategy: Derive a bound on $|L_i|$ (running time is polynomial in $|L_i|$)

iteration i runs in $O(|L_i|)$

Analysis of APPROX-SUBSET-SUM

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Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is polynomial in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \underline{\geq 1 + \epsilon/(2n)}$

Analysis of Approx-Subset-Sum

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Approx-Subset-Sum is a FPTAS for the subset-sum problem.

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- Strategy: Derive a bound on $|L_i|$ (running time is polynomial in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq \boxed{1 + \epsilon/(2n)}$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{\boxed{1+\epsilon/(2n)}} t \rfloor$ additional values.

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$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1+\epsilon/(2n))} + 2$$
$$\leq \frac{2n(1+\epsilon/(2n)) \ln t}{\epsilon} + 2$$

For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

$\Leftrightarrow x > 0 \quad \ln(x) \geq 1 - \frac{1}{x}$

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$$\left(1 + \frac{\varepsilon}{2n}\right) \leq \frac{3}{2}$$

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Need $\log(t)$ bits to represent t and n bits to represent S.

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i\colon\, x_i \in S'} x_i \leq t$.

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- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t

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A more general problem than Subset-Sum.

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("weighted" version of Subset-Sum)

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Algorithm very similar to APPROX-SUBSET-SUM.

Theorem

There is a FPTAS for the Knapsack problem.

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$

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- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

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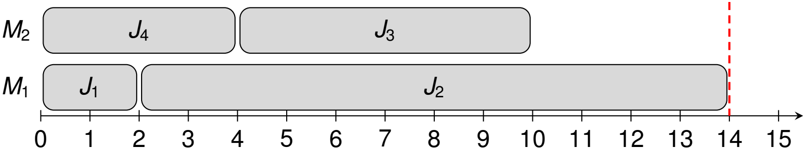

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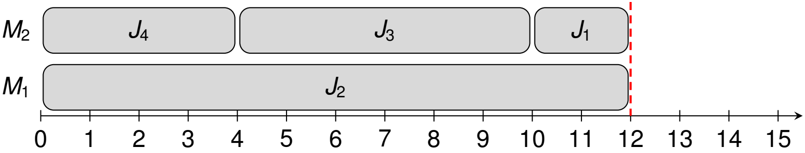

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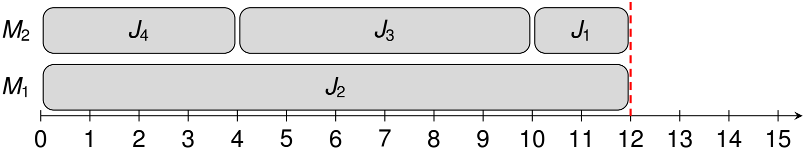
For the analysis, it will be convenient to denote by $\underline{C_i}$ the completion time of a machine $\underline{i}$.



NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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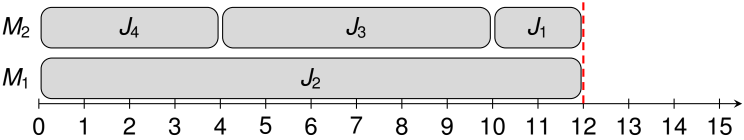

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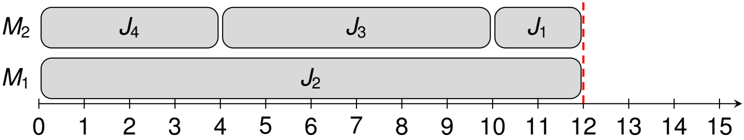

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    LIST SCHEDULING($J_1, J_2, \dots, J_n, m$)
1: while there exists an unassigned job
2:     Schedule job on the machine with the least load
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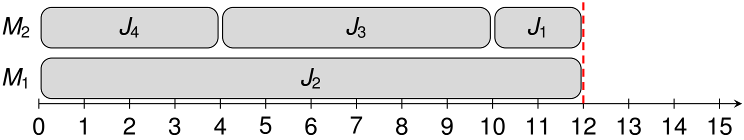

Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

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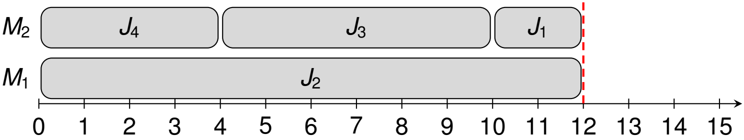

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How good is this most basic Greedy Approach?

List Scheduling Analysis (Observations)

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k .$$

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b. ~~a.~~ The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

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Proof:

b. ~~a.~~ The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

~~b.~~ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ □

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

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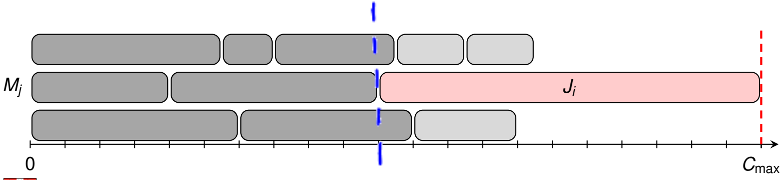

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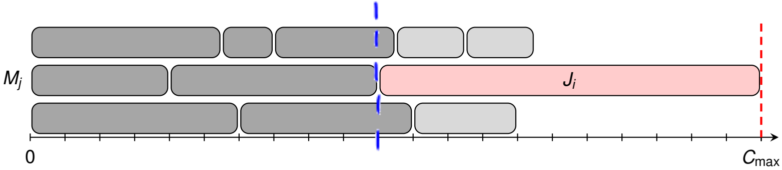

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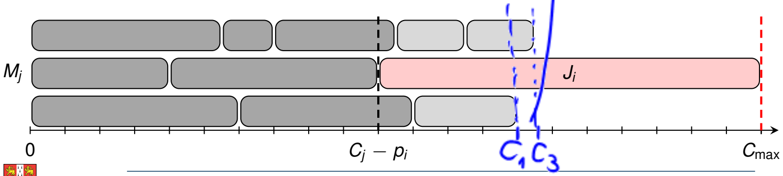
"Greedy"-Property



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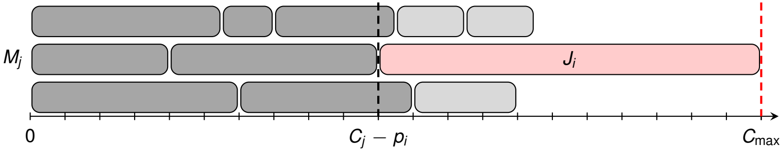

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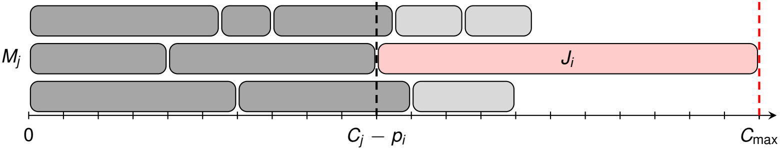

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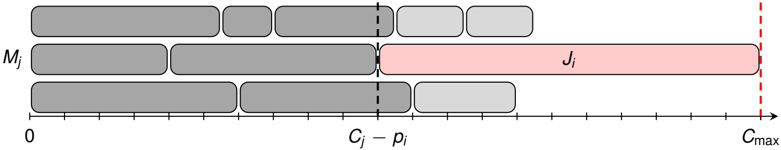

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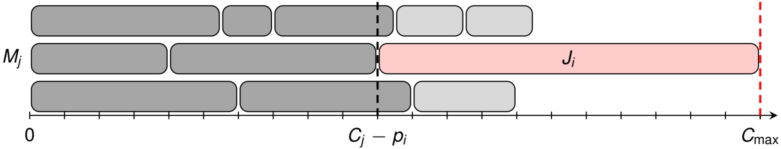

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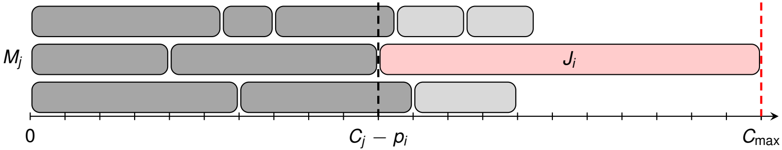

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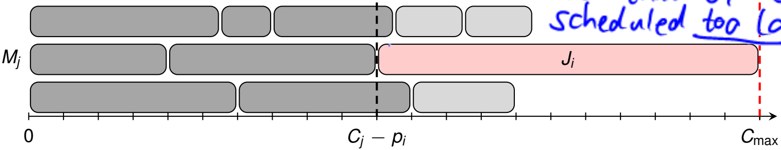
▨

Idea: Problem is that J_i is scheduled too late!



Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

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give them "higher priority"

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```
LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)
Sort jobs decreasingly in their processing times
for $i = 1$ to $m$
    $C_i = 0$
    $S_i = \emptyset$
end for
for $j = 1$ to $n$
    $i = \text{argmin}_{1 \leq k \leq m} C_k$
    $S_i = S_i \cup \{j\}$, $C_i = C_i + p_j$
end for
return $S_1, \ldots, S_m$
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Runtime:

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Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting the minimum (use priority queue).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

exactly!!

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The LPT algorithm has an approximation ratio of $\underline{4/3 - 1/(3m)}$.

$\hookrightarrow$ a bit easier to prove

Proof (of approximation ratio $\underline{3/2}$).

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- Observation 1: If there are at most m jobs, then the solution is optimal.

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- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.

↓

$\exists$ machine which has to process two jobs from $J_1, J_2, \ldots, J_{m+1}$

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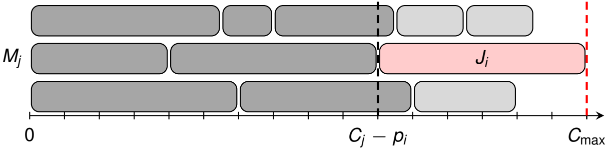

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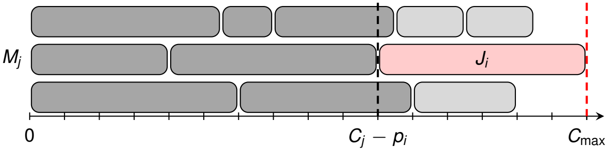

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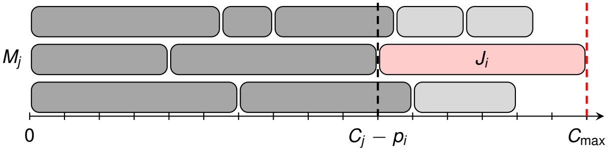
$$C_{max} = C_j = \boxed{(C_j - p_i)} + \boxed{p_i} \leq \boxed{C^*_{\max}} + \boxed{\frac{1}{2} C^*_{\max}} \qquad \boxed{C_j - p_i \leq C_k \ \forall k}$$

This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



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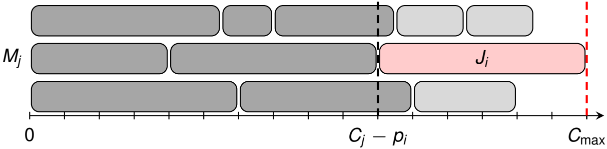

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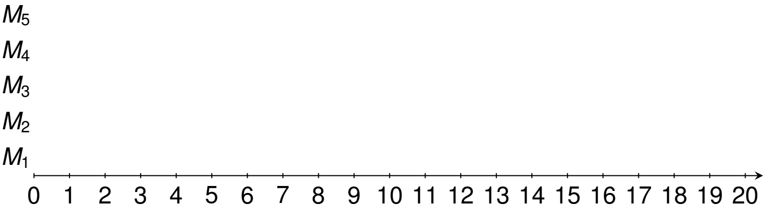

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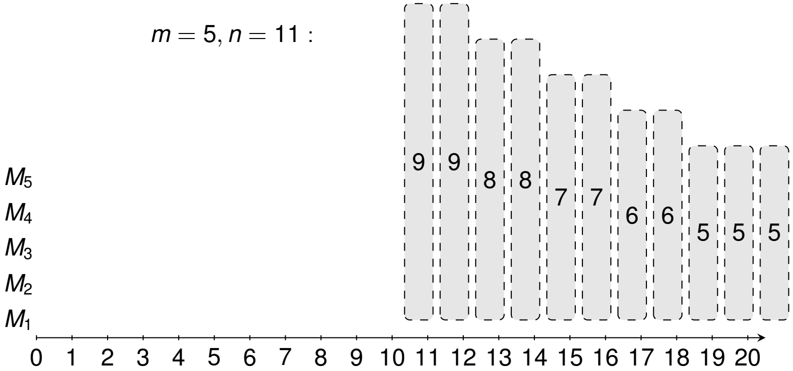

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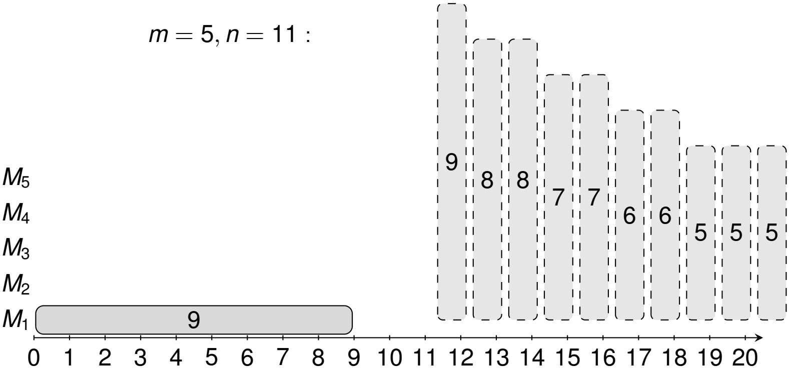

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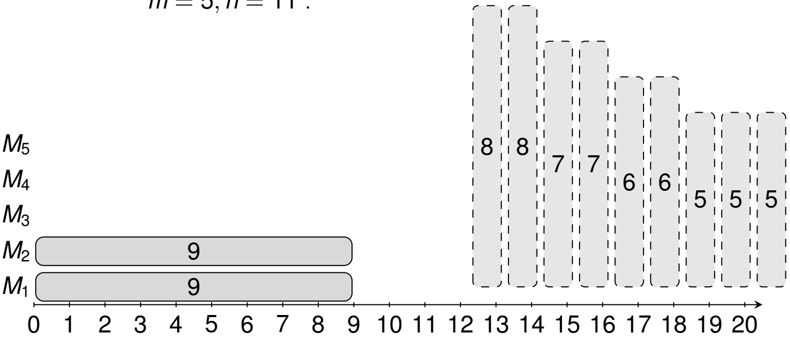

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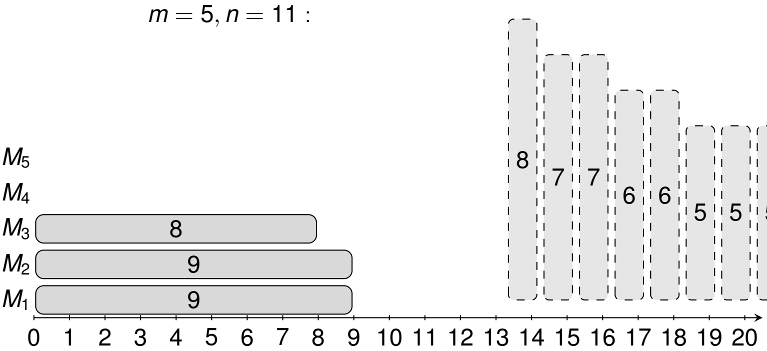

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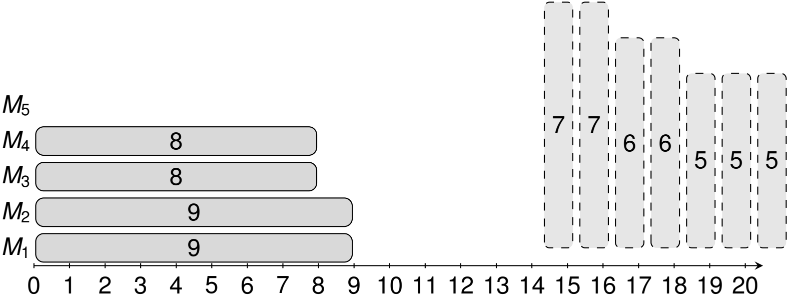

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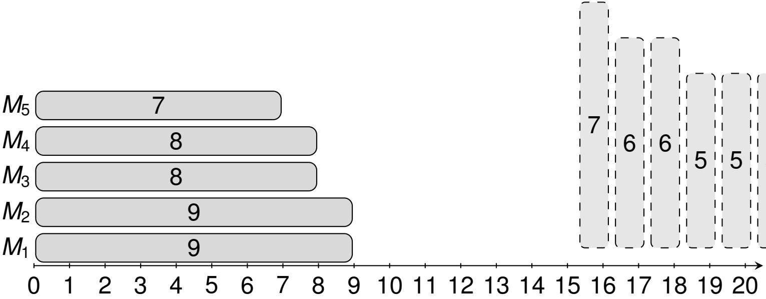

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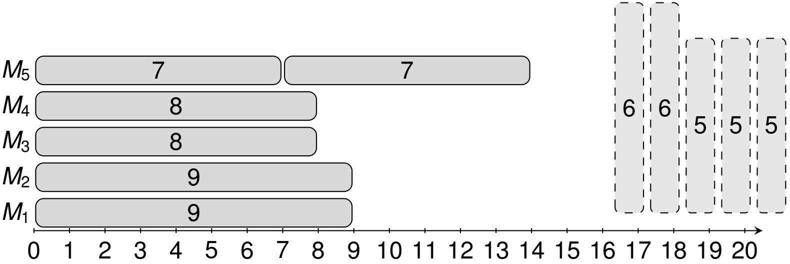

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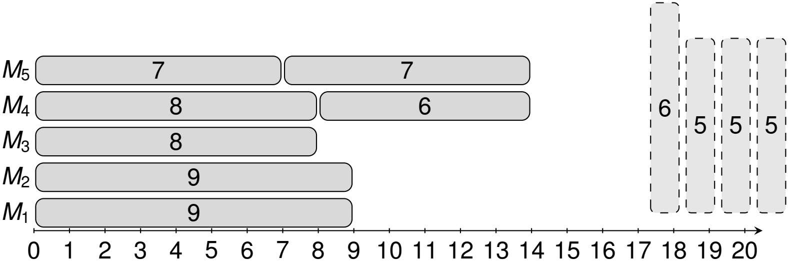

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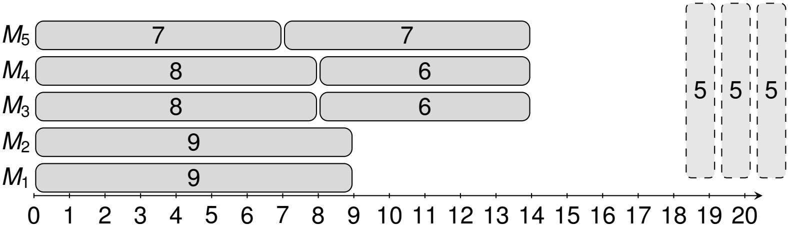

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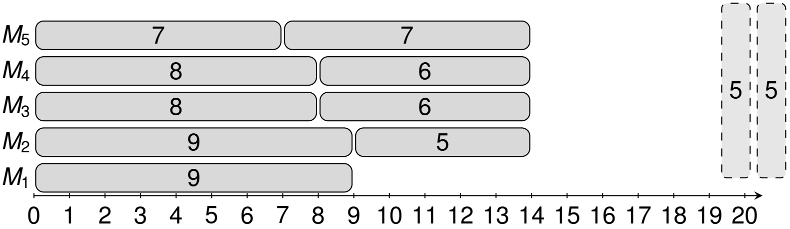

