Mathematical Foundations of Computer Networking

by

S. Keshav

To Nicole, my foundation

Introduction

Motivation

Graduate students, researchers, and practitioners in the field of computer networking often require a firm conceptual understanding of one or more of its theoretical foundations. Knowledge of optimization, information theory, game theory, control theory, and queueing theory is assumed by research papers in the field. Yet these subjects are not taught in a typical computer science undergraduate curriculum. This leaves only two alternatives: either to study these topics on one's own from standard texts or take a remedial course. Neither alternative is attractive. Standard texts pay little attention to computer networking in their choice of problem areas, making it a challenge to map from the text to the problem at hand. And it is inefficient to require students to take an entire course when all that is needed is an introduction to the topic.

This book addresses these problems by providing a single source to learn about the mathematical foundations of computer networking. Assuming only a rudimentary grasp of calculus, it provides an intuitive yet rigorous introduction to a wide range of mathematical topics. The topics are covered in sufficient detail so that the book will usually serve as both the first and ultimate reference. Note that the topics are selected to be *complementary* to those found in a typical undergraduate computer science curriculum. The book, therefore, does not cover network foundations such as discrete mathematics, combinatorics, or graph theory.

Each concept in the book is described in four ways: intuitively; using precise mathematical notation; with a carefully chosen numerical example; and with a numerical exercise to be done by the reader. This progression is designed to gradually deepen understanding. Nevertheless, the depth of coverage provided here is not a substitute for that found in standard textbooks. Rather, I hope to provide enough intuition to allow a student to grasp the essence of a research paper that uses these theoretical foundations.

Organization

The chapters in this book fall into two broad categories: foundations and theories. The first five foundational chapters cover probability, statistics, linear algebra, optimization, and signals, systems and transforms. These chapters provide the basis for the four theories covered in the latter half of the book: queueing theory, game theory, control theory, and information theory. Each chapter is written to be as self-contained as possible. Nevertheless, some dependencies do exist, as shown in Figure 1, where light arrows show weak dependencies and bold arrows show strong dependencies.

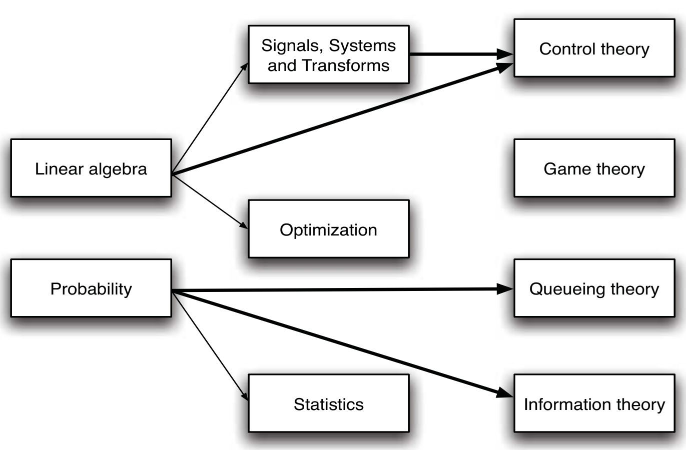

FIGURE 1. **Chapter organization**

Using this book

The material in this book can be completely covered in a sequence of two graduate courses, with the first course focussing on the first five chapters and the second course on the latter four. For a single-semester course, some posible alternatives are to cover:

- probability, statistics, queueing theory, and information theory
- linear algebra, signals, systems and transforms, control theory and game theory
- linear algebra, signals, systems and transforms, control theory, selected portions of probability, and information theory
- linear algebra, optimization, probability, queueing theory, and information theory

This book is designed for self-study. Each chapter has numerous solved examples and exercises to reinfornce concepts. My aim is to ensure that every topic in the book should be accessible to the perservering reader.

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CHAPTER 1 *Probability*

1.1 Introduction

The concept of probability pervades every aspect of our life. Weather forecasts are couched in probabilistic terms, as are economic predictions and even outcomes of our own personal decisions. Designers and operators of computer networks need to often think probabilistically, for instance, when anticipating future traffic workloads or computing cache hit rates. From a mathematical standpoint, a good grasp of probability is a necessary foundation to understanding statistics, game theory, and information theory. For these reasons, the first step in our excursion into the mathematical foundations of computer networking is to study the concepts and theorems of probability.

This chapter is a self-contained introduction to the theory of probability. We begin by introducing the elementary concepts of outcomes, events, and sample spaces, which allows us to precisely define the conjunctions and disjunctions of events. We then discuss concepts of conditional probability and Bayes' rule. This is followed by a description of discrete and continuous random variables, expectations and other moments of a random variable, and the Moment Generating Function. We discuss some standard discrete and continuous distributions and conclude with some useful theorems of probability and a description of Bayesian networks.

Note that in this chapter, as in the rest of the book, the solved examples are an essential part of the text. They provide a concrete grounding for otherwise abstract concepts and are necessary to understand the material that follows.

1.1.1 Outcomes

The mathematical theory of probability uses terms such as 'outcome' and 'event' with meanings that differ from those in common practice. Therefore, we first introduce a standard set of terms to precisely discuss probabilistic processes. These terms are shown in bold font below. We will use the same convention to introduce other mathematical terms in the rest of the book.

Probability measures the degree of uncertainty about the potential **outcomes** of a **process**. Given a set of **distinct** and **mutually exclusive** outcomes of a process, denoted $\{o_1, o_2, \ldots\}$, called the **sample space *S***, the probability of any outcome, denoted $P(o_i)$, is a real number between 0 and 1, where 1 means that the outcome will surely occur, 0 means that it surely will not occur, and intermediate values reflect the degree to which one is confident that the outcome will or will not occur[1]. We assume that it is certain that *some* element in S occurs. Hence, the elements of S describe all possible outcomes and the sum of probability of all the elements of S is always 1.

1. Strictly speaking, S must be a measurable σ field.

EXAMPLE 1: SAMPLE SPACE AND OUTCOMES

Imagine rolling a six-faced die numbered 1 through 6. The process is that of rolling a die and an outcome is the number shown on the upper horizontal face when the die comes to rest. Note that the outcomes are distinct and mutually exclusive because there can be only one upper horizontal face corresponding to each throw.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$ which has a size $|S| = 6$. If the die is fair, each outcome is equally likely and the probability of each outcome is $\frac{1}{|S|} = \frac{1}{6}$.

■

EXAMPLE 2: INFINITE SAMPLE SPACE AND ZERO PROBABILITY

Imagine throwing a dart at random on to a dartboard of unit radius. The process is that of throwing a dart and the outcome is the point where the dart penetrates the dartboard. We will assume that this is point is vanishingly small, so that it can be thought of as a point on a two-dimensional real plane. Then, the outcomes are distinct and mutually exclusive.

The sample space S is the infinite set of points that lie within a unit circle in the real plane. If the dart is thrown truly randomly, every outcome is equally likely; because there are an infinity of outcomes, every outcome has a **probability of zero**. We need special care in dealing with such outcomes. It turns out that, in some cases, it is necessary to interpret the probability of the occurrence of such an event as being vanishingly small rather than exactly zero. We consider this situation in greater detail in Section 1.1.5 on page 4. Note that although the probability of any particular outcome is zero, the probability associated with any *subset* of the unit circle with area a is given by $\frac{a}{\pi}$, which tends to zero as a tends to zero.

■

1.1.2 Events

The definition of probability naturally extends to any subset of elements of S, which we call an **event**, denoted E. If the sample space is discrete, then every event E is an element of the power set of S, which is the set of all possible subsets of S. The probability associated with an event, denoted $P(E)$, is a real number $0 \le P(E) \le 1$ and is the sum of the probabilities associated with the outcomes in the event.

EXAMPLE 3: EVENTS

Continuing with Example 1, we can define the event "the roll of a die results in a odd-numbered outcome." This corresponds to the set of outcomes {1,3,5}, which has a probability of $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$. We write $P(\{1,3,5\}) = 0.5$.

■

1.1.3 Disjunctions and conjunctions of events

Consider an event E that is considered to have occurred if either or both of two other events E_1 or E_2 occur, where both events are defined in the same sample space. Then, E is said to be the **disjunction** or logical OR of the two events denoted $E = E_1 \vee E_2$ and read "E_1 or E_2."

EXAMPLE 4: DISJUNCTION OF EVENTS

Continuing with Example 1, we define the events E_1 = "the roll of a die results in an odd-numbered outcome" and E_2 = "the roll of a die results in an outcome numbered less than 3." Then, $E_1 = \{1, 3, 5\}$ and $E_2 = \{1, 2\}$ and $E = E_1 \vee E_2 = \{1, 2, 3, 5\}$.

■

In contrast, consider event E that is considered to have occurred only if *both* of two other events E_1 or E_2 occur, where both are in the same sample space. Then, E is said to be the **conjunction** or logical AND of the two events denoted $E = E_1 \wedge E_2$ and read "E_1 and E_2 .". When the context is clear, we abbreviate this to $E = E_1 E_2$.

EXAMPLE 5: CONJUNCTION OF EVENTS

Continuing with Example 4, $E = E_1 \wedge E_2 = E_1 E_2 = \{1\}$.

■

Two events E_i and E_j in S are **mutually exclusive** if only one of the two may occur simultaneously. Because the events have no outcomes in common, $P(E_i \wedge E_j) = P(\{\ \}) = 0$. Note that outcomes are *always* mutually exclusive but events need not be so.

1.1.4 Axioms of probability

One of the breakthroughs in modern mathematics was the realization that the theory of probability can be derived from just a handful of intuitively obvious axioms. Several variants of the axioms of probability are known. We present the axioms as stated by Kolmogorov to emphasize the simplicity and elegance that lie at the heart of probability theory:

1. $0 \le P(E) \le 1$, that is, the probability of an event lies between 0 and 1.
2. $P(S) = 1$, that is, it is certain that at least some event in S will occur.
3. Given a potentially infinite set of *mutually exclusive* events E_1, E_2,...

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \quad \textbf{(EQ 1)}$$

That is, the probability that any *one* of the events in the set of mutually exclusive events occurs is the sum of their individual probabilities. For any finite set of n mutually exclusive events, we can state the axiom equivalently as:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \quad \textbf{(EQ 2)}$$

An alternative form of Axiom 3 is:

$$P(E_1 \vee E_2) = P(E_1) + P(E_2) - P(E_1 \wedge E_2) \quad \textbf{(EQ 3)}$$

This alternative form applies to non-mutually exclusive events.

EXAMPLE 6: PROBABILITY OF UNION OF MUTUALLY EXCLUSIVE EVENTS

Continuing with Example 1, we define the mutually exclusive events {1,2} and {3,4} which both have a probability of 1/3. Then, $P(\{1, 2\} \cup \{3, 4\}) = P(\{1, 2\}) + P(\{3, 4\}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

■

EXAMPLE 7: PROBABILITY OF UNION OF NON-MUTUALLY EXCLUSIVE EVENTS

Continuing with Example 1, we define the non-mutually exclusive events {1,2} and {2,3} which both have a probability of 1/3. Then, $P(\{1,2\} \cup \{2,3\}) = P(\{1,2\}) + P(\{2,3\}) - P(\{1,2\} \wedge \{2,3\}) = \frac{1}{3} + \frac{1}{3} - P(\{2\}) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$.

■

1.1.5 Subjective and objective probability

The axiomatic approach is indifferent as to *how* the probability of an event is determined. It turns out that there are two distinct ways in which to determine the probability of an event. In some cases, the probability of an event can be derived from counting arguments. For instance, given the roll of a fair die, we know that there are only six possible outcomes, and that all outcomes are equally likely, so that the probability of rolling, say, a 1, is 1/6. This is called its **objective** probability. Another way of computing objective probabilities is to define the probability of an event as being the limit of a counting process, as the next example shows.

EXAMPLE 8: PROBABILITY AS A LIMIT

Consider a measurement device that measures the packet header types of every packet that crosses a link. Suppose that during the course of a day the device samples 1,000,000 packets and of these 450,000 packets are UDP packets, 500,000 packets are TCP packets, and the rest are from other transport protocols. Given the large number of underlying observations, to a first approximation, we can consider that the probability that a randomly selected packet uses the UDP protocol to be 450,000/1,000,000 = 0.45. More precisely, we state: $P(UDP) = \underset{t \to \infty}{Lim} (UDPCount(t)) / (TotalPacketCoun(t))$

where *UDPCount*(t) is the number of UDP packets seen during a measurement interval of duration t, and *TotalPacketCount*(t) is the total number of packets seen during the same measurement interval. Similarly $P(TCP) = 0.5$.

Note that in reality the mathematical limit cannot be achieved because no packet trace is infinite. Worse, over the course of a week or a month the underlying workload could change, so that the limit may not even exist. Therefore, in practice, we are forced to choose 'sufficiently large' packet counts and hope that the ratio thus computed corresponds to a probability. This approach is also called the **frequentist** approach to probability.

■

In contrast to an objective assessment of probability, we can also use probabilities to characterize events **subjectively**.

EXAMPLE 9: SUBJECTIVE PROBABILITY AND ITS MEASUREMENT

Consider a horse race where a favoured horse is likely to win, but this is by no means assured. We can associate a subjective probability with the event, say 0.8. Similarly, a doctor may look at a patient's symptoms and associate them with a 0.25 probability of a particular disease. Intuitively, this measures the degree of confidence that an event will occur, based on expert knowledge of the situation that is not (or cannot be) formally stated.