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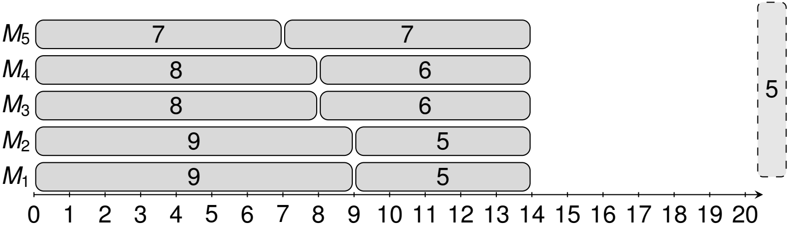

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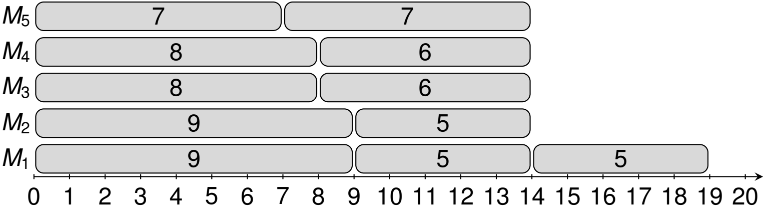

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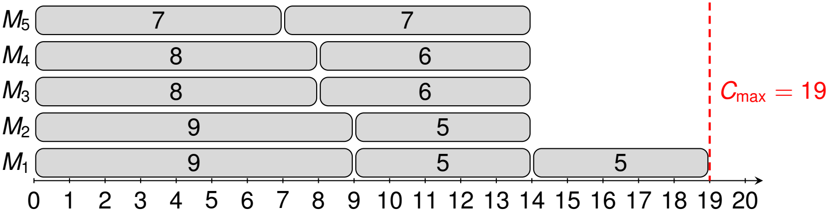

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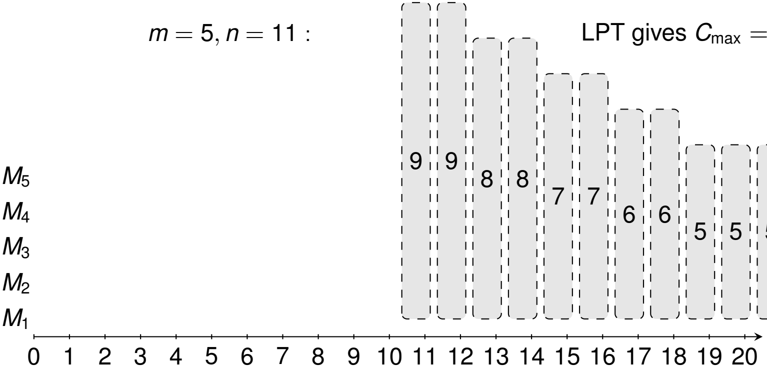

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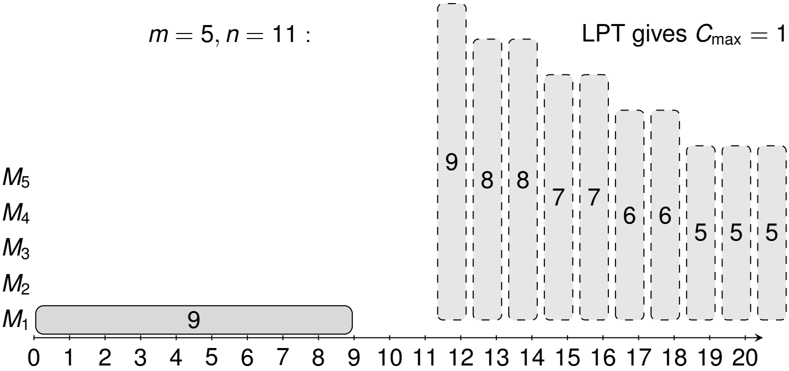

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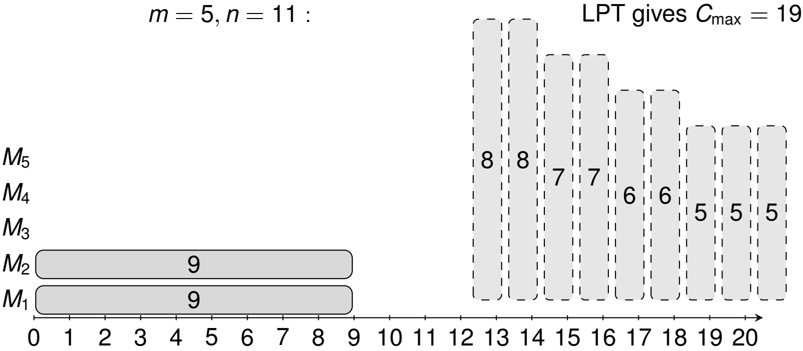

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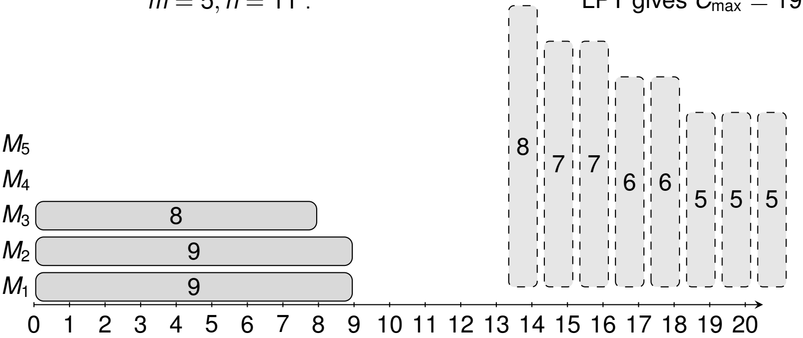

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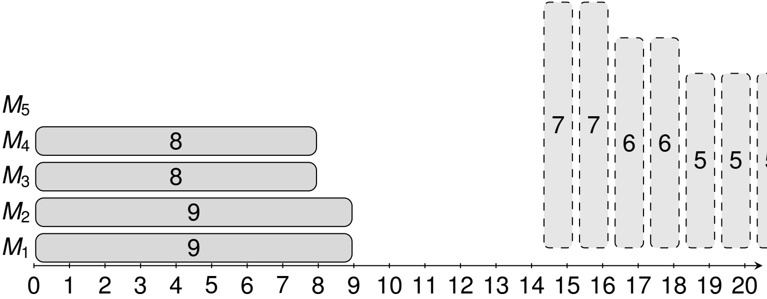

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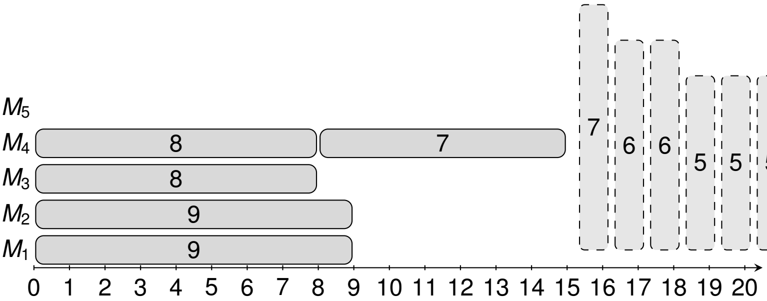

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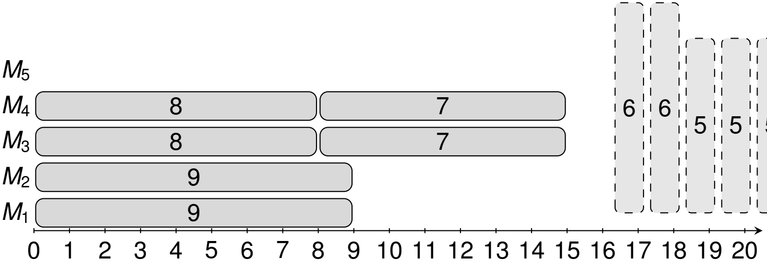

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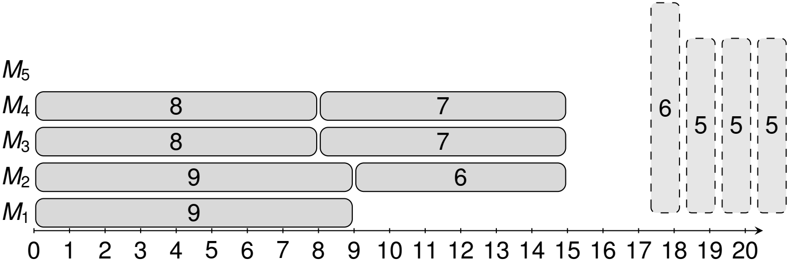

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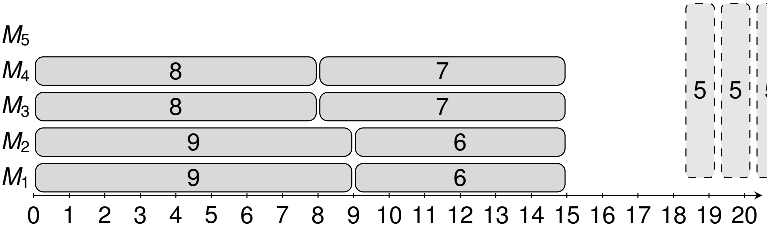

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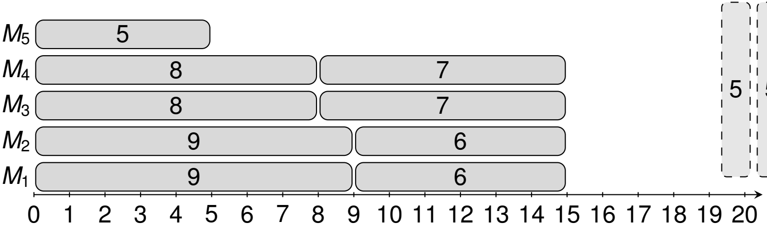

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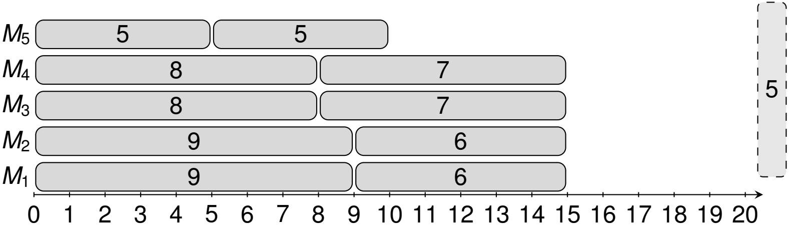

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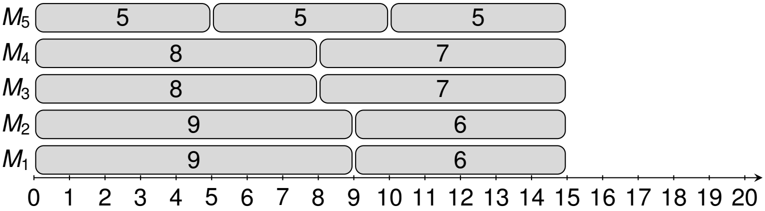

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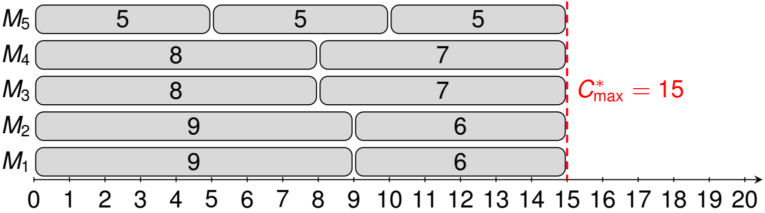

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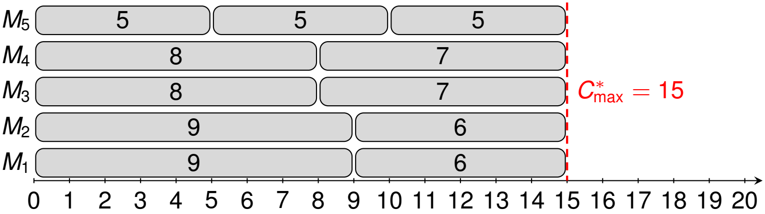

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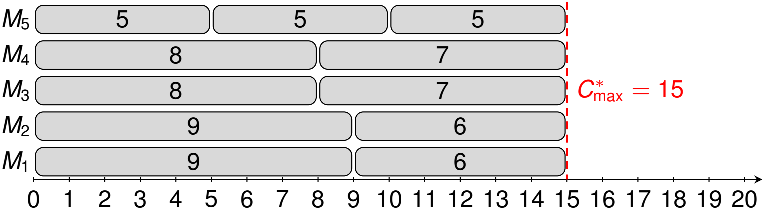

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

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$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \qquad \Rightarrow \qquad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the “well-known” formula

Implementation of Subroutine

```
SUBROUTINE($J_1, J_2, \ldots, J_n, m, T$)
1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$
```

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

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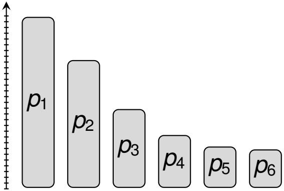

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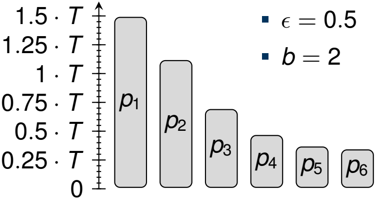

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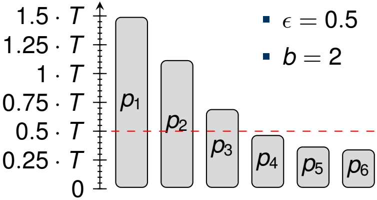

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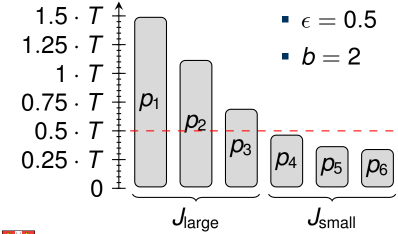

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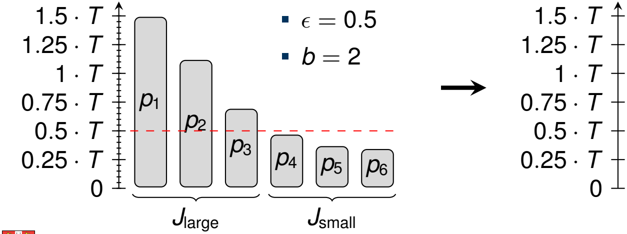

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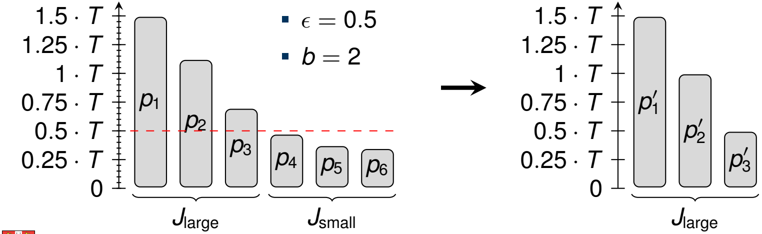

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Can assume there are no jobs with $p_j \geq T$!

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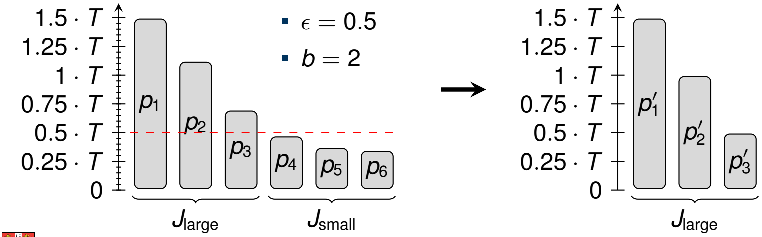

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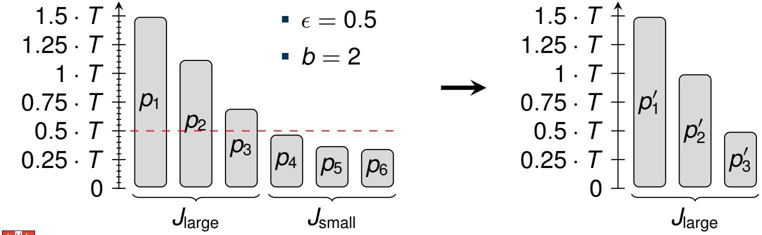

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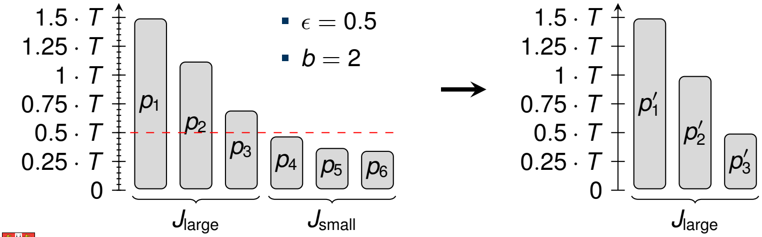
Assignments to one machine with makespan $\leq T$.



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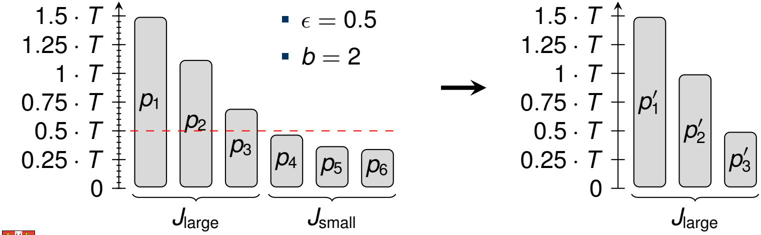

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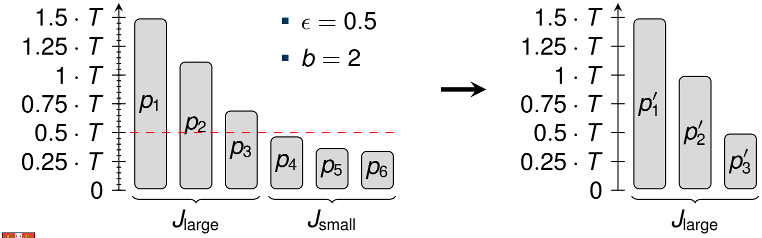

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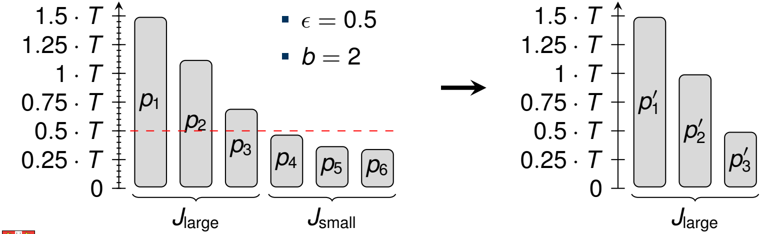

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Assign some jobs to one machine, and then use as few machines as possible for the rest.