How is subjective probability to be determined? A common approach is to measure the odds that a knowledgeable person would bet on that event. Continuing with the example, if a bettor really thought that the favourite would win with a probability of 0.8, then the bettor should be willing to bet $1 under the terms: if the horse wins, the bettor gets $1.25, and if the horse loses, the bettor gets $0. With this bet, the bettor expects to not lose money, and if the reward is greater than $1.25, the bettor will expect to make money. So, we can elicit the implicit subjective probability by offering a high reward, and then lowering it until the bettor is just about to walk away, which would be at the $1.25 mark.

■

The subjective and frequentist approaches interpret zero-probability events differently. Consider an infinite sequence of successive events. Any event that occurs only a finite number of times in this infinite sequence will have a frequency that can be made arbitrarily small. In number theory, we do not and cannot differentiate between a number that can be made arbitrarily small and zero. So, from this perspective, such an event can be considered to have a probability of occurrence of zero *even though it may occur a finite number of times* in the sequence.

From a subjective perspective, a zero-probability event is defined as an event E such that a rational person would be willing to bet an arbitrarily large but finite amount that E will not occur. More concretely, suppose this person were to receive a reward of $1 if E did not occur but would have to forfeit a sum of $$F$ if E occurred. Then, the bet would be taken for any finite value of F.

1.2 Joint and conditional probability

Thus far, we have defined the terms used in studying probability and considered single events in isolation. Having set this foundation, we now turn our attention to the interesting issues that arise when studying **sequences of events**. In doing so, it is very important to keep track of the sample space in which the events are defined: a common mistake is to ignore the fact that two events in a sequence may be defined on different sample spaces.

1.2.1 Joint probability

Consider two processes with sample spaces S_1 and S_2 that occur one after the other. The two processes can be viewed as a single **joint process** whose outcomes are the tuples chosen from the **product space** $S_1 \times S_2$. We refer to the subsets of the product space as **joint events.** Just as before, we can associate probabilities with outcomes and events in the product space. To keep things straight, in this section, we denote the sample space associated with a probability as a subscript, so that $P_{S_1}(E)$ denotes the probability of event E defined over sample space S_1 and $P_{S_1 \times S_2}(E)$ is an event defined over the product space $S_1 \times S_2$.

EXAMPLE 10: JOINT PROCESS AND JOINT EVENTS

Consider sample space $S_1 = \{1, 2, 3\}$ and sample space $S_2 = \{a, b, c\}$. Then, the product space is given by $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$. If these events are equiprobable, then the probability of each tuple is $\frac{1}{9}$. Let $E = \{1, 2\}$ be an event in S_1 and $F = \{b\}$ be an event in S_2. Then the event EF is given by the tuples $\{(1, b), (2, b)\}$ and has probability $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$.

■

We will return to the topic of joint processes in Section 1.8 on page 31. We now turn our attention to the concept of conditional probability**.**

1.2.2 Conditional probability

Common experience tells us that if a sky is sunny, there is no chance of rain in the immediate future, but that if it is cloudy, it may or may not rain soon. Knowing that the sky is cloudy, therefore, increases the chance that it may rain soon, compared to the situation when it is sunny. How can we formalize this intuition?

To keep things simple, first consider the case when two events *E* and *F* share a common sample space S and occur one after the other. Suppose that the probability of *E* is $P_S(E)$ and the probability of *F* is $P_S(F)$. Now, suppose that we are informed that event *E* actually occurred. By definition, the **conditional probability** of the event *F* conditioned on the occurrence of event *E* is denoted $P_{S \times S}(F|E)$ (read "the probability of *F* given *E*") and computed as:

$$P_{S \times S}(F|E) = \frac{P_{S \times S}(E \wedge F)}{P_S(E)} = \frac{P_{S \times S}(EF)}{P_S(E)} \quad \text{(EQ 4)}$$

If knowing that *E* occurred does not affect the probability of *F*, *E* and *F* are said to be **independent** and

$$P_{S \times S}(EF) = P_S(E)P_S(F)$$

EXAMPLE 11: CONDITIONAL PROBABILITY OF EVENTS DRAWN FROM THE SAME SAMPLE SPACE

Consider sample space $S = \{1, 2, 3\}$ and events $E = \{1\}$ and $F = \{3\}$. Let $P_S(E) = 0.5$ and $P_S(F) = 0.25$. Clearly, the space $S \times S = \{(1, 1), (1, 2), \ldots, (3, 2), (3, 3)\}$. The joint event $EF = \{(1, 3)\}$. Suppose $P_{S \times S}(EF) = 0.3$. Then, $P_{S \times S}(F|E) = \frac{P_{S \times S}(EF)}{P_S(E)} = \frac{0.3}{0.5} = 0.6$. We interpret this to mean that if event *E* occurred, then the probability that event *F* occurs is 0.6. This is higher than the probability of *F* occurring on its own (which is 0.25). Hence, the fact the *E* occurred improves the chances of *F* occurring, so the two events are not independent. This is also clear from the fact that $P_{S \times S}(EF) = 0.3 \neq P_S(E)P_S(F) = 0.125$.

■

The notional of conditional probability generalizes to the case where events are defined on more than one sample space. Consider a sequence of two processes with sample spaces S_1 and S_2 that occur one after the other (this could be the condition of the sky now, for instance, and whether or not it rains after two hours). Let event *E* be a subset of S_1 and let event *F* be a subset of S_2. Suppose that the probability of *E* is $P_{S_1}(E)$ and the probability of *F* is $P_{S_2}(F)$. Now, suppose that we are informed that event *E* actually occurred. We define the probability $P_{S_1 \times S_2}(F|E)$ as the **conditional probability** of the event *F* conditional on the occurrence of *E* as:

$$P_{S_1 \times S_2}(F|E) = \frac{P_{S_1 \times S_2}(EF)}{P_{S_1}(E)} \quad \text{(EQ 5)}$$

If knowing that *E* occurred does not affect the probability of *F*, *E* and *F* are said to be **independent** and

$$P_{S_1 \times S_2}(EF) = P_{S_1}(E) \times P_{S_2}(F) \quad \text{(EQ 6)}$$

EXAMPLE 12: CONDITIONAL PROBABILITY OF EVENTS DRAWN FROM DIFFERENT SAMPLE SPACES

Consider sample space $S_1 = \{1, 2, 3\}$ and sample space $S_2 = \{a, b, c\}$ with product space $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$. Let $E = \{1, 2\}$ be an event in S_1 and $F = \{b\}$ be an event in S_2. Also, let $P_{S_1}(E) = 0.5$ and let $P_{S_1 \times S_2}(EF) = P_{S_1 \times S_2}(\{(1, b), (2, b)\}) = 0.05$.

If *E* and *F* are independent then:

$$P_{S_1 \times S_2}(EF) = P_{S_1 \times S_2}(\{(1, b), (2, b)\}) = P_{S_1}(\{1, 2\}) \times P_{S_2}(\{b\})$$

$$0.05 = 0.5 \times P_{S_2}(\{b\})$$

$$P_{S_2}(\{b\}) = 0.1$$

Otherwise,

$$P_{S_1 \times S_2}(F|E) = \frac{P_{S_1 \times S_2}(EF)}{P_{S_1}(E)} = \frac{0.05}{0.5} = 0.1$$

■

It is important not to confuse $P(F|E)$ and $P(F)$. The conditional probability is defined in the product space $S_1 \times S_2$ and the unconditional probability in the space S_2. Explicitly keeping track of the underlying sample space can help avoid apparent contradictions such as the one discussed in Example 14.

EXAMPLE 13: USING CONDITIONAL PROBABILITY

Consider a device that samples packets on a link, as in Example 8. Suppose that measurements show that 20% of the UDP packets have a packet size of 52 bytes. Let $P(UDP)$ denote the probability that the packet is of type UDP and let $P(52)$ denote the probability that the packet is of length 52 bytes. Then, $P(52|UDP) = 0.2$. In Example 8, we computed that $P(UDP) = 0.45$. Therefore, $P(UDP$ AND $52) = P(52|UDP) * P(UDP) = 0.2 * 0.45 = 0.09$. That is, if we were to pick a packet at random from the sample, there is a 9% chance that is a UDP packet of length 52 bytes (but it has a 20% chance of being of length 52 bytes if we know already that it is a UDP packet).

■

EXAMPLE 14: THE MONTY HALL PROBLEM

Consider a television show (loosely modelled on a similar show hosted by Monty Hall) where three identical doors hide two goats and a luxury car. You, the contestant, can pick any door and obtain the prize behind it. Assume that you prefer the car to the goat. If you did not have any further information, your chance of picking the winning door is clearly 1/3. Now, suppose that after you pick one of the doors, say Door 1, the host opens one of the other doors, say Door 2, and reveals a goat behind it. Should you switch your choice to Door 3 or stay with Door 1?

Solution:

We can view the Monty Hall problem as a sequence of three processes. The first process is the placement of a car behind one of the doors. The second is the selection of a door by the contestant and the third process is the revelation of what lies behind one of the other doors. The sample space for the first process is {Door 1, Door 2, Door 3} abbreviated {1, 2, 3}, as are the sample spaces for the second and third processes. So, the product space is {(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1),..., (3, 3, 3)}.

Without loss of generality, assume that you pick Door 1. The game show host's hand is now forced: he has to pick either Door 2 or Door 3. Without loss of generality, suppose that the host picks Door 2, so that the set of possible outcomes that constitutes the reduced sample space is {(1, 1, 2), (2, 1, 2), (3, 1, 2)}. However, we know that the game show host will never open a door with a car behind it - only a goat. Therefore, the outcome (2, 1, 2) is not possible. So, the reduced sample space is just the set {(1, 1, 2), (3, 1, 2)}. What are the associated probabilities?

To determine this, note that the initial probability space is {1, 2, 3} with equiprobable outcomes. Therefore, the outcomes {(1, 1, 2), (2, 1, 2), (3, 1, 2)} are also equiprobable. When the game show host makes his move to open Door 2, he reveals private information that the outcome (2, 1, 2) is impossible, so the probability associated with this outcome is 0. The show host's forced move cannot affect the probability of the outcome (1, 1, 2) because the host never had the choice of opening Door 1 once you selected it. Therefore, its probability in the reduced sample space continues to be 1/3. This means that $P(\{(3, 1, 2)\} = 2/3$, so it doubles your chances for you to switch doors.

One way to understand this somewhat counterintuitive result is to realize that the game show host's actions reveal private information, that is, the location of the car. Two-thirds of the time, the prize is behind the door you did not choose. The host always opens a door that does not have a prize behind it. Therefore, the residual probability (2/3) must all be assigned to Door 3. Another way to think of it is that if you repeat a large number of experiments with two contestants, one who never switches doors and the other who always switches doors, then the latter would win twice as often.

■

1.2.3 Bayes' rule

One of the most widely used rules in the theory of probability is due to an English country minister Thomas Bayes. Its significance is that it allows us to infer 'backwards' from effects to causes, rather than from causes to effects. The derivation of his rule is straightforward, though its implications are profound.

We begin with the definition of conditional probability (Equation 4):

$$P_{S \times S}(F|E) = \frac{P_{S \times S}(EF)}{P_S(E)}$$

If the underlying sample spaces can be assumed to be implicitly known, we can rewrite this as:

$$P(EF) = P(F|E)P(E) \tag{EQ 7}$$

We interpret this to mean that the probability that both E and F occur is the product of the probabilities of two events, first, that E occurs, and second, that conditional on E, F occurs.

Recall that $P(F|E)$ is defined in terms of the event F following event E. Now, consider the converse: F is known to have occurred–what is the probability that E occurred? This is similar to the problem: if there is fire, there is smoke, but if we see smoke, what is the probability that it was due to a fire? The probability we want is $P(E|F)$. Using the definition of conditional probability it is given by:

$$P(E|F) = \frac{P(EF)}{P(F)} \tag{EQ 8}$$

Substituting for $P(F)$ from Equation 7, we get

$$P(E|F) = \frac{P(F|E)}{P(F)}P(E) \tag{EQ 9}$$

which is **Bayes' rule**. One way of interpreting this is that it allows us to compute the degree to which some effect or **posterior** F can be attributed to some cause or **prior** E.

EXAMPLE 15: BAYES' RULE

Continuing with Example 13, we want to compute the following quantity: Given that a packet is 52 bytes long, what is the probability that it is a UDP packet?

Solution:

From Bayes' rule:

$$P(UDP|52) = \frac{P(52|UDP)P(UDP)}{P(52)} = \frac{0.2(0.45)}{0.54} = 0.167$$

■

We can generalize Bayes' rule when a posterior can be attributed to more than one prior. Consider a posterior *F* that is due to some set of *n* priors E_i such that the priors are mutually exclusive and exhaustive (that is, at least one of them occurs and only one of them can occur). This implies that $\sum_{i=1}^{n} P(E_i) = 1$. Then,

$$P(F) = \sum_{i=1}^{n} P(FE_i) = \sum_{i=1}^{n} P(F|E_i)P(E_i) \qquad \text{(EQ 10)}$$

This is also called the **Law of Total Probability**.