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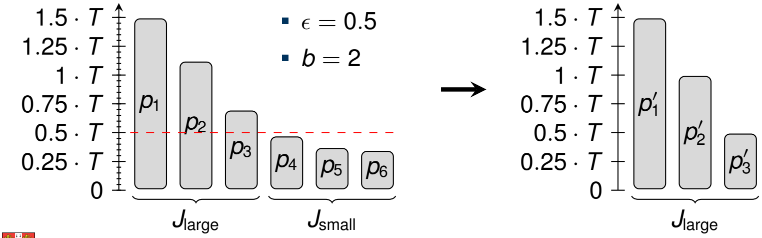

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Final Remarks

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List scheduling has an approximation ratio of 2.

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The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

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Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{\underline{O(1/\epsilon^2)}} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

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There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Can we find a FPTAS (for polynomially bounded processing times)?

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Can we find a FPTAS (for polynomially bounded processing times)? **No!**

Because for sufficiently small approximation ratio $1+\epsilon$, the computed solution has to be optimal,

and: Par. Mach. Sched. is strongly NP-hard!

VI. Approximation Algorithms: Travelling Salesman Problem

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

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Introduction

General TSP

Metric TSP

The Traveling Salesman Problem (TSP)

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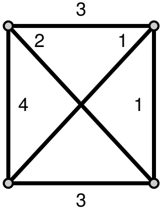

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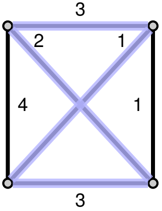

$3 + 2 + 1 + 3 = 9$

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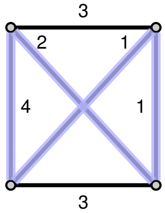

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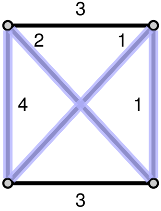
Solution space consists of n! possible tours!



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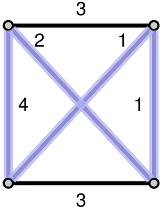

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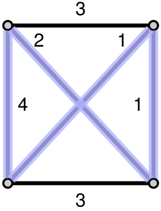

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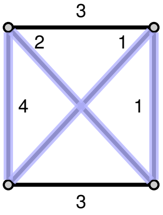

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$$\underline{\forall u, v, w \in V}: \qquad c(u, w) \leq c(u, v) + c(v, w).$$

↳ this "enforces" complete graph

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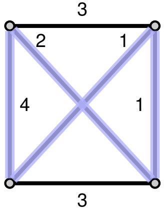

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- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their Euclidean distance → if irrational, need to be approximated by rationals numbers and then scaled to integers.

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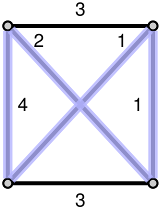

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Even this version is NP hard (Ex. 35.2-2)

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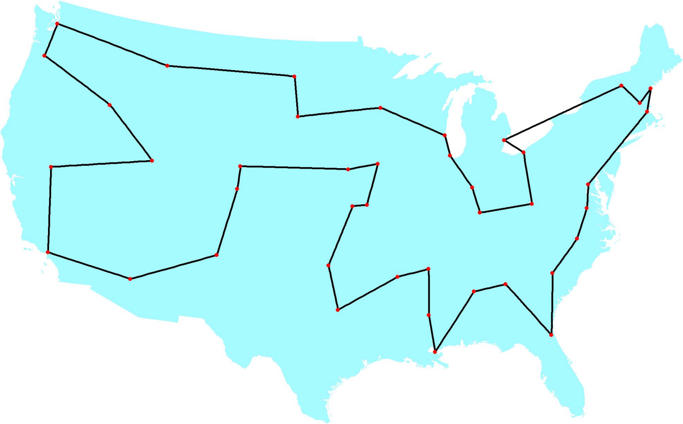
History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

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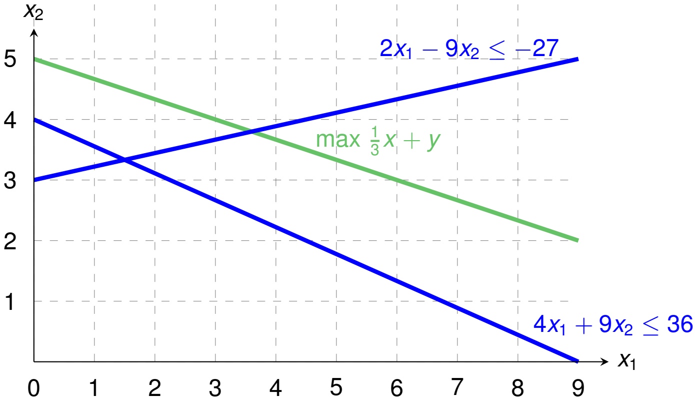

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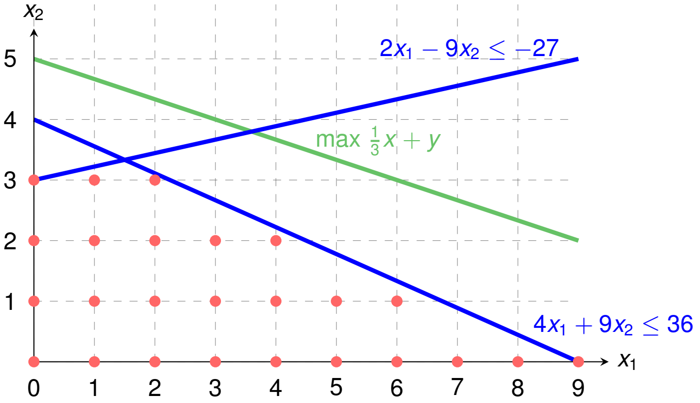

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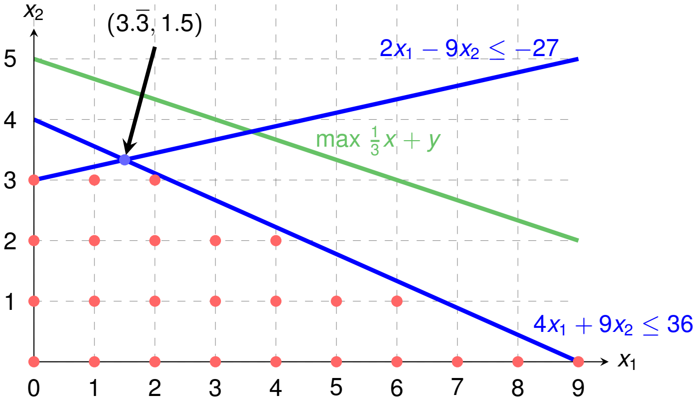

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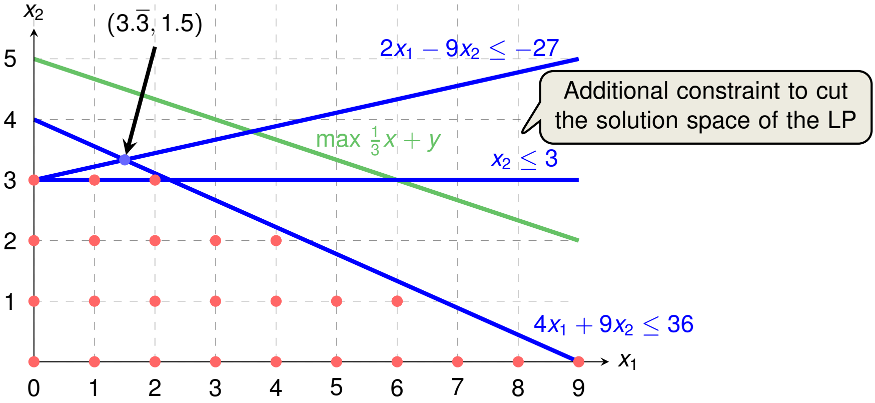

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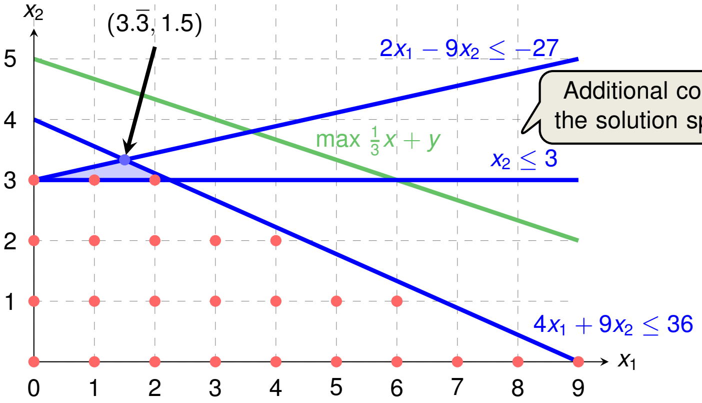

Additional constraint to cut the solution space of the LP

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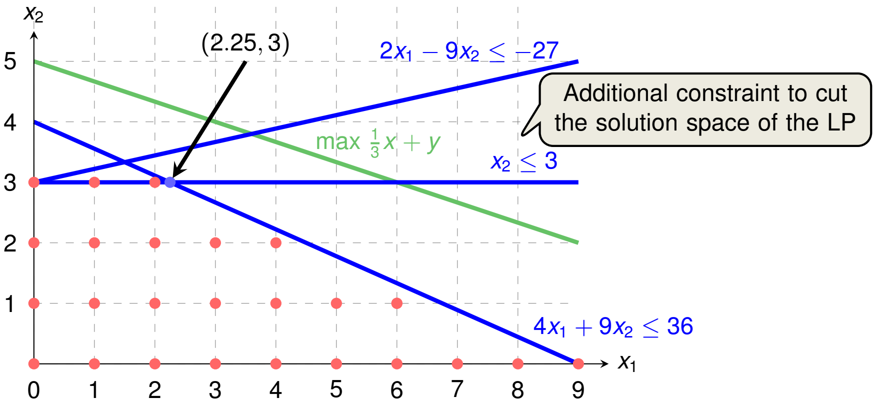

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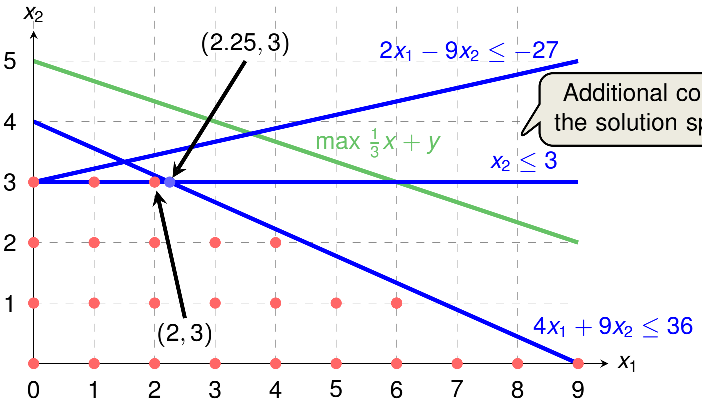

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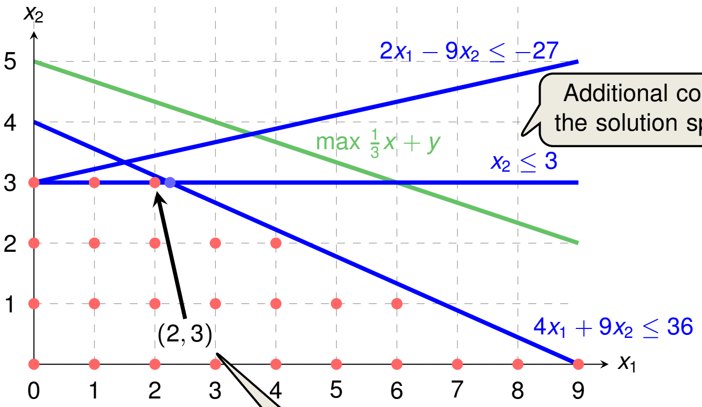

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More cuts are needed to find integral solution

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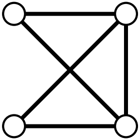

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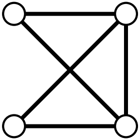

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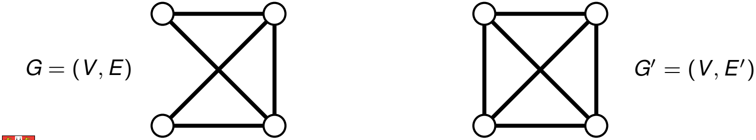

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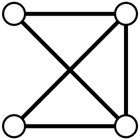

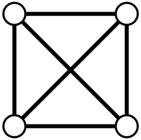

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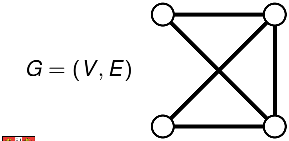

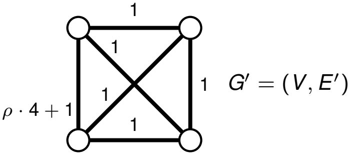

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Large weight will render this edge useless!

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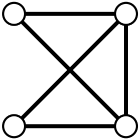

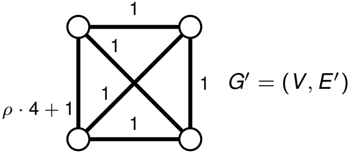

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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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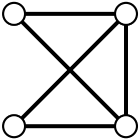

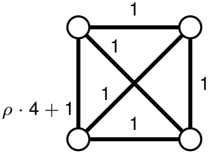

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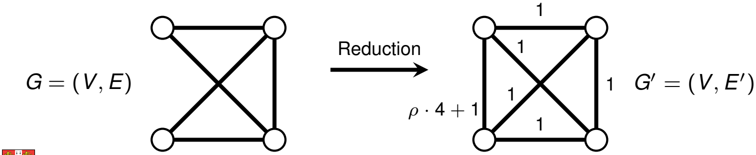

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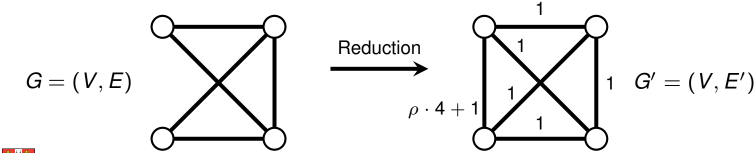
- If G has a hamiltonian cycle H, then (G', c) contains a tour of cost $|V|$



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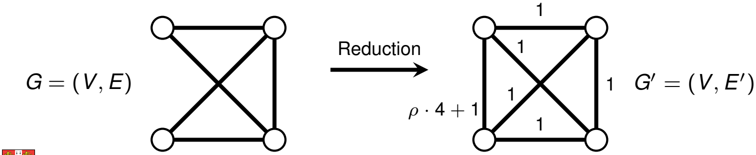

Hardness of Approximation

Theorem 35.3

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

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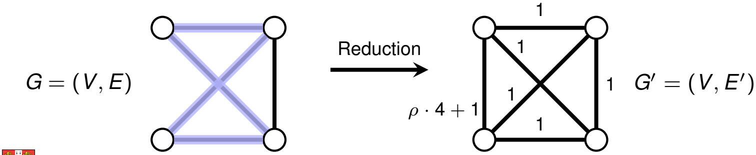

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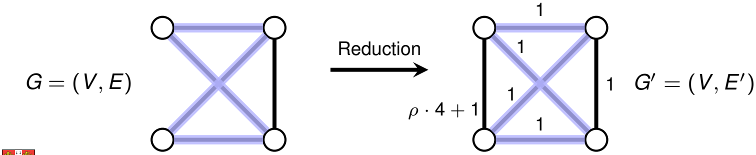

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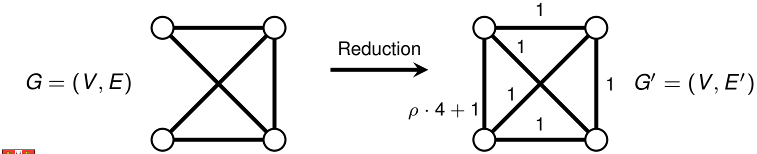

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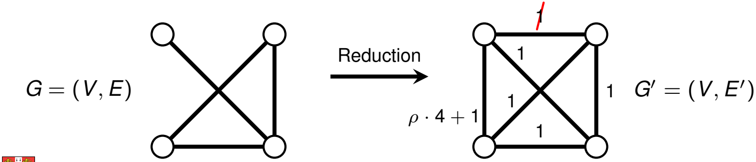

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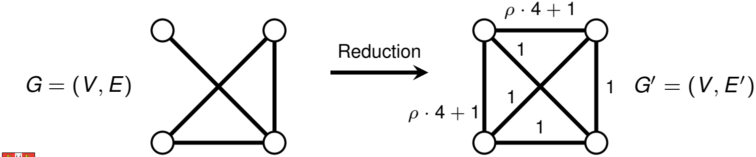

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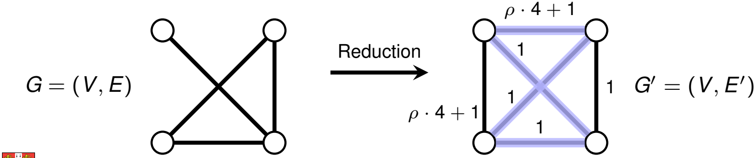

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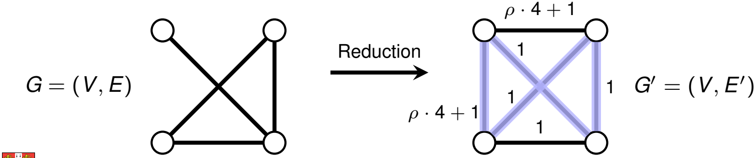

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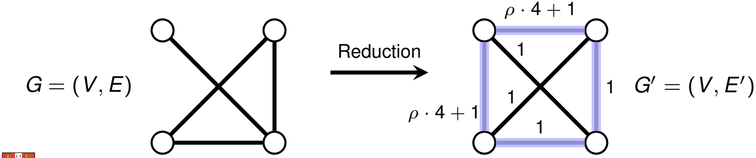

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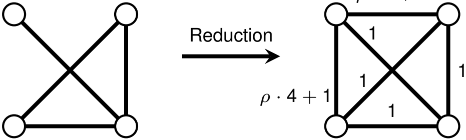
$G = (V, E)$



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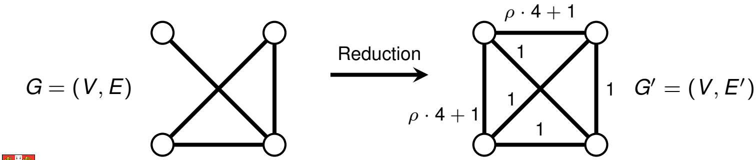

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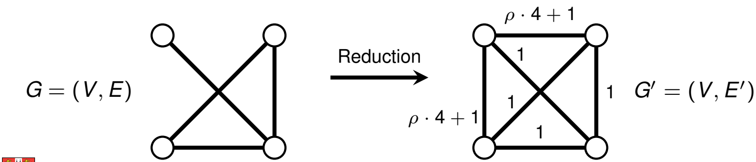

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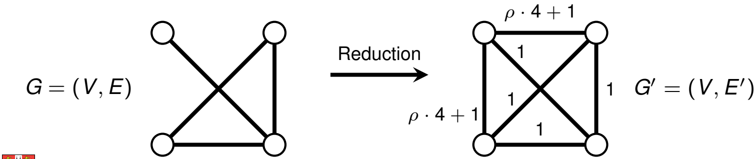