EXAMPLE 16: LAW OF TOTAL PROBABILITY

Continuing with Example 13, let us compute *P*(*52*), that is, the probability that a packet sampled at random has a length of 52 bytes. To compute this, we need to know the packet sizes for all other traffic types. For instance, if *P*(52| *TCP*) = 0.9 and all other packets were known to be of length other than 52 bytes, then *P*(52) = *P*(52|*UDP*) * *P*(*UDP*) + *P*(52|*TCP*) * *P*(*TCP*) + *P*(52|*other*) * *P*(*other*) = 0.2 * 0.45 + 0.9 * 0.5 + 0 = 0.54.

■

The law of total probability allows one further generalization of Bayes' rule to obtain **Bayes' Theorem**. From the definition of conditional probability, we have:

$$P(E_i|F) = \frac{P(E_iF)}{P(F)}$$

From Equation 7,

$$P(E_i|F) = \frac{P(F|E_i)P(E_i)}{P(F)}$$

Substituting Equation 10, we get

$$P(E_i|F) = \frac{P(F|E_i)P(E_i)}{\left(\sum_{i=1}^{n} P(F|E_i)P(E_i)\right)} \qquad \text{(EQ 11)}$$

This is called the **generalized Bayes' rule** or Bayes' Theorem. It allows us to compute the probability of any one of the priors E_i, conditional on the occurrence of the posterior *F*. This is often interpreted as follows: we have some set of mutually exclusive and exhaustive hypotheses E_i. We conduct an experiment, whose outcome is *F*. We can then use Bayes' formula to compute the revised estimate for each hypothesis.

EXAMPLE 17: BAYES' THEOREM

Continuing with Example 15, consider the following situation: we pick a packet at random from the set of sampled packets and find that its length is *not* 52 bytes. What is the probability that it is a UDP packet?

Solution:

As in Example 6, let 'UDP' refer to the event that a packet is of type UDP and '52' refer to the event that the packet is of length 52 bytes. Denote the complement of the latter event, that is, that the packet is not of length 52 bytes by '52^c'.

From Bayes' rule:

$$P(UDP|52^c) = \frac{P(52^c|UDP)P(UDP)}{P(52^c|UDP)P(UDP) + P(52^c|TCP)P(TCP) + P(52^c|other)P(other)}$$
$$= \frac{0.8(0.45)}{0.8(0.45) + 0.1(0.5) + 1(0.05)}$$
$$= 0.78$$

Thus, if we see a packet that is *not* 52 bytes long, it is quite likely that it is a UDP packet. Intuitively, this must be true because most TCP packets are 52 bytes long, and there aren't very many non-UDP and non-TCP packets.

■

1.3 Random variables

So far, we have restricted our consideration to studying events, which are collections of outcomes of experiments or observations. However, we are often interested in abstract quantities or outcomes of experiments that are derived from events and observations, but are not themselves events or observations. For example, if we throw a fair die, we may want to compute the probability that the square of the face value is smaller than 10. This is random and can be associated with a probability, and, moreover, depends on some underlying random events. Yet, it is neither an event nor an observation: it is a **random variable**. Intuitively, a random variable is a quantity that can assume any one of a set of values (called its **domain *D***) and whose value can only be stated probabilistically. In this section, we will study random variables and their distributions.

More formally, a **real random variable** (the one most commonly encountered in applications having to do with computer networking) is a mapping from events in a sample space *S* to the domain of real numbers. The probability associated with each value assumed by a real random variable[2] is the probability of the underlying event in the sample space as illustrated in Figure 1.

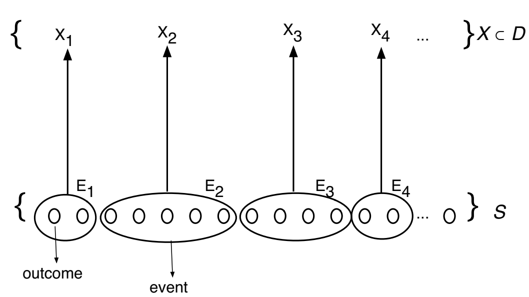


Figure 1. The random variable *X* takes on values from the domain *D*. Each value taken on by the random variable is associated with a probability corresponding to an event *E*, which is a subset of outcomes in the sample space *S*.

A random variable is **discrete** if the set of values it can assume is finite and countable. The elements of *D* should be *mutually exclusive* (that is, the random variable cannot simultaneously take on more than one value) and *exhaustive* (the random variable cannot assume a value that is not an element of *D*).

EXAMPLE 18: A DISCRETE RANDOM VARIABLE

2. We deal with only real random variables in this text so will drop the qualifier 'real' at this point.

Consider a random variable I defined as the size of an IP packet rounded up to closest kilobyte. Then, I assumes values from the domain $D = \{1,2,3,..., 64\}$. This set is both mutually exclusive and exhaustive. The underlying sample space S is the set of potential packet sizes and is therefore identical to D. The probability associated with each value of I is the probability of seeing an IP packet of that size in some collection of IP packets, such as a measurement trace.

■

A random variable is **continuous** if the values it can take on are a subset of the real line.

EXAMPLE 19: A CONTINUOUS RANDOM VARIABLE

Consider a random variable T defined as the time between consecutive packet arrivals at a port of a switch (also called the packet interarrival time). Although each packet's arrival time is quantized by the receiver's clock, so that the set of interarrival times are finite and countable, given the high clock speeds of modern systems, modelling T as a continuous random variable is a good approximation of reality. The underlying sample space S is the subset of the real line that spans the smallest and largest possible packet interarrival times. As in the previous example, the sample space is identical to the domain of T.

■

1.3.1 Distribution

In many cases, we are not interested in the actual value taken by a random variable, but in the probabilities associated with each such value that it can assume. To make this more precise, consider a discrete random variable X_d that assumes distinct values $D = \{x_1, x_2,..., x_n\}$. We define the value $p(x_i)$ to be the probability of the event that results in X_d assuming the value x_i. The function $p(X_d)$, which characterizes the probability that X_d will take on each value in its domain is called the **probability mass function** of X_d[3]. It is also sometimes called the **distribution** of X_d.

EXAMPLE 20: PROBABILITY MASS FUNCTION

Consider a random variable H defined as 0 if fewer than 100 packets are received at a router's port in a particular time interval T and 1 otherwise. The sample space of outcomes consists of all possible numbers of packets that could arrive at the router's port during T, which is simply the set $S = \{1, 2, 3, \ldots, M\}$ where M is the maximum number of packets that can be received in time T. Assuming $M > 99$, we define two events $E_0 = \{0, 1, 2, \ldots, 99\}$ and $E_1 = \{100, 101, \ldots, M\}$. Given the probability of each outcome in S, we can compute the probability of each event, $P(E_0)$ and $P(E_1)$. By definition, $p(H = 0) = p(0) = P(E_0)$ and $p(H = 1) = p(1) = P(E_1)$. The set $\{p(0), p(1)\}$ is the probability mass function of H. Notice how the probability mass function is closely tied to events in the underlying sample space.

■

Unlike a discrete random variable, which has non-zero probability of taking on any particular value in its domain, the probability that a continuous real random variable X_c will take on any specific value in its domain is 0. Nevertheless, in nearly all cases of interest in the field of computer networking, we will be able to assume that we can define the **density** function $f(x)$ of X_c as follows: the probability that X_c takes on a value between two reals x_1 and x_2, $p(x_1 \le x \le x_2)$ is given by the integral $\int_{x_1}^{x_2} f(x)dx$. Of course, we need to ensure that $\int_{-\infty}^{\infty} f(x)dx = 1$. Alternatively, we can think of $f(x)$ being implicitly

3. Note the subtlety in this standard notation. Recall that $P(E)$ is the probability of an event E. In contrast, $p(X)$ refers to the distribution of a random variable X, and $p(X = x_i) = p(x_i)$ refers to the probability that random variable X takes on the value x_i.

defined by the statement that a variable x chosen randomly in the domain of X_c has probability $f(a)\Delta$ of lying in the range $\left[a - \frac{\Delta}{2}, a + \frac{\Delta}{2}\right]$ when Δ is very small.

EXAMPLE 21: DENSITY FUNCTION

Suppose we know that packet interarrival times are distributed *uniformly* in the range [0.5s, 2.5s]. The corresponding density function is a constant c over the domain. It is easy to see that c = 0.5 because we require $\int_{-\infty}^{\infty} f(x)dx = \int_{0.5}^{2.5} c dx = 2c = 1$. The probability that the interarrival time is in the interval $\left[1 - \frac{\Delta}{2}, 1 + \frac{\Delta}{2}\right]$ is therefore 0.5Δ.

■

1.3.2 Cumulative density function

The domain of a discrete real random variable X_d is totally ordered (that is, for any two values x_1 and x_2 in the domain, either $x_1 > x_2$ or $x_2 > x_1$). We define the **cumulative density function** $F(X_d)$ by:

$$F(x) = \sum_{i | x_i \le x} p(x_i) = p(X_d \le x) \qquad \textbf{(EQ 12)}$$

Note the difference between $F(X_d)$, which denotes the cumulative distribution of random variable X_d, and $F(x)$, which is the value of the cumulative distribution for the value $X_d = x$.

Similarly, the cumulative density function of a continuous random variable X_c, denoted $F(X_c)$ is given by:

$$F(x) = \int_{-\infty}^{x} f(y)dy = p(X_c \le x) \qquad \textbf{(EQ 13)}$$

By definition of probability, in both cases $0 \le F(X_d) \le 1$, $0 \le F(X_c) \le 1$

EXAMPLE 22: CUMULATIVE DENSITY FUNCTIONS

Consider a discrete random variable D that can take on values {1, 2, 3, 4, 5} with probabilities {0.2, 0.1, 0.2, 0.2, 0.3} respectively. The latter set is also the probability mass function of D. Because the domain of D is totally ordered, we compute the cumulative density function $F(D)$ as $F(1) = 0.2$, $F(2) = 0.3$, $F(3) = 0.5$, $F(4) = 0.7$, $F(5) = 1.0$.

Now, consider a continuous random variable C defined by the density function $f(x) = 1$ in the range [0,1]. The cumulative density function $F(C) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x} dy = y\big|_0^x = x$. We see that, although, for example, $f(0.1) = 1$, this does not mean that the value 0.1 is certain!

Note that, by definition of cumulative density function, it is necessary that it achieve a value of 1 at right extreme value of the domain.

■

1.3.3 Generating values from an arbitrary distribution

The cumulative density function $F(X)$, where X is either discrete or continuous, can be used to generate values drawn from the underlying discrete or continuous distribution $p(X_d)$ or $f(X_c)$ as illustrated in Figure 2.

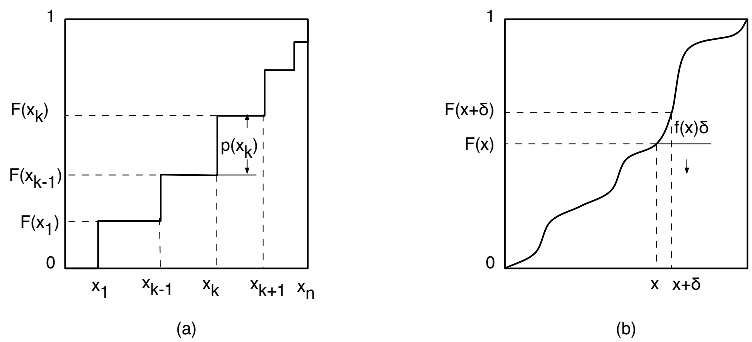


Figure 2. Generating values from an arbitrary (a) discrete or (b) continuous distribution.

Consider a discrete random variable X_d that takes on values $x_1, x_2, \ldots, x_n$ with probabilities $p(x_i)$. By definition, $F(x_k) = F(x_{k-1}) + p(x_k)$. Moreover, $F(X_d)$ always lies in the range [0,1]. Therefore, if we were to generate a random number u with uniform probability in the range [0,1], the probability that u lies in the range $[F(x_{k-1}), F(x_k)]$ is $p(x_k)$. Moreover, $x_k = F^{-1}(u)$. Therefore, the procedure to generate values from the discrete distribution $p(X_d)$ is as follows: first, generate a random variable u uniformly in the range [0,1]; second, compute $x_k = F^{-1}(u)$.

We can use a similar approach to generate values from a continuous random variable X_c with associated density function $f(X_c)$. By definition, $F(x + \delta) = F(x) + f(x)\delta$ for very small values of δ. Moreover, $F(X_c)$ always lies in the range [0,1]. Therefore, if we were to generate a random number u with uniform probability in the range [0,1], the probability that u lies in the range $[F(x), F(x + \delta)]$ is $f(x)\delta$, which means that $x = F^{-1}(u)$ is distributed according to the desired density function $f(X_c)$. Therefore, the procedure to generate values from the continuous distribution $f(X_c)$ is as follows: first, generate a random variable u uniformly in the range [0,1]; second, compute $x = F^{-1}(u)$.

1.3.4 Expectation of a random variable

The **expectation**, **mean** or **expected value** $E[X_d]$ of a discrete random variable X_d that can take on n values x_i with probability $p(x_i)$ is given by:

$$E[X_d] = \sum_{i=1}^{n} x_i p(x_i) \qquad \text{(EQ 14)}$$

Similarly, the expectation $E[X_c]$ of a continuous random variable X_c with density function $f(x)$ is given by

$$E[X_c] = \int_{-\infty}^{\infty} x f(x) dx \qquad \text{(EQ 15)}$$

Intuitively, the expected value of a random variable is the value we expect it to take, knowing nothing else about it. For instance, if you knew the distribution of the random variable corresponding to the time it takes for you to travel from your home to work, then, on a typical day, you expect your commute time to be the expected value of this random variable.