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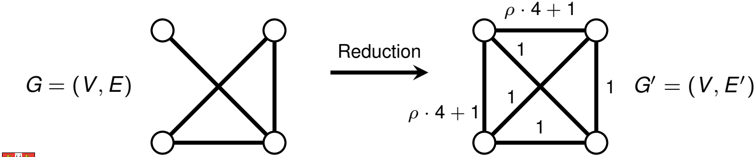
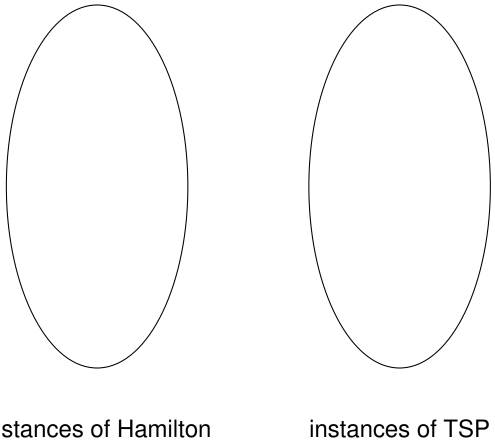


Proof of Theorem 35.3 from a higher perspective



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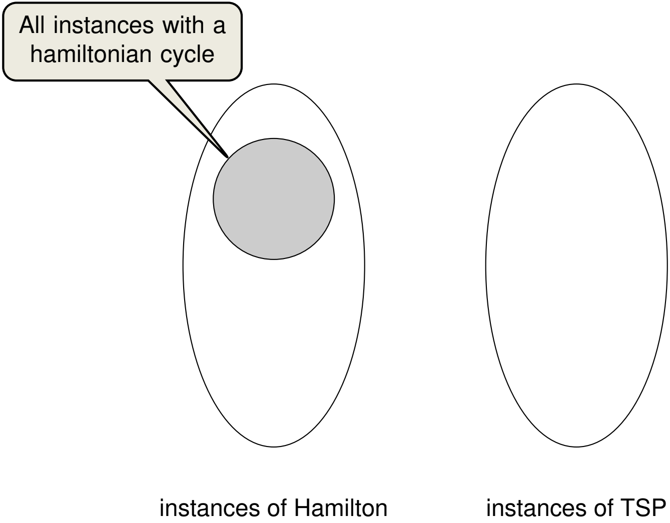

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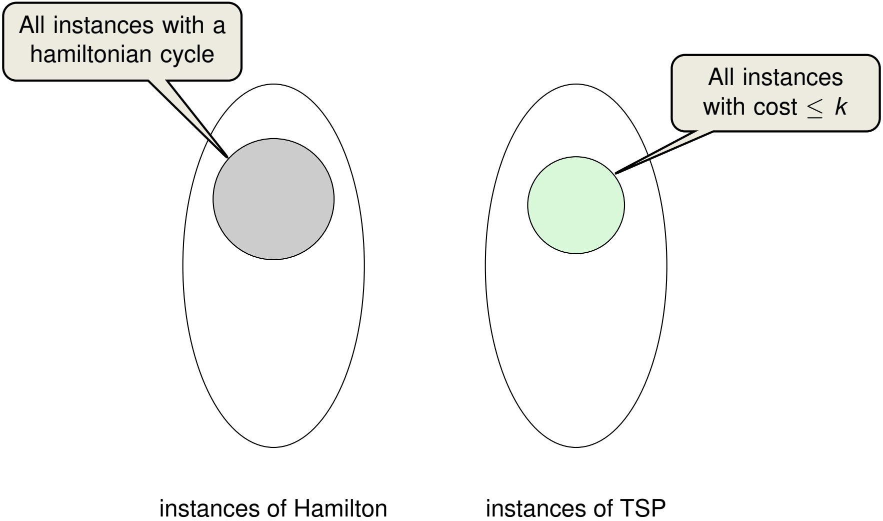
All instances with a hamiltonian cycle
All instances with cost $\leq k$
instances of Hamilton
instances of TSP

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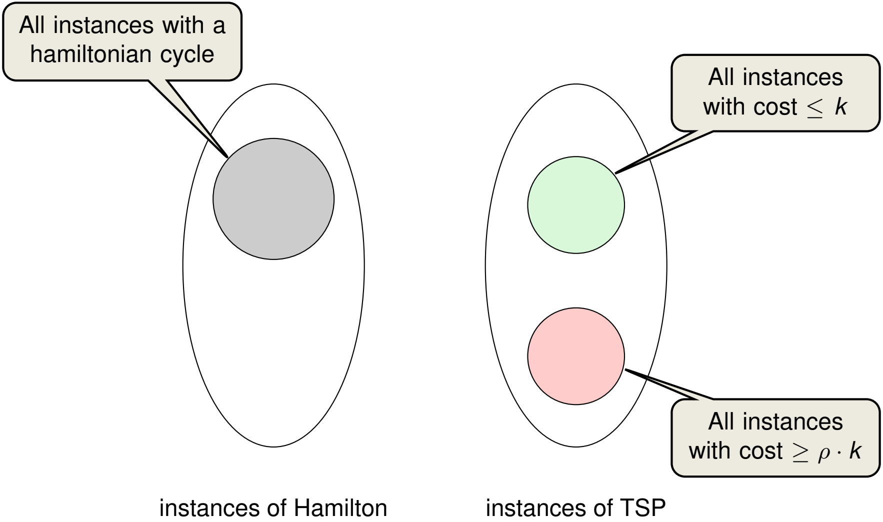
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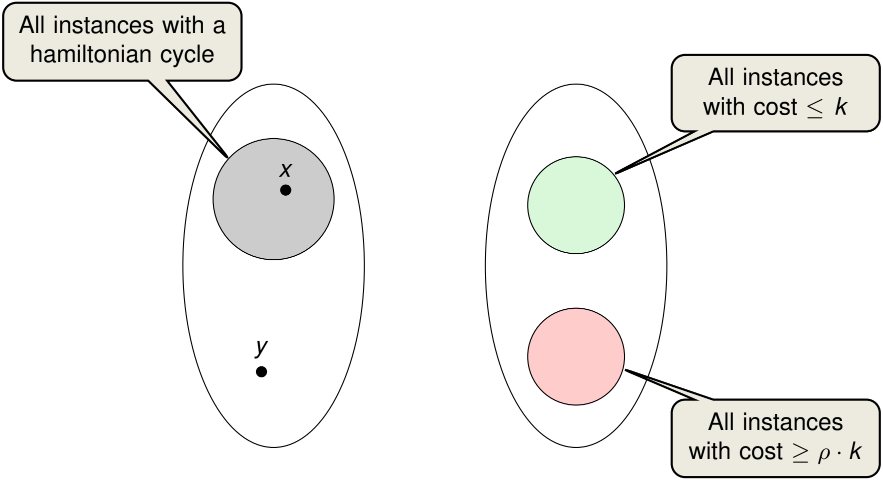
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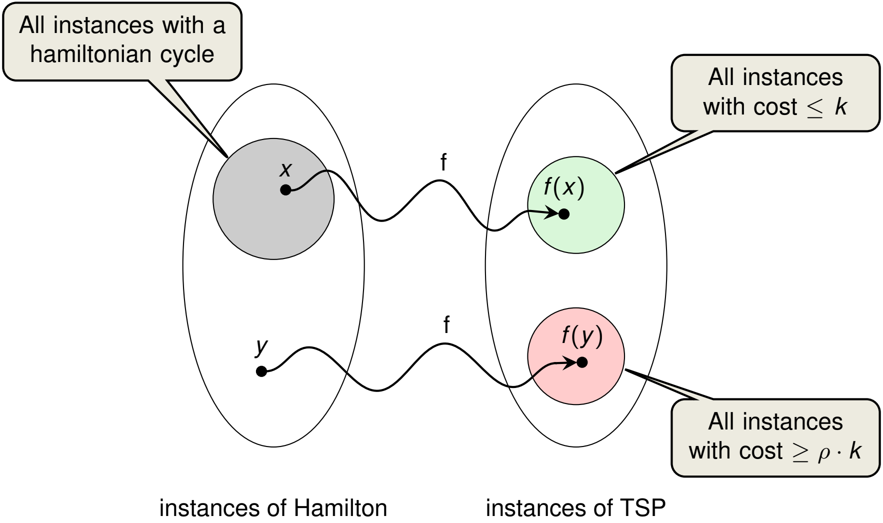
All instances with a hamiltonian cycle
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f
$f(x)$
All instances with cost $\leq k$
y
f
$f(y)$
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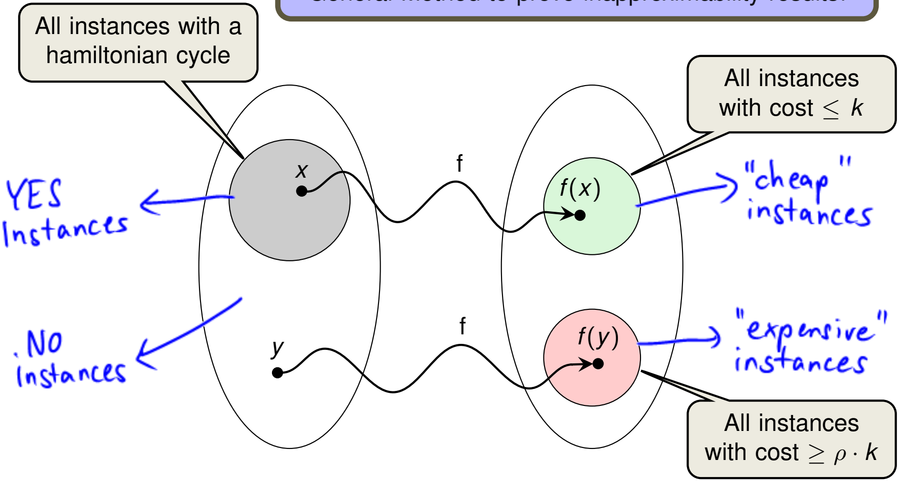
General Method to prove inapproximability results!



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Outline

Introduction

General TSP

Metric TSP

The TSP Problem with the Triangle Inequality

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR$(G, c)$
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T$ for $G$ from root $r$
    using MST-PRIM$(G, c, r)$
let $H$ be a list of vertices, ordered according to when they are first visited
    in a preorder tree walk of $T$
return the hamiltonian cycle $H$
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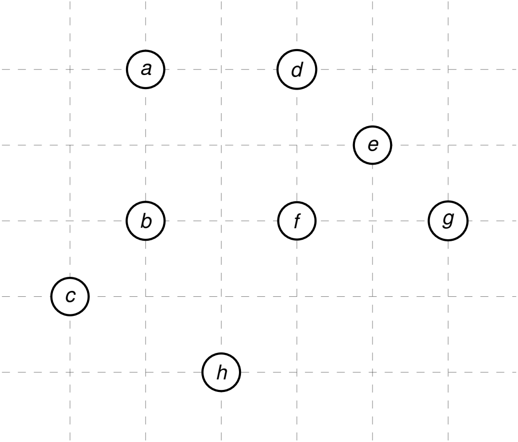
Runtime is dominated by MST-PRIM, which is $\Theta(V^2)$.

number of edges is V^2, as G is a complete graph.

Run of APPROX-TSP-TOUR



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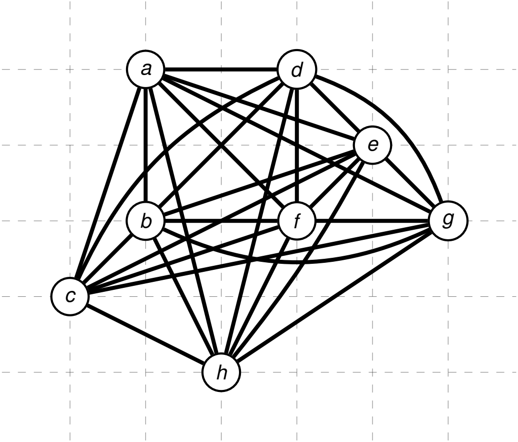
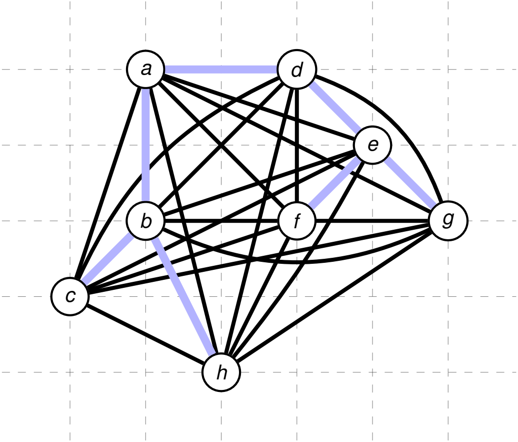


1. Compute MST

Run of APPROX-TSP-TOUR



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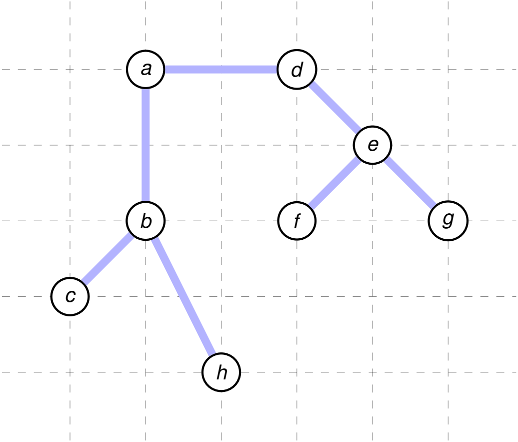

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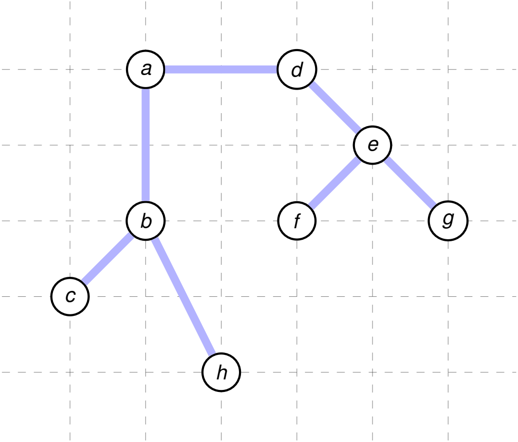
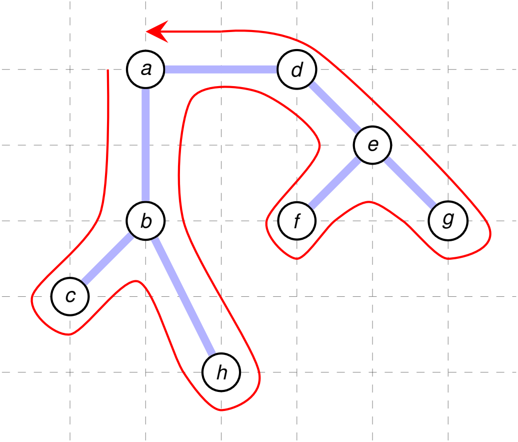


1. Compute MST ✓
2. Perform preorder walk on MST

Run of APPROX-TSP-TOUR



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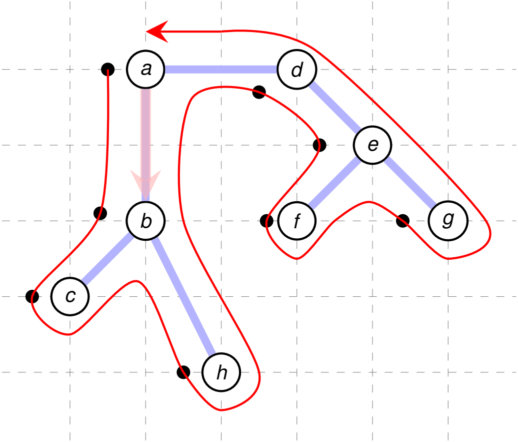
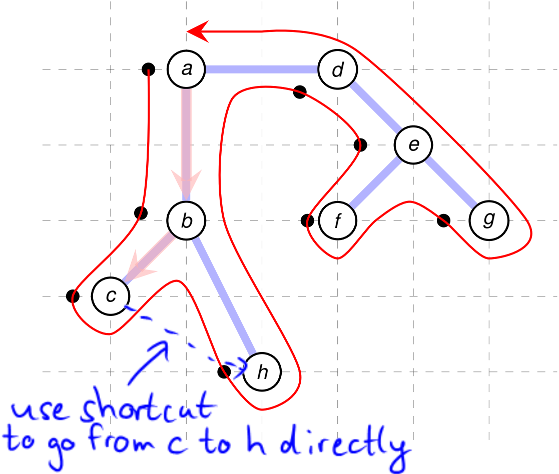


1. Compute MST ✓
2. Perform preorder walk on MST ✓
3. Return list of vertices according to the preorder tree walk

Run of APPROX-TSP-TOUR



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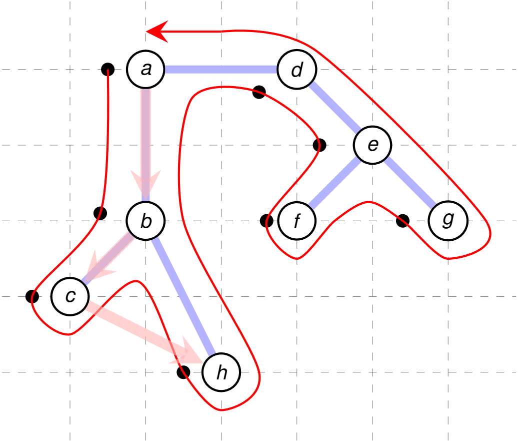

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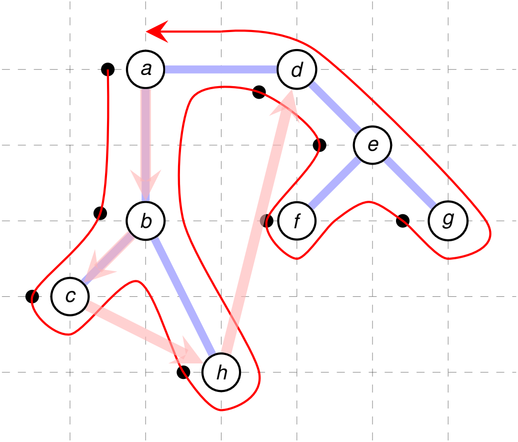

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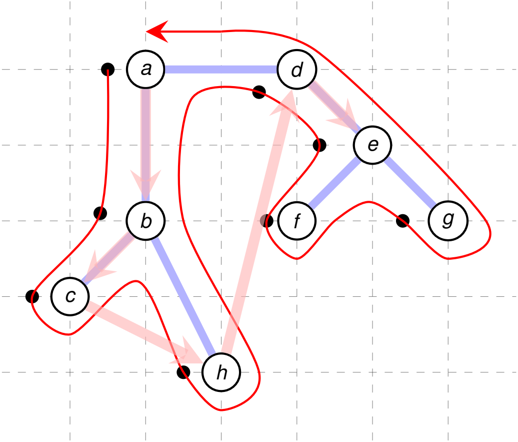