EXAMPLE 23: EXPECTATION OF A DISCRETE AND A CONTINUOUS RANDOM VARIABLE

Continuing with the random variables *C* and *D* defined in Example 22, we find

$E[D]$ = 1*0.2 + 2*0.1 + 3*0.2 + 4*0.2 + 5*0.3 = 0.2 + 0.2 + 0.6 + 0.8 + 1.5 = 3.3.

Note that the expected value of *D* is actually a value it cannot assume! This is often true of discrete random variables. One way to interpret this is that *D* will take on values 'close' to its expected value, in this case, 3 or 4.

Similarly,

$$E[C] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{1} xdx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

C is the *uniform* distribution and its expected value is the midpoint of the domain, i.e. 0.5.

■

The expectation of a random variable gives us a reasonable idea of how it behaves in the long run. It is important to remember, however, that two random variables with the same expectation can have rather different behaviours.

We now state, without proof, some useful properties of expectations.

1. For constants *a* and *b*:

$$E[aX + b] = aE[X] + b \qquad \text{(EQ 16)}$$

2. $E[X+Y] = E[X] + E[Y]$, or, more generally, for any set of random variables X_i:

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] \qquad \text{(EQ 17)}$$

3. For a discrete random variable X_d with probability mass function $p(x_i)$ and any function $g(.)$,

$$E[g(X_d)] = \sum_i g(x_i)p(x_i) \qquad \text{(EQ 18)}$$

4. For a continuous random variable X_c with density function $f(x)$, and any function $g(.)$,

$$E[g(C)] = \int_{-\infty}^{\infty} g(x)f(x)dx \qquad \text{(EQ 19)}$$

Note that, in general, $E[g(X)]$ is not the same as $g(E[X])$, that is, a function cannot be 'taken out' of the expectation.

EXAMPLE 24: EXPECTED VALUE OF A FUNCTION OF A VARIABLE

Consider a discrete random variable *D* that can take on values {1, 2, 3, 4, 5} with probabilities {0.2, 0.1, 0.2, 0.2, 0.3} respectively. Then, $E[e^D] = 0.2e^1 + 0.1e^2 + 0.2e^3 + 0.2e^4 + 0.3e^5 = 60.74$.

■

EXAMPLE 25: EXPECTED VALUE OF A FUNCTION OF A VARIABLE

Let X be a random variable that has equal probability of lying anywhere in the interval [0,1]. Then, $f(x) = 1; \quad 0 \le x \le 1$.

$$E[X^2] = \int_0^1 x^2 f(x)dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

■

1.3.5 Variance of a random variable

The **variance** of a random variable is defined by $V(X) = E[(X-E[X])^2]$. Intuitively, it shows how ‘far away’ the values taken on by a random variable would be from its expected value. We can express the variance of a random variable in terms of two expectations as $V(X) = E[X^2] - E[X]^2$. For

$$V[X] = E[(X-E[X])^2]$$

$$= E[X^2 - 2XE[X] + E[X]^2]$$

$$= E[X^2] - 2E[XE[X]] + E[X]^2$$

$$= E[X^2] - 2E[X]E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2$$

In practical terms, the distribution of a random variable over its domain D (this domain is also called the **population**) is not usually known. Instead, the best that we can do is to sample the values it takes on by observing its behaviour over some period of time. We can estimate the variance of the random variable from the array of sample values by keeping running counters for $\sum x_i$ and $\sum x_i^2$. Then, $V[X] \approx \left(\frac{\sum x_i^2 - (\sum x_i)^2}{n}\right)$, where this approximation improves with n, the size of the sample as a consequence of the law of large numbers, discussed in Section 1.7.4 on page 29.

The following properties of the varianceof a random variable can be easily shown for both discrete and continuous random variables.

1. For constant a,

$$V[X+a] = V[X] \quad \text{(EQ 20)}$$

2. For constant a,

$$V[aX] = a^2V[X] \quad \text{(EQ 21)}$$

3. If X and Y are independent random variables,

$$V[X+Y] = V[X] + V[Y] \quad \text{(EQ 22)}$$

1.4 Moments and moment generating functions

We have focussed thus far on elementary concepts of probability. To get to the next level of understanding, it is necessary to dive into the somewhat complex topic of moment generating functions. The moments of a distribution generalize its mean and variance. In this section we will see how we can use a moment generating function (abbreviated MGF) to compactly represent *all* the moments of a distribution. The moment generating function is interesting not only because it allows us to prove some useful results, such as the Central Limit Theorem, but also because it is similar in form to the Fourier and Laplace transforms that are discussed in Chapter 5.

1.4.1 Moments

The **moments** of a distribution are a set of parameters that summarize it. Given a random variable X, its first **moment about the origin** denoted M_0^1 is defined to be $E[X]$. Its **second moment about the origin**, denoted M_0^2 is defined as the expected value of the random variable X^2, i.e., $E[X^2]$. In general, the r^{th} moment of X about the *origin*, denoted M_0^r, is defined as $M_0^r = E[X^r]$.

We can similarly define the ***rth* moment about the *mean*,** denoted M_μ^r, by $E[(X-\mu)^r]$. Note that the **variance** of the distribution, denoted by σ^2 or $V[X]$ is the same as M_μ^2. The third moment about the mean M_μ^3 is used to construct a measure of **skewness** (which describes whether the probability mass is more to the left or the right of the mean, compared to a normal distribution) and the fourth moment about the mean M_μ^4 is used to construct a measure of peakedness or **kurtosis**, which measures the ‘width’ of a distribution.

The two definitions of a moment are related. For example, we have already seen that the variance of X, denoted $V[X]$, can be computed as $V[X] = E[X^2] - (E[X])^2$. Therefore, $M_\mu^2 = M_0^2 - (M_0^1)^2$. Similar relationships can be found between the higher moments by writing out the terms of the binomial expansion of $(X-\mu)^r$.

1.4.2 Moment generating functions

Except under some pathological conditions, a distribution can be thought to be uniquely represented by its moments. That is, if two distributions have the same moments, then, except under some rather unusual circumstances, they will be identical. Therefore, it is convenient to have an expression (or ‘fingerprint’) that compactly represents all the moments of a distribution. Such an expression should have terms corresponding to $\mu_r{}'$ for all values of r.