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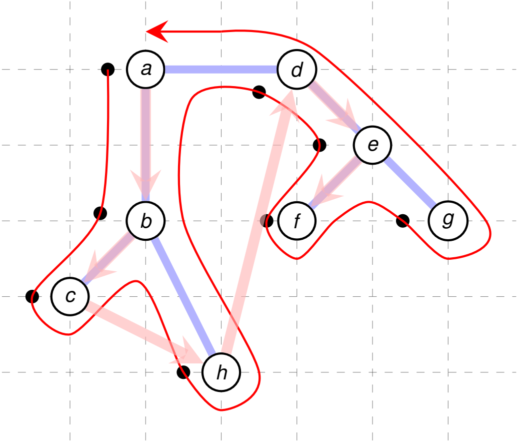

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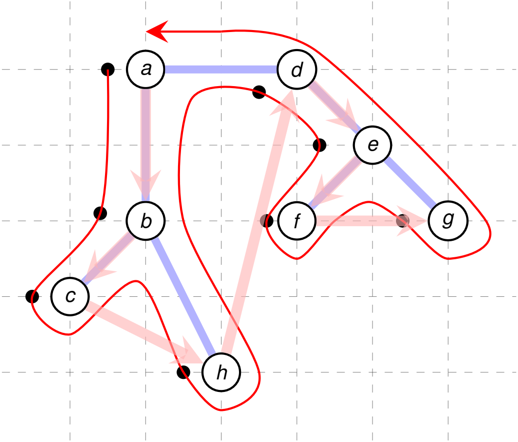

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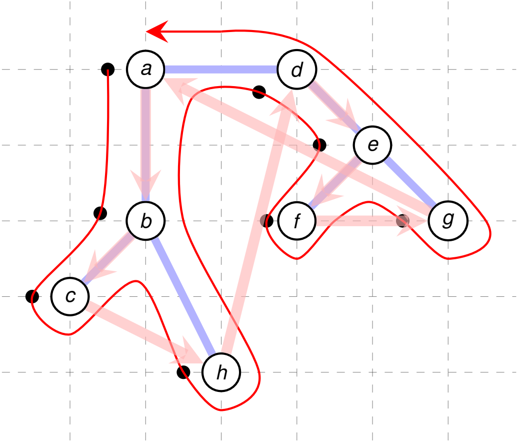

1. Compute MST ✓
2. Perform preorder walk on MST ✓
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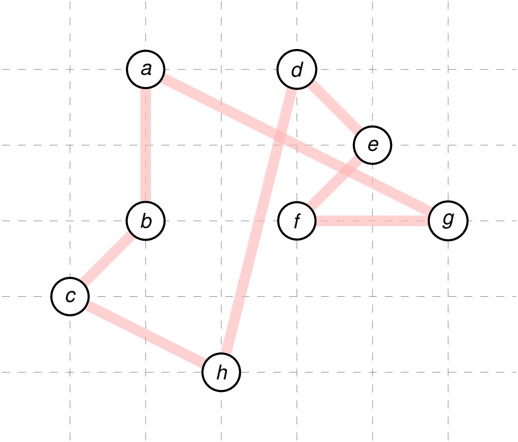

1. Compute MST ✓
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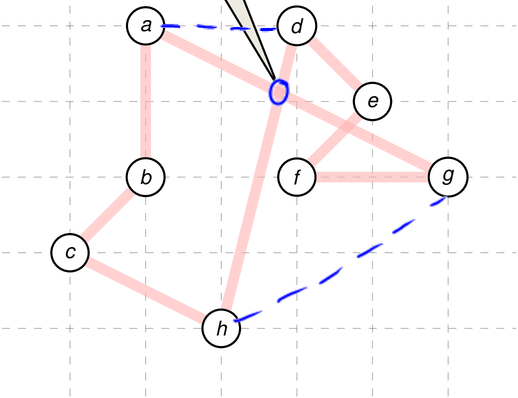
Run of APPROX-TSP-TOUR

Solution has cost ≈ 19.704 - not optimal!



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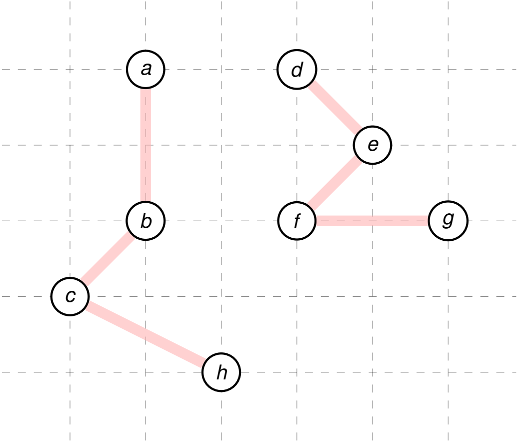

1. Compute MST ✓
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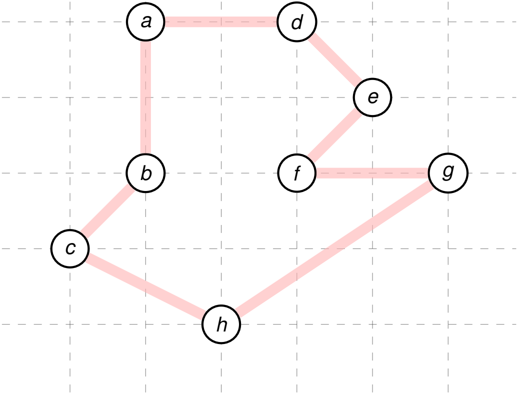
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR

Better solution, yet still not optimal!



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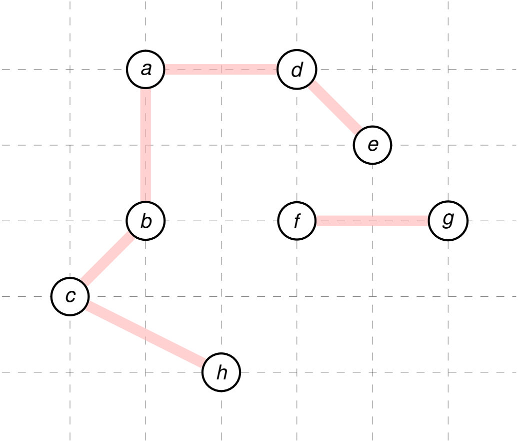
Run of APPROX-TSP-TOUR



1. Compute MST ✓
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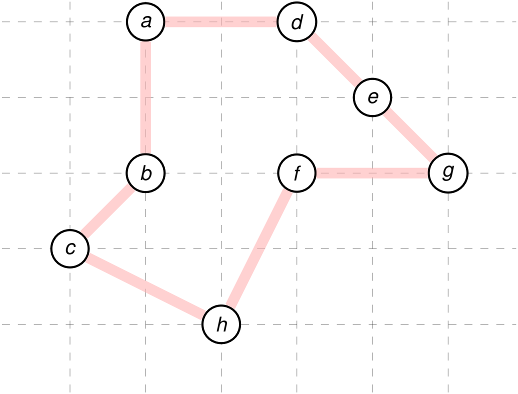
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR

This is the optimal solution (cost $\approx$ 14.715).



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Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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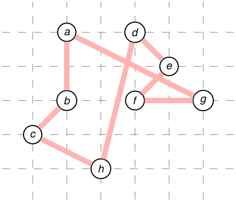

solution H of APPROX-TSP

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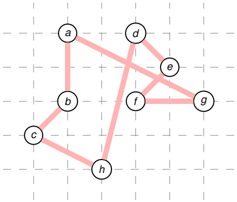

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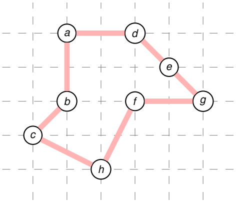

optimal solution H^*

Proof of the Approximation Ratio

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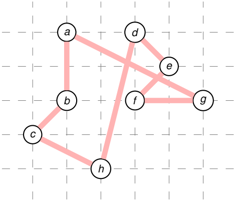
Proof:

- Consider the optimal tour H^* and remove one edge



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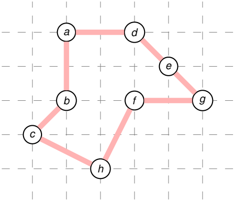

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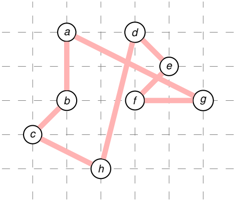

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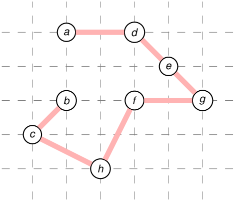

spanning tree as a subset of H^*

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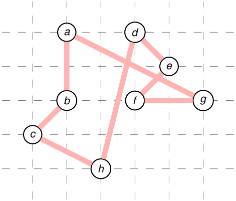

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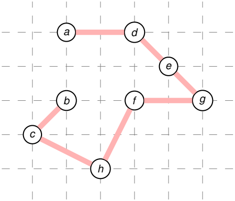

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$\Rightarrow$ yields a spanning tree and therefore $c(T) \leq c(H^*)$

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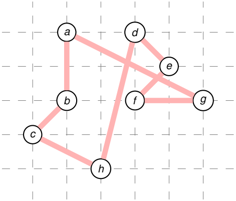

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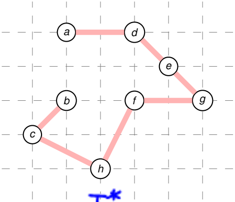

T^*

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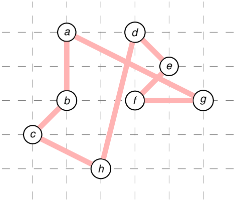
exploiting that all edge costs are non-negative!



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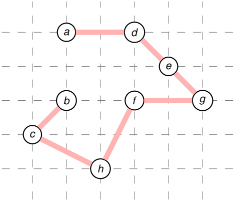

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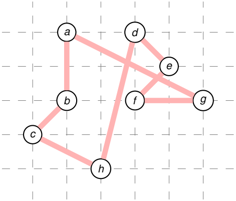

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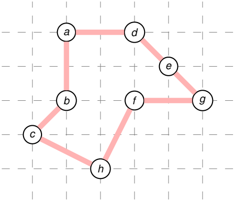

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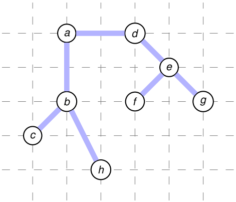
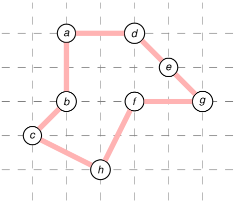


minimum spanning tree T



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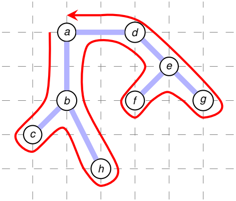
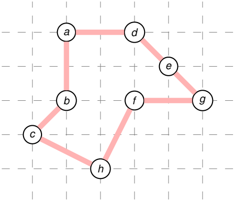


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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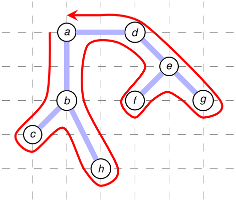

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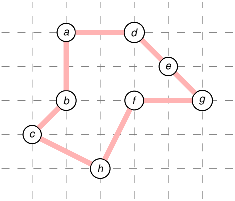

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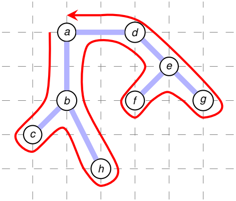

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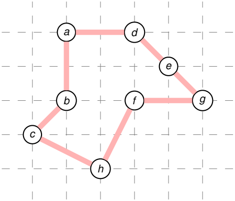

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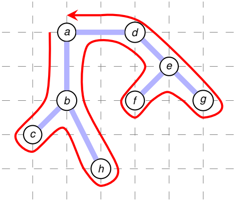

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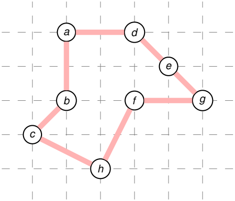

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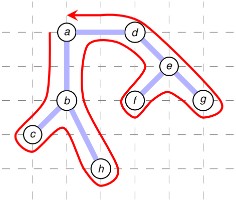

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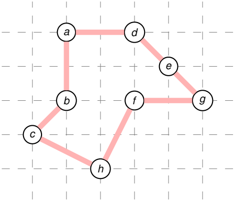

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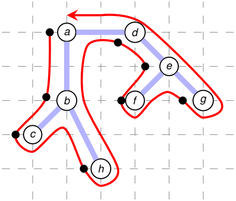

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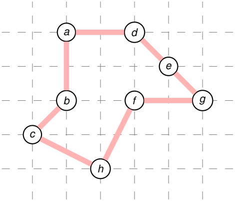

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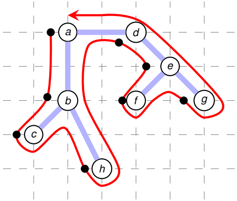
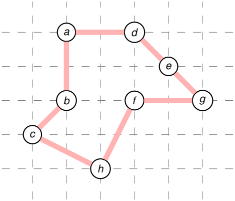


Walk $W = (a, b, c, \cancel{b}, h, \cancel{b}, \cancel{a}, d, e, f, \cancel{e}, g, \cancel{e}, \cancel{d}, a)$



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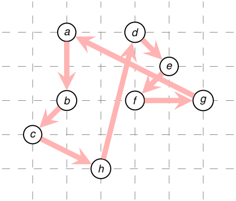
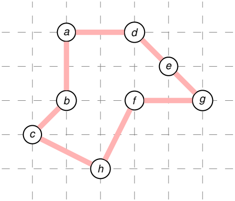


Tour $H = (a, b, c, h, d, e, f, g, a)$



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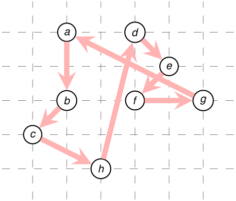

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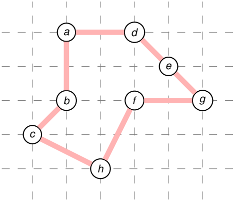

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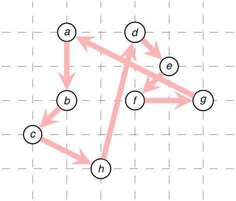
$$c(H) \leq c(W)$$



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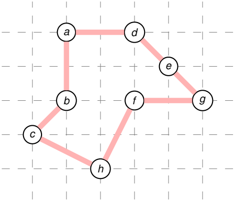

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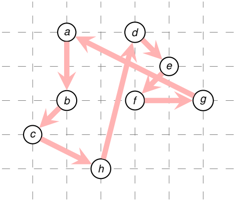

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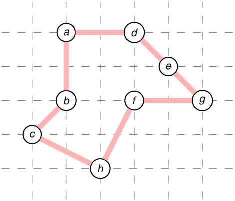

optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour H^* and remove one edge

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exploiting that all edge costs are non-negative!

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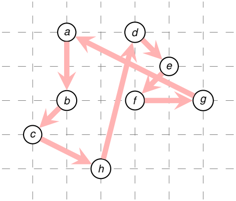

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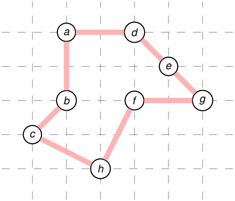

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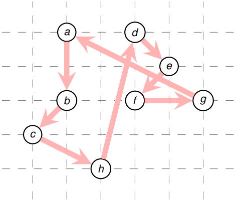
$$c(H) \leq c(W) \leq 2c(H^*) \qquad \square$$



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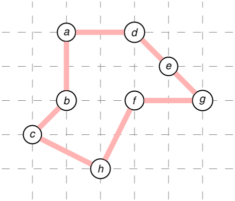

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Christofides Algorithm

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Can we get a better approximation ratio?

```
CHRISTOFIDES($G, c$)
select a vertex $r \in G.V$ to be a “root” vertex
compute a minimum spanning tree $T$ for $G$ from root $r$
        using MST-PRIM($G, c, r$)
compute a perfect matching $M$ with minimum weight in the complete graph
        over the odd-degree vertices in $T$
let $H$ be a list of vertices, ordered according to when they are first visited
        in a Eulearian circuit of $T \cup M$
return $H$
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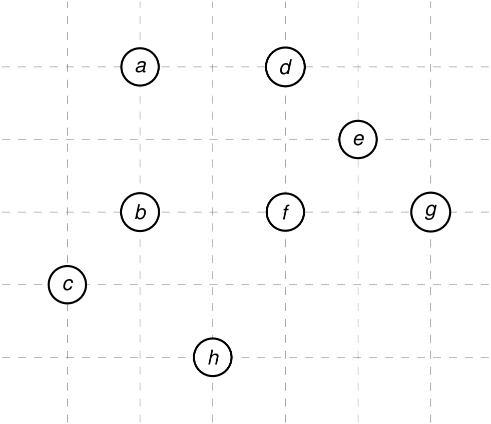
Theorem (Christofides’76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES



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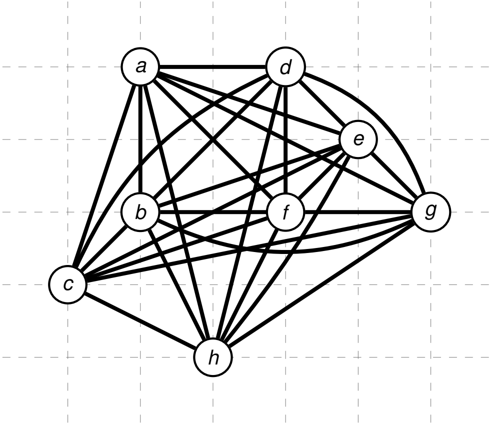
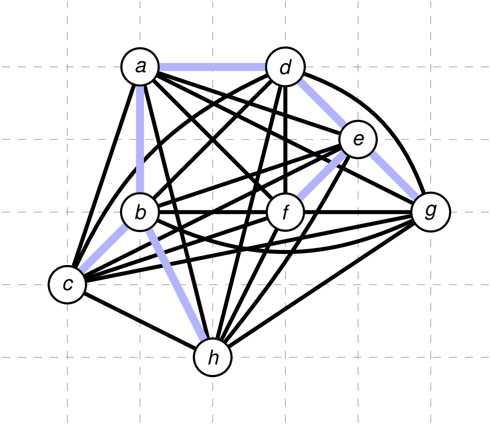


1. Compute MST

Run of CHRISTOFIDES



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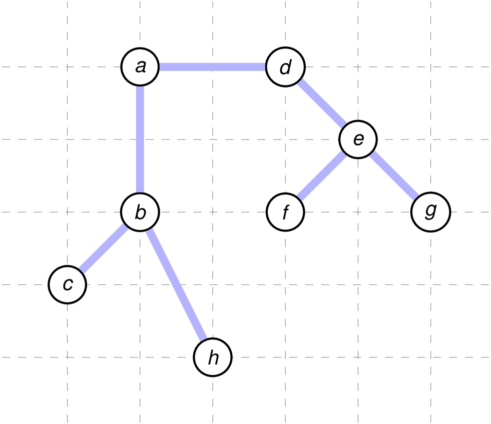

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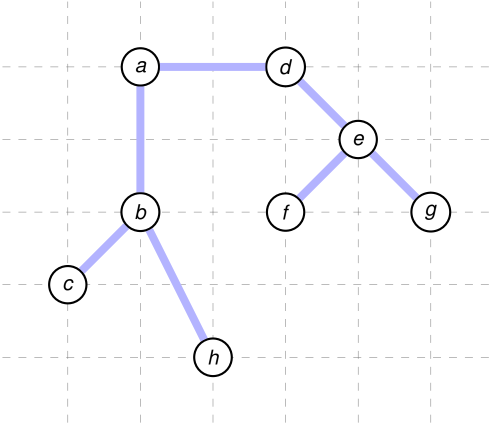
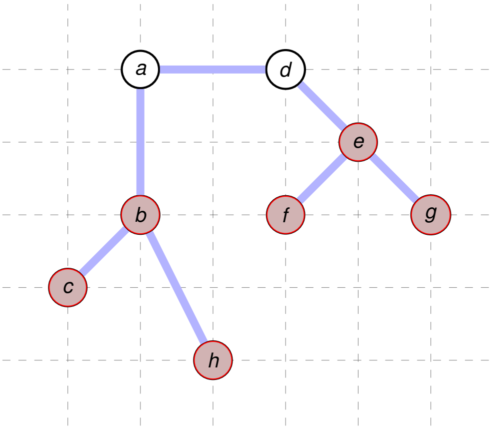
Run of CHRISTOFIDES



1. Compute MST ✓
2. Add a minimum-weight perfect matching M of the odd vertices in T

Run of CHRISTOFIDES



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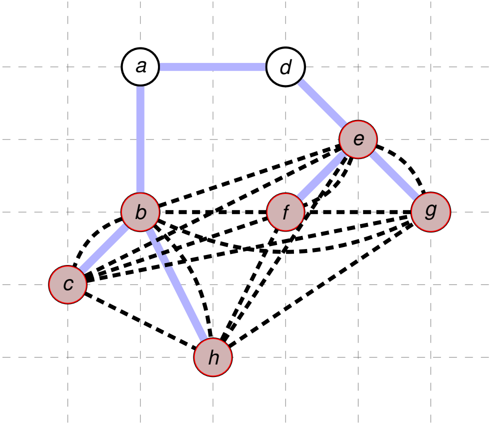

1. Compute MST ✓
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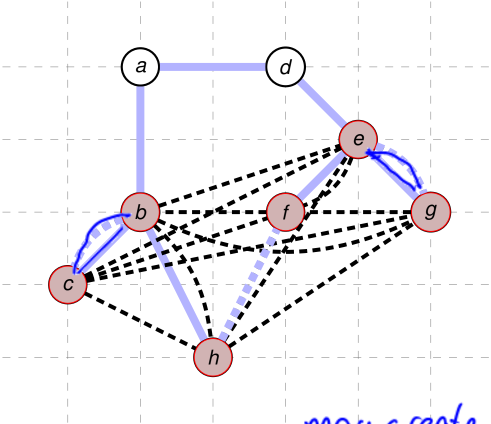
Run of CHRISTOFIDES



→ may create a multigraph!

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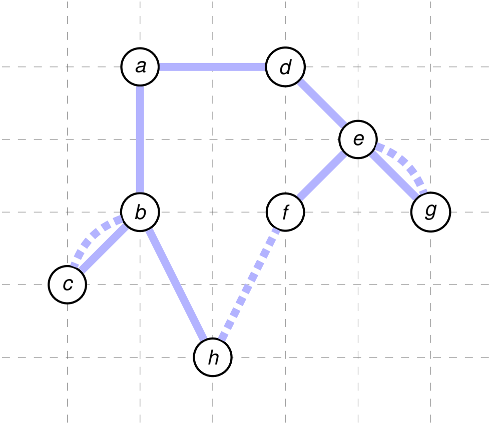
Run of CHRISTOFIDES



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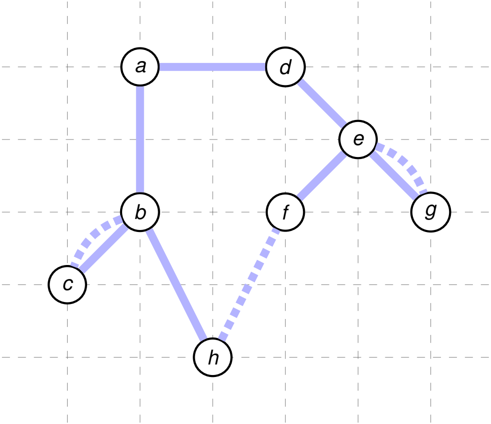
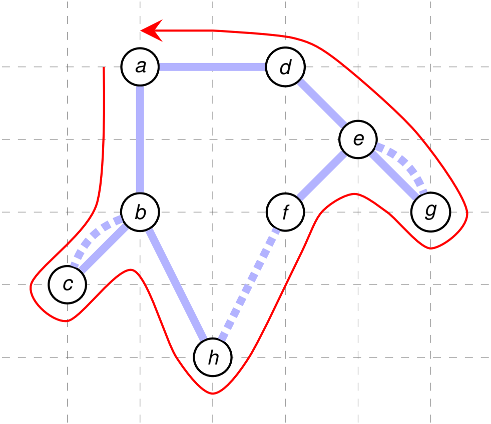


1. Compute MST ✓
2. Add a minimum-weight perfect matching M of the odd vertices in T✓
3. Find an Eulerian Circuit (all vertices in $T \cup M$ have even degree)

Run of CHRISTOFIDES



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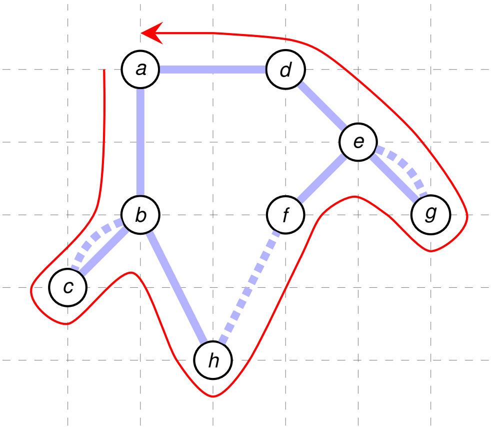

1. Compute MST ✓
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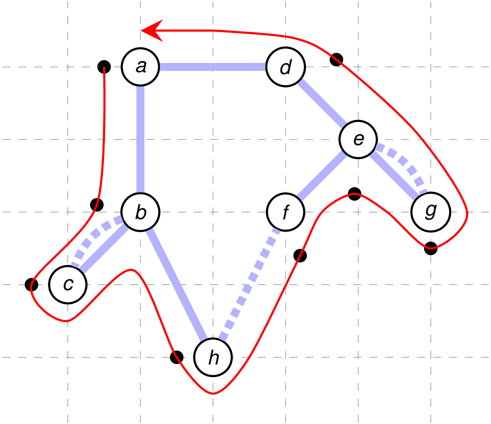
4. Transform the Circuit into a Hamiltonian Cycle

Run of CHRISTOFIDES



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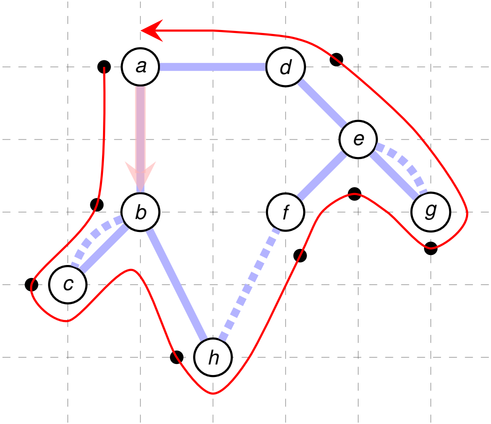

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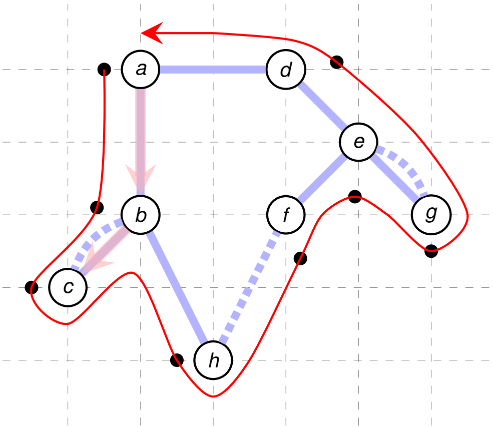

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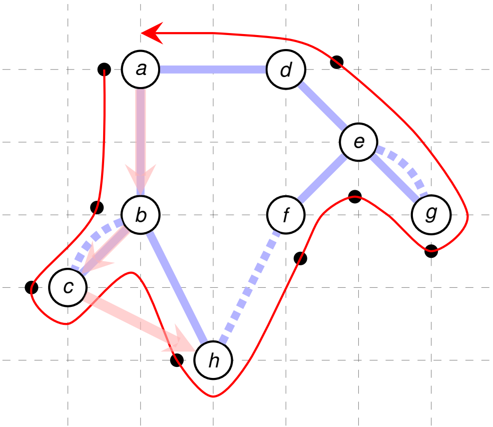

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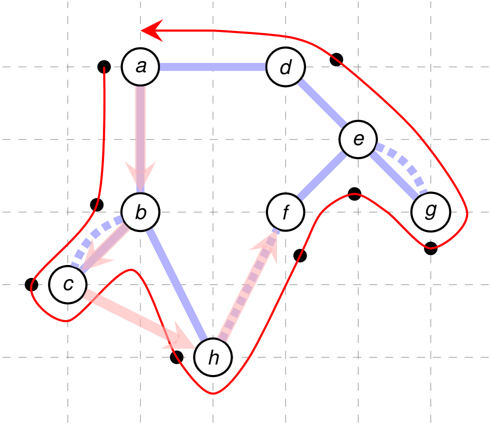

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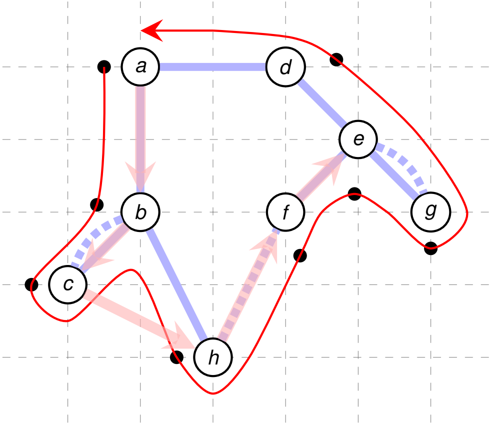

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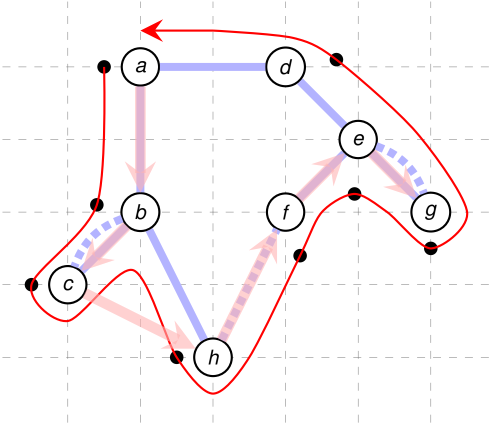

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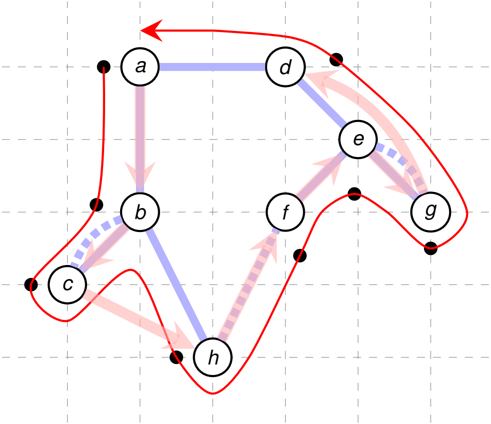

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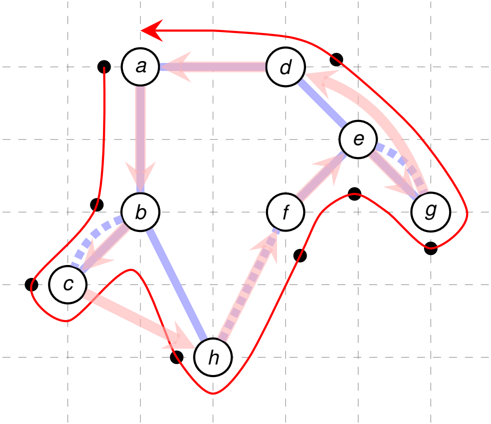

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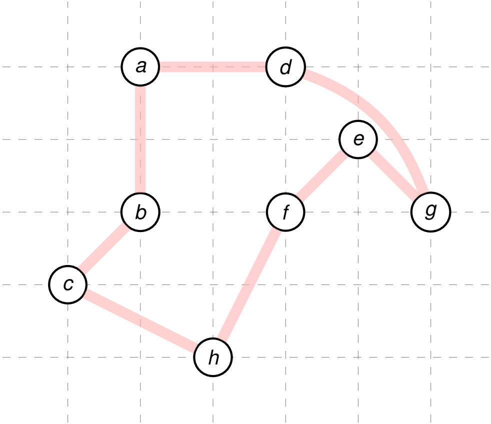

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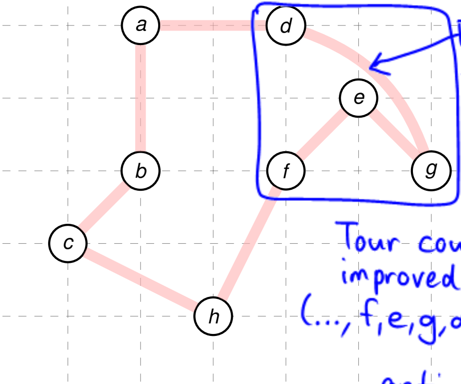
Solution has cost ≈ 15.54 - within 10% of the optimum!



Despite its drawing, this edge goes straight through e, i.e. tour crosses itself

Tour could be further improved by changing (..., f, e, g, d, ...) to (.., f, g, e, d, ..)

⇓

optimal tour

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Concluding Remarks

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There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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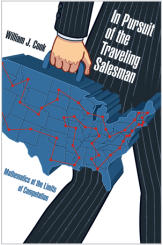
William J. Cook
In Pursuit of the Traveling Salesman
Mathematics at the Limits of Computation

VII. Approximation Algorithms: Randomisation and Rounding

Thomas Sauerwald

Easter 2015

UNIVERSITY OF CAMBRIDGE

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

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- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n.
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extends in the natural way to randomised algorithms

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Weighted Vertex Cover

Weighted Set Cover

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- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$

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Example:

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Idea: What about assigning each variable independently at random?

Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Proof:

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Proof:

- For every clause $i = 1, 2, \ldots, m$, define a random variable:

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maximum number of satisfiable clauses is m

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Interesting Implications

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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For any instance of MAX-3-CNF, there exists an assigment which satisfies at least $\frac{7}{8}$ of all clauses.

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$7 \cdot \frac{7}{8} > 6$

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Follows from the previous Corollary.

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One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid \underline{x_1 = 1}\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid \underline{x_1 = 0}\,].$$

Y is defined as in the previous proof.

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Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

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```
GREEDY-3-CNF($\phi, n, m$)
for $\boxed{j} = 1, 2, \ldots, n$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, \boxed{x_j = 1}\,]$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, \boxed{x_j = 0}\,]$
    Let $x_j = v_j$ so that the conditional expectation is maximized
return the assignment $v_1, v_2, \ldots, v_n$
```

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $\underline{8/7}$-approximation.

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This algorithm is deterministic.

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random assignment to
3 (or less) literals in one
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↓ x_j is still random

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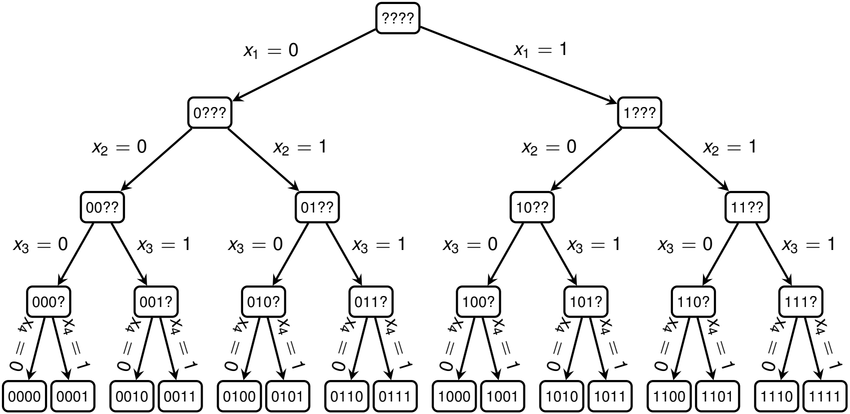
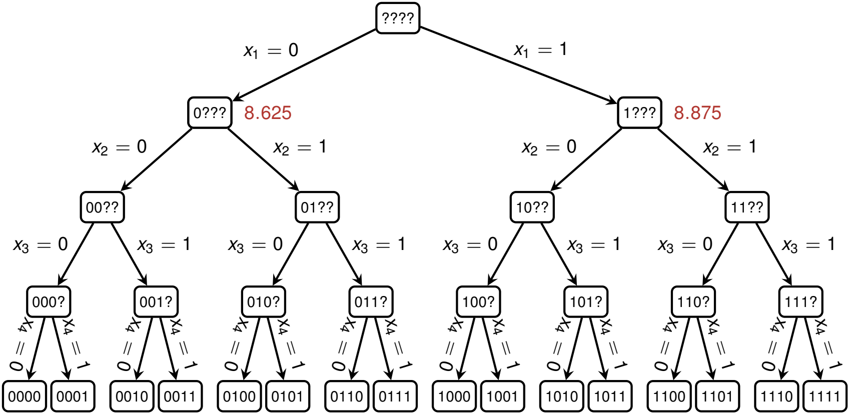
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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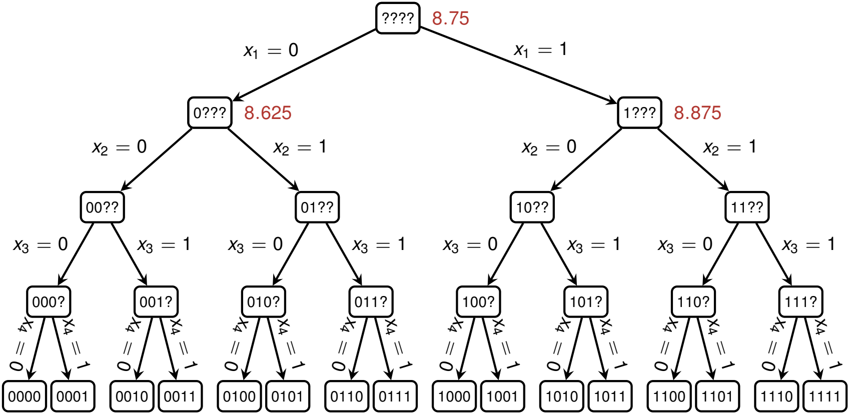
$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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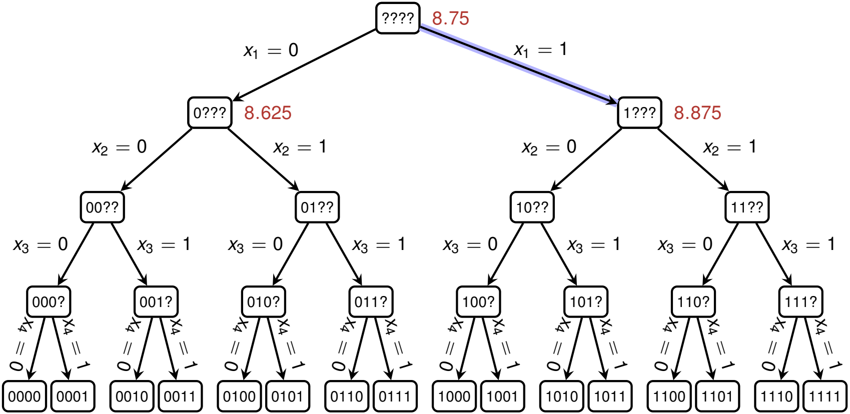
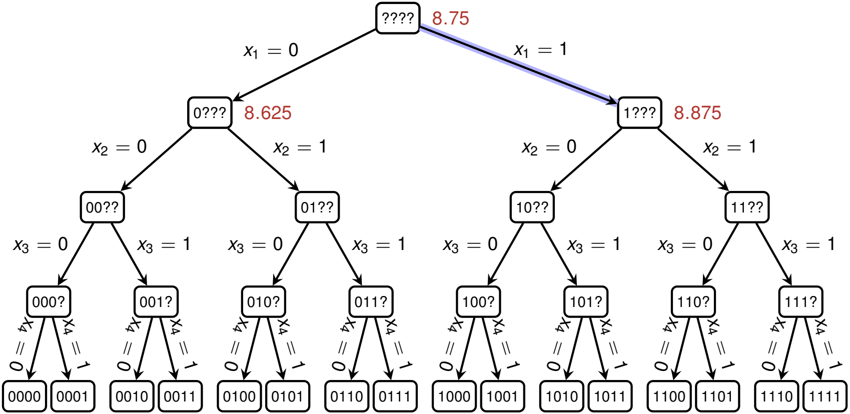
$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



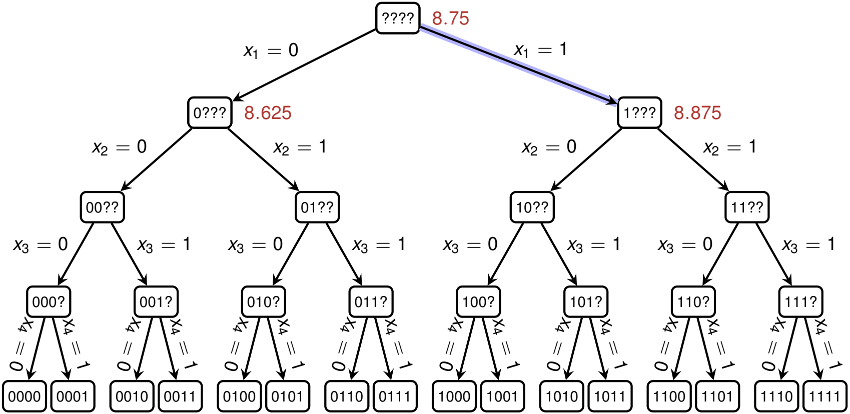
Run of GREEDY-3-CNF(φ, n, m)

$\cancel{(x_1 \vee x_2 \vee \overline{x_3})} \wedge \cancel{(x_1 \vee \overline{x_2} \vee \overline{x_4})} \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_4})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_3} \vee x_4) \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_3})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\cancel{\overline{x_1}} \vee x_2 \vee x_3) \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee x_3) \wedge \cancel{(x_1 \vee x_3 \vee \overline{x_4})} \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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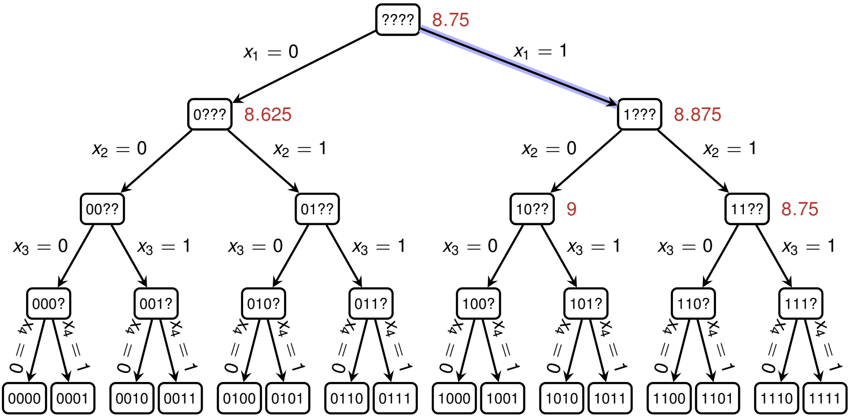

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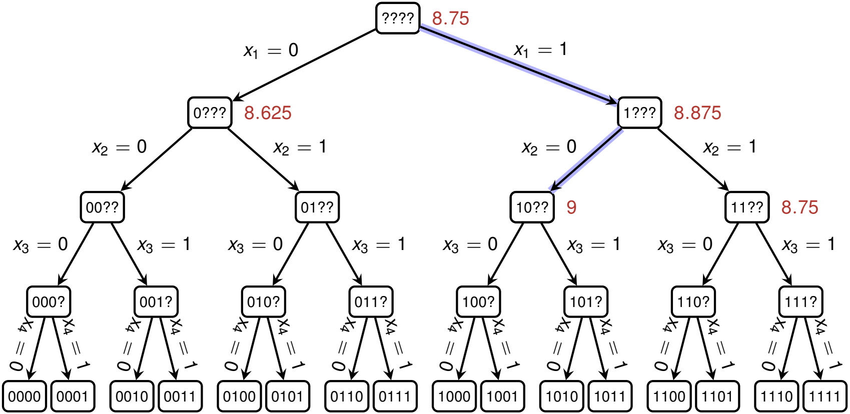

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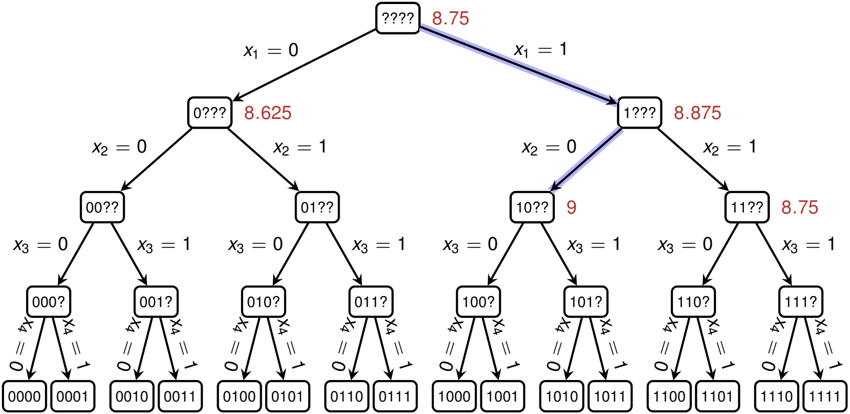

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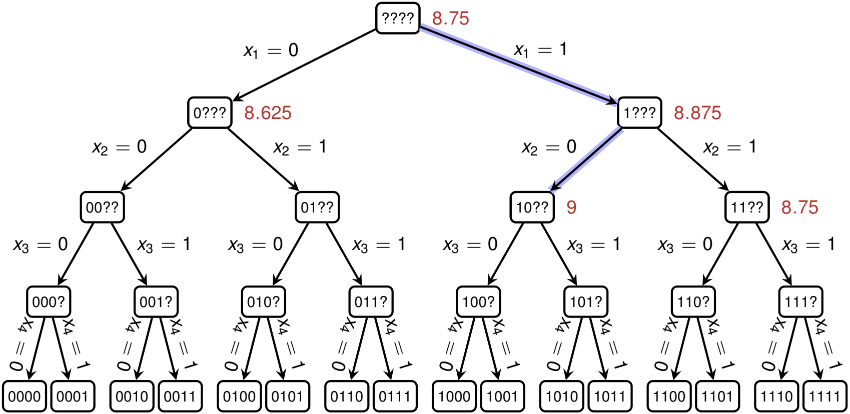

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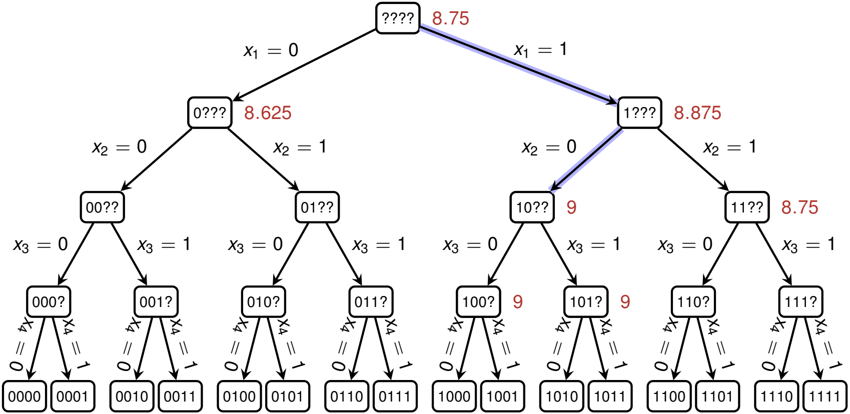

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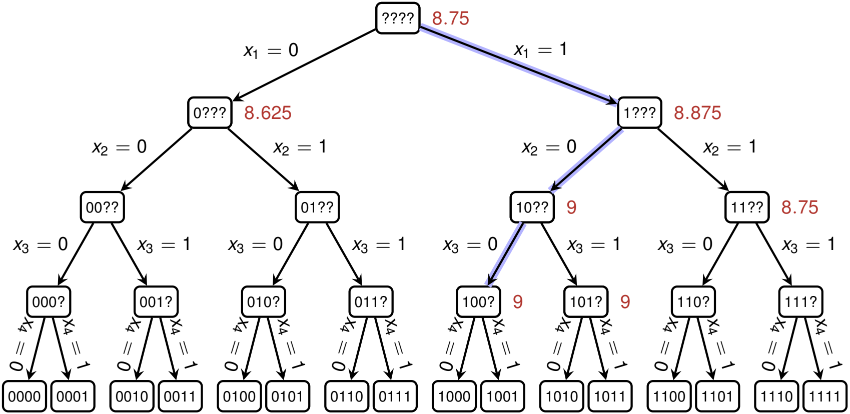

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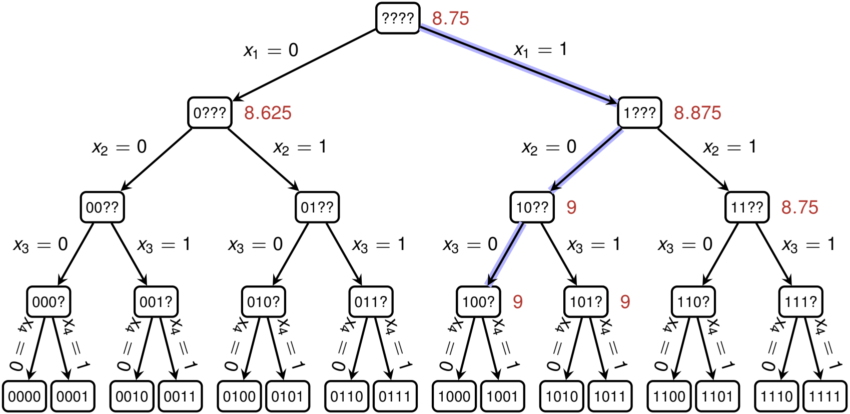
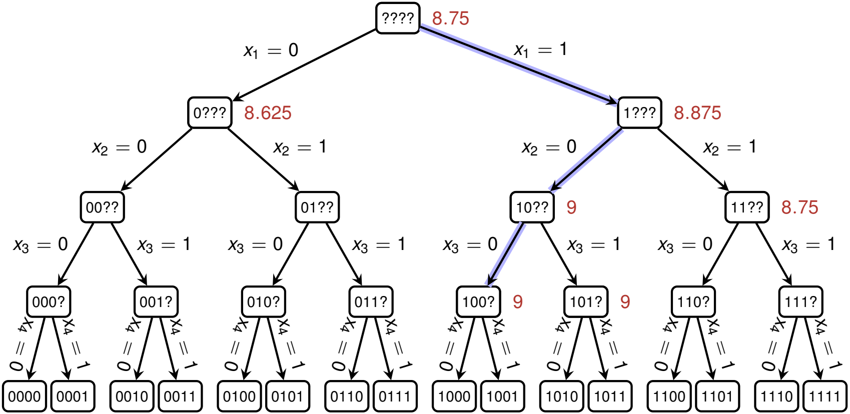


Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



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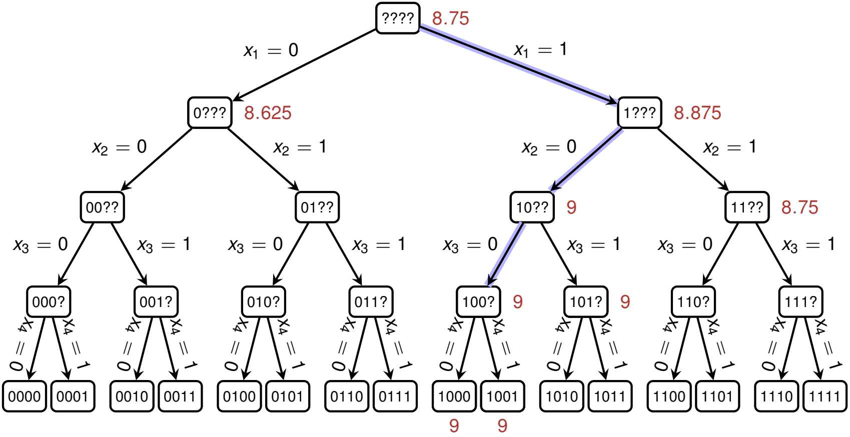

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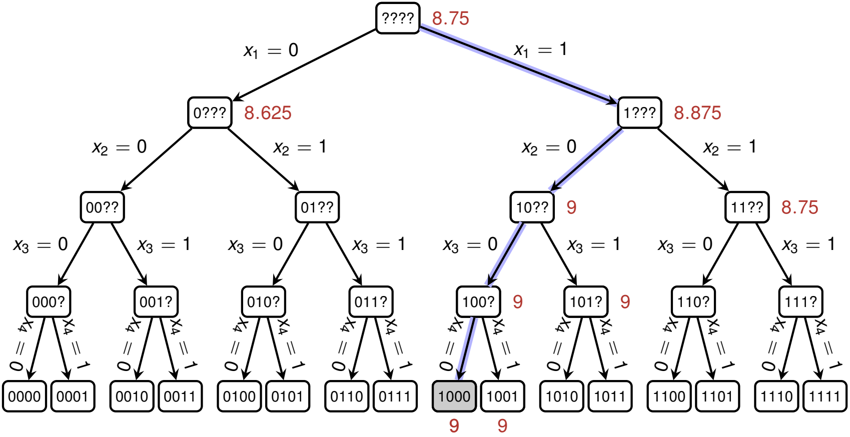

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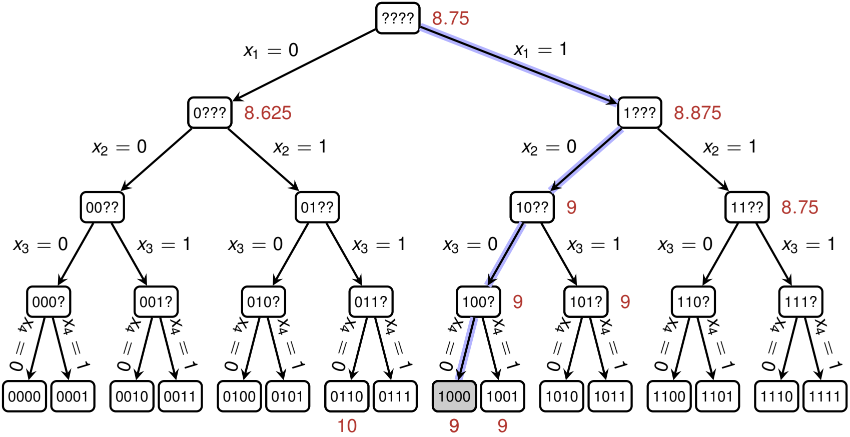

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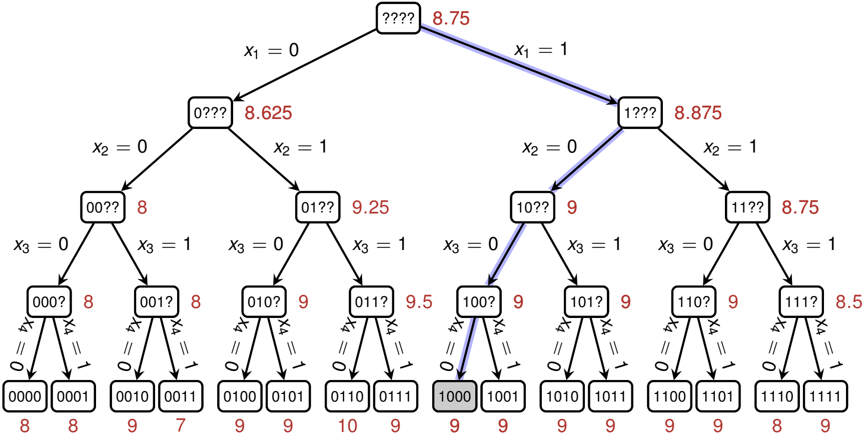

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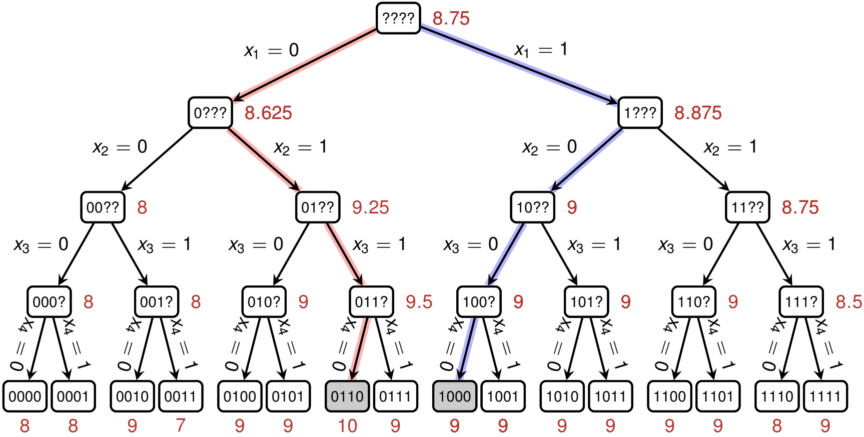

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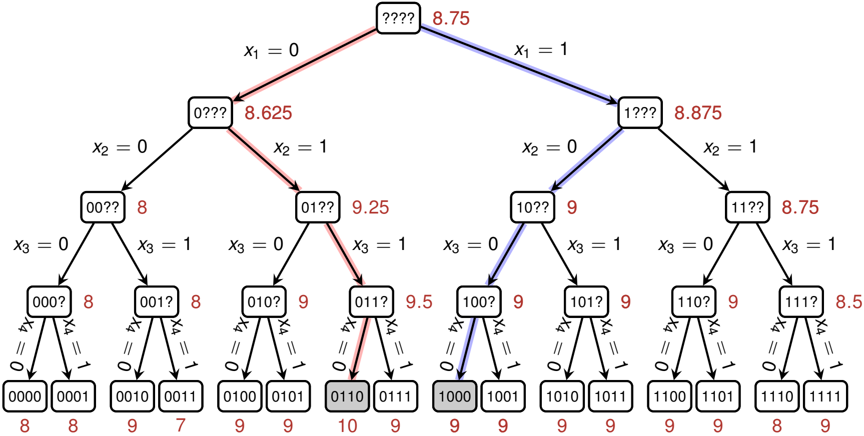

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Theorem (Hastad'97)

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Roughly speaking, there is nothing smarter than just guessing.

Outline

Randomised Approximation

MAX-3-CNF

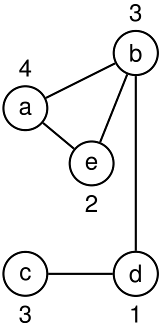
Weighted Vertex Cover

Weighted Set Cover

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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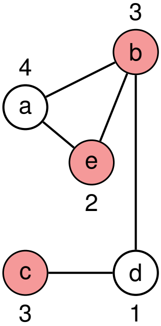

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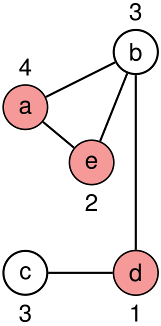
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



The Weighted Vertex-Cover Problem

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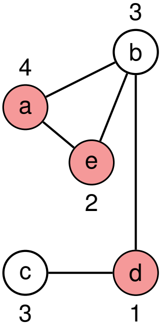

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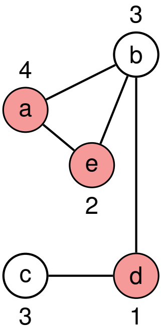

Applications:

The Weighted Vertex-Cover Problem

Vertex Cover Problem

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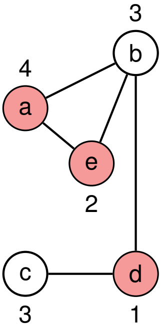

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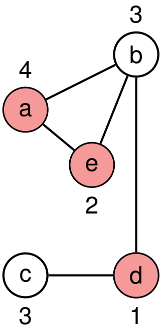

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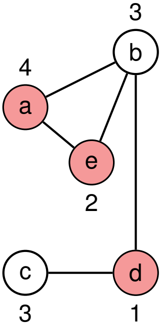

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

```
APPROX-VERTEX-COVER$(G)$
$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
    let $(u, v)$ be an arbitrary edge of $E'$
    $C = C \cup \{u, v\}$
    remove from $E'$ every edge incident on either $u$ or $v$
return $C$
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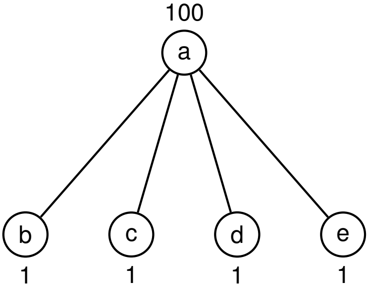

The Greedy Approach from (Unweighted) Vertex Cover

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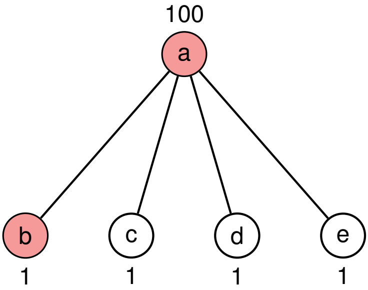
APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```

ignores all vertex-weights



Computed solution has weight 101.

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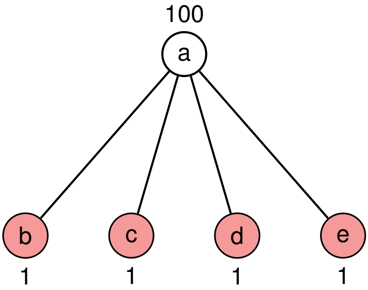

Optimal solution has weight 4.

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & & \\
\text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u, v) \in E \\
 & x(v) & \in & \{0, 1\} & \text{for each } v \in V
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Linear Program

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 & \underbrace{x(v) \in [0,1]}_{\to \text{ more formally, } 0 \leq x(v) \leq 1} & & & \text{for each } v \in V
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optimum is a lower bound on the optimal weight of a minimum weight-cover.

Linear Program

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

```
APPROX-MIN-WEIGHT-VC$(G, w)$
$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return $C$
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same as the Greedy for unweighted

Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

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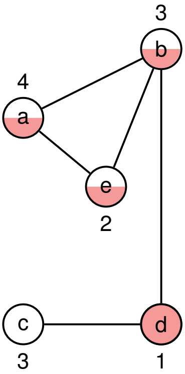
Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$



fractional solution of LP
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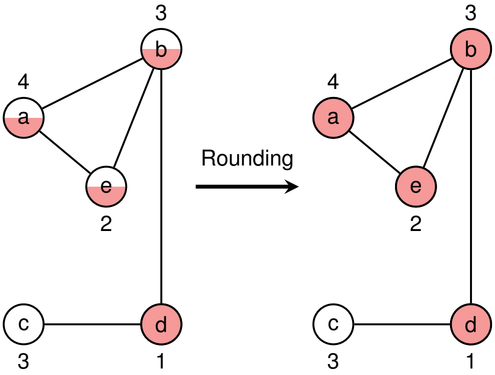

fractional solution of LP
with weight = 5.5

rounded solution of LP
with weight = 10

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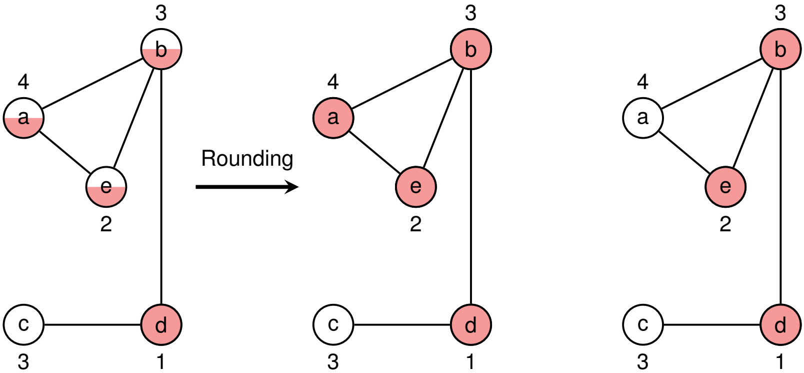

fractional solution of LP
with weight $= 5.5$

rounded solution of LP
with weight $= 10$

optimal solution
with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2):

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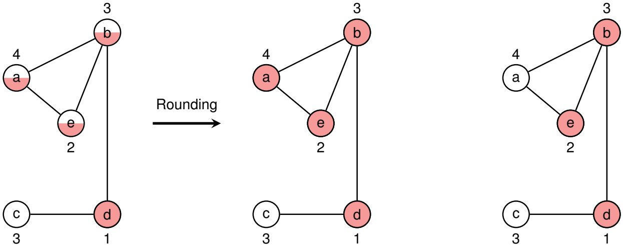

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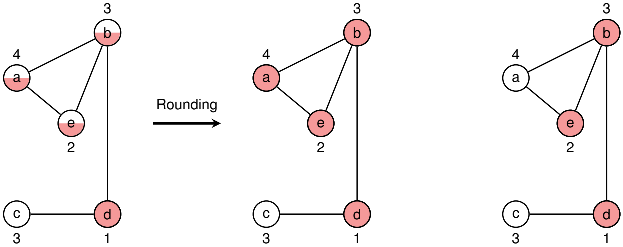

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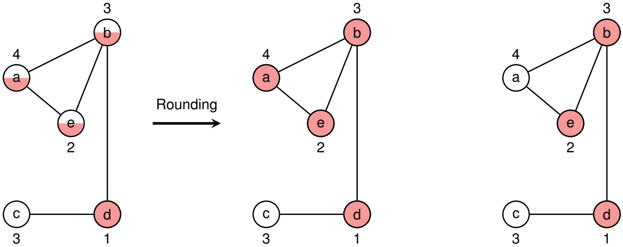

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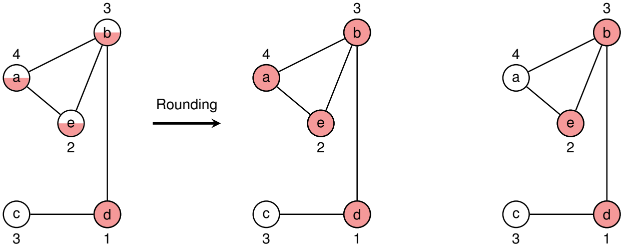

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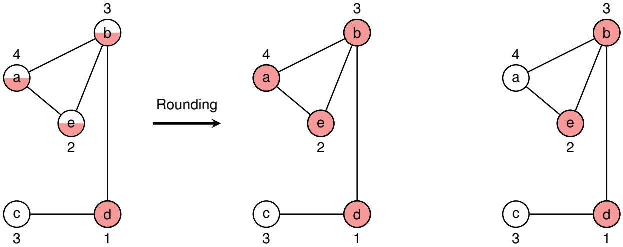

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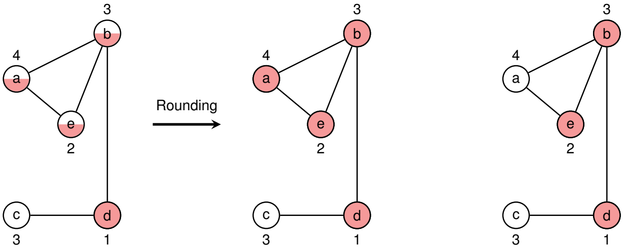

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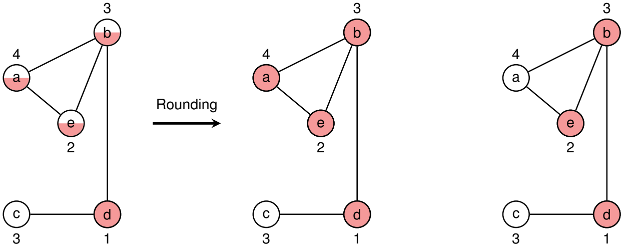

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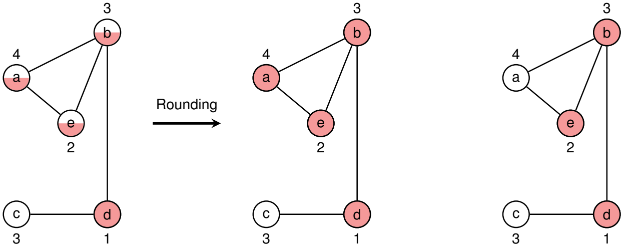

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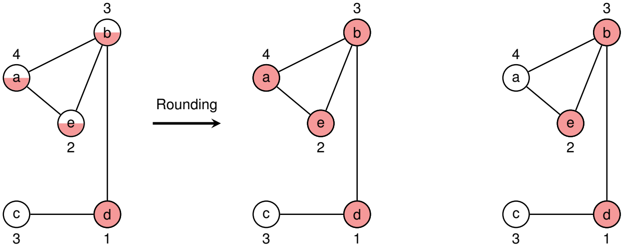

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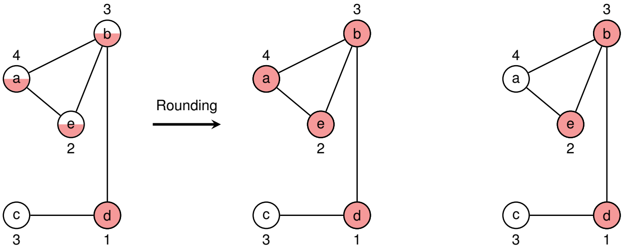

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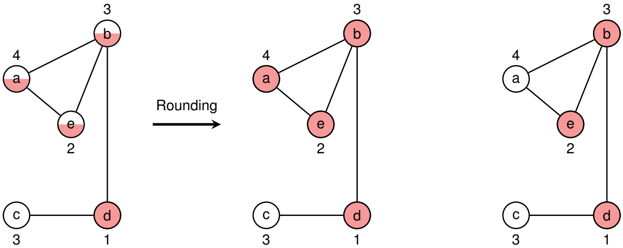

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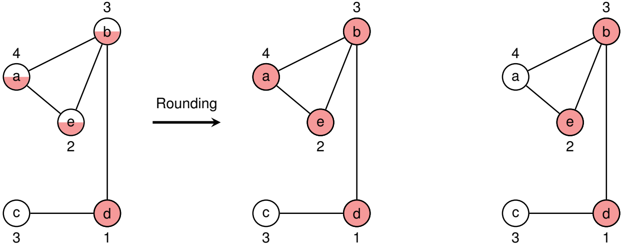

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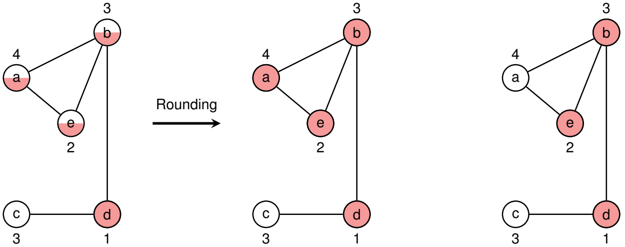

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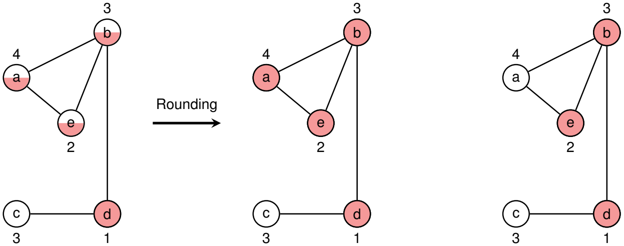

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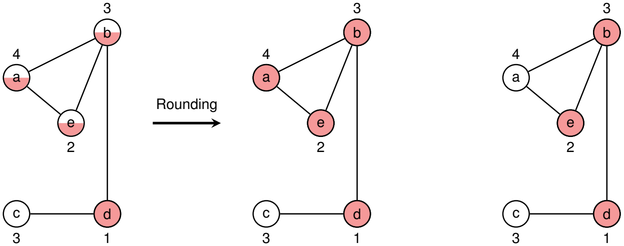

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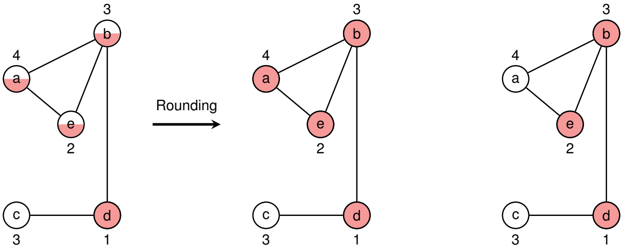
$$w(C^*) \geq z^* = \sum_{v \in V} w(v)\overline{x}(v) \geq \sum_{v \in V:\ \overline{x}(v) \geq 1/2} w(v) \cdot \frac{1}{2} = \frac{1}{2} w(C). \quad \square$$



Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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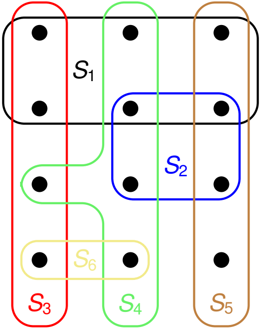

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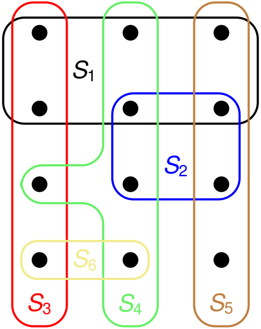

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c :	2	3	3	5	1	2

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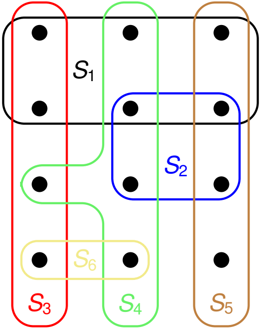

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

Setting up an Integer Program

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
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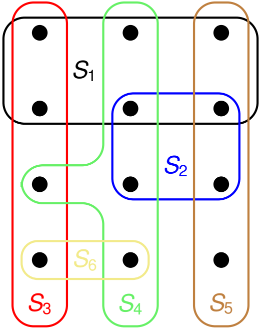
Back to the Example



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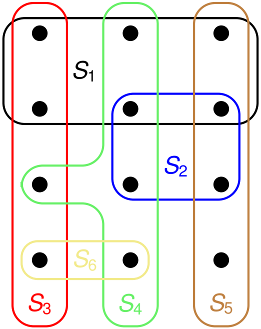

	S_1	S_2	S_3	S_4	S_5	S_6
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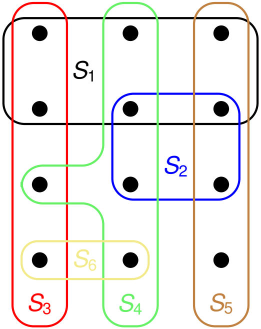

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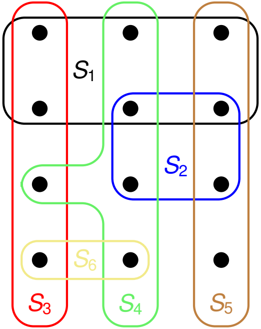

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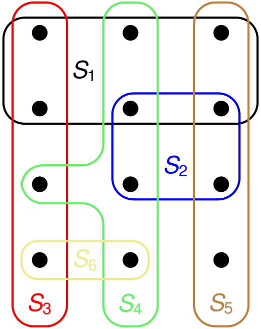

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Even worse: If all y's were below $1/2$, we would not even return a valid cover!

Randomised Rounding

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Idea: Interpret the y-values as probabilities for picking the respective set.

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Lemma

→ may or may not be feasible

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $y(S)$.

in other words, if $y(.)$ is the LP solution, then we obtain an IP solution $y'(.)$ by:

$$\forall S \in \mathcal{F}: \ y'(S) = \begin{cases} 1 & \text{with prob. } y(S) \\ 0 & \text{with prob. } 1 - y(S) \end{cases}$$

$$\Rightarrow E[y'(S)] = y(S)$$

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this is a random subset!

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how we can ↗ apply linearity of expectations

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$1 + x \leq e^x$ for any $x \in \mathbb{R}$

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$$= e^{-\boxed{\sum_{S \in \mathcal{F}:\ x \in S} y(S)}} \geq 1$$

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clearly runs in polynomial-time!

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$$\mathbf{Pr}\left[\, X = \cup_{S \in \mathcal{C}} S \,\right] = 1 - \mathbf{Pr}\left[\, \bigcup_{x \in X} \{x \notin \cup_{S \in \mathcal{C}} S\} \,\right]$$

$\mathbf{Pr}[\, A \cup B \,] \leq \mathbf{Pr}[\, A \,] + \mathbf{Pr}[\, B \,]$

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- **Step 2**: The expected approximation ratio ✓
  - By previous lemma, the expected cost of one iteration is $\sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
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Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Proof:

- **Step 1**: The probability that $\mathcal{C}$ is a cover ✓
  - By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

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Hence with probability at least $1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}$, solution is within a factor of $4\ln(n)$ of the optimum.

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Typical Approach for Designing Approximation Algorithms based on LPs

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Thank you and Best Wishes for the Exam!