We can get a hint regarding a suitable representation from the expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \textbf{(EQ 23)}$$

We see that there is one term for each power of x. This motivates the definition of the **moment generating function** (MGF) of a random variable X as the expected value of e^{tX}, where t is an auxiliary variable:

$$M(t) = E(e^{tX}) \quad \textbf{(EQ 24)}$$

To see how this represents the moments of a distribution, we expand $M(t)$ as

$$\begin{aligned} M(t) = E(e^{tX}) &= E\left(1 + (tX) + \left(\frac{t^2X^2}{2!}\right) + \left(\frac{t^3X^3}{3!}\right) + \ldots\right) \\ &= 1 + E(tX) + E\left(\frac{t^2X^2}{2!}\right) + E\left(\frac{t^3X^3}{3!}\right) + \ldots \\ &= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \ldots \\ &= 1 + tM_0^1 + \frac{t^2}{2!}M_0^2 + \frac{t^3}{3!}M_0^3 + \ldots \end{aligned} \quad \textbf{(EQ 25)}$$

Thus, the MGF represents all the moments of the random variable X in a single compact expression. Note that the MGF of a distribution is undefined if one or more of its moments are infinite.

We can extract all the moments of the distribution from the MGF as follows: if we differentiate $M(t)$ once, the only term that is not multiplied by t or a power of t is M_0^1. So, $\left.\frac{dM(t)}{dt}\right|_{t=0} = M_0^1$. Similarly, $\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = M_0^2$. Generalizing, it is easy to show that to get the r^{th} moment of a random variable X about the origin, we only need to differentiate its MGF r times with respect to t and then set t to 0.

It is important to remember that the 'true' form of the MGF is the series expansion in Equation 25. The exponential is merely a convenient representation that has the property that operations on the series (as a whole) result in corresponding operations being carried out in the compact form. For example, it can be shown that the series resulting from the product of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ is $1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \ldots = e^{x+y}$. This simplifies the computation of operations on the series. However, it is sometimes necessary to revert to the series representation for certain operations. In particular, if the compact notation of $M(t)$ is not differentiable at $t = 0$, then we must revert to the series to evaluate $M(0)$, as shown next.

EXAMPLE 26: MGF OF A STANDARD UNIFORM DISTRIBUTION

Let X be a uniform random variable defined in the interval [0,1]. This is also called a **standard uniform distribution.** We would like to find all its moments. We find that $M(t) = E[e^{tX}] = \int_0^1 e^{tx}dx = \left.\frac{1}{t}e^{tx}\right|_0^1 = \frac{1}{t}[e^t - 1]$. However, this function is not defined—and therefore not differentiable—at $t = 0$. Instead, we revert to the series:

$$\frac{1}{t}[e^t - 1] = \frac{1}{t}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right] = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots$$

which *is* differentiable term by term. Differentiating r times and setting t to 0, we find that $M_0^r = 1/(r+1)$. So, $M_0^1 = \mu = 1/(1+1) = 1/2$ is the mean, and $M_0^2 = 1/(1+2) = 1/3 = E(X^2)$. Note that we found the expression for $M(t)$ using the compact notation, but reverted to the series for differentiating it. The justification is that the integral of the compact form is identical to the summation of the integrals of the individual terms.

■

1.4.3 Properties of moment generating functions

We now prove some useful properties of MGFs.

(a) If X and Y are two independent random variables, the MGF of their sum is the product of their MGFs. If their individual MGFs are $M_1(t)$ and $M_2(t)$ respectively, the MGF of their sum is:

$$M(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] \text{ (from independence)}$$
$$= M_1(t).M_2(t) \qquad \text{(EQ 26)}$$

EXAMPLE 27: MGF OF THE SUM

Find the MGF of the sum of two independent [0,1] uniform random variables.

Solution:

From Example 26, the MGF of a standard uniform random variable is $\frac{1}{t}[e^t - 1]$, so the MGF of random variable X defined as the sum of two independent uniform variables is $\frac{1}{t^2}[e^t - 1]^2$.

■

(b) If random variable X has MGF $M(t)$ then the MGF of random variable $Y = a+bX$ is $e^{at}M(bt)$. This is because:

$$E[e^{tY}] = E[e^{t(a+bX)}] = E[e^{at}e^{bXt}] = e^{at}E[e^{btX}]$$

$$= e^{at}M(bt) \qquad \text{(EQ 27)}$$

As a corollary, if $M(t)$ is the MGF of a random variable X, then the MGF of $(X$-$\mu)$ is given by $e^{-\mu t}M(t)$. The moments about the origin of $(X$-$\mu)$ are the moments about the mean of X. So, to compute the r^{th} moment about the mean for a random variable X, we can differentiate $e^{-\mu t}M(t)$ r times with respect to t and set t to 0.

EXAMPLE 28: VARIANCE OF A STANDARD UNIFORM RANDOM VARIABLE

The MGF of a standard uniform random variable X is $\frac{1}{t}[e^t - 1]$, so, the MGF of $(X$-$\mu)$ is given by $\frac{e^{-\mu t}}{t}[e^t - 1]$. To find the variance of a standard uniform random variable, we need to differentiate twice with respect to t and then set t to 0. Given the t in the denominator, it is convenient to rewrite the expression as $\left(1 - \mu t + \frac{\mu^2 t^2}{2!} - \ldots\right)\left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots\right)$, where the ellipses refer to terms with third and higher powers of t, which will reduce to 0 when t is set to 0. In this product, we need only consider the coefficient of t^2 (why?), which is $\frac{1}{3!} - \frac{\mu}{2!} + \frac{\mu^2}{2!}$. Differentiating the expression twice results in multiplying the coefficient by 2, and when we set t to zero, we obtain $E[(X\text{-}\mu)^2] = V[X] = 1/12$.

■

These two properties allow us to compute the MGF of a complex random variable that can be decomposed into the linear combination of simpler variables. In particular, it allows us to compute the MGF of independent, identically distributed (i.i.d) random variables, a situation that arises frequently in practice.

1.5 Standard discrete distributions

We now present some discrete distributions that frequently arise when studying networking problems.

1.5.1 Bernoulli distribution

A discrete random variable X is called a **Bernoulli** random variable if it can take only two values, 0 or 1, and its probability mass function is defined as $p(0) = 1\text{-}p$ and $p(1) = p$. We can think of X as representing the result of some experiment, with X=1 being 'success,' with probability p. The expected value of a Bernoulli random variable is p and variance is $p(1\text{-}p)$.

1.5.2 Binomial distribution

Consider a series of n Bernoulli experiments where the result of each experiment is *independent* of the others. We would naturally like to know the number of successes in these n trials. This can be represented by a discrete random variable X with parameters (n,p) and is called a **binomial** random variable. The probability mass function of a binomial random variable with parameters (n,p) is given by:

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad \text{(EQ 28)}$$

If we set $q = 1\text{-}p$, then these are just the terms of the expansion $(p+q)^n$. The expected value of a variable that is binomially distributed with parameters (n,p) is np.

EXAMPLE 29: BINOMIAL RANDOM VARIABLE

Consider a local area network with 10 stations. Assume that, at a given moment, each node can be active with probability p= 0.1. What is the probability that: a) one station is active, b) five stations are active, c) all 10 stations are active?

Solution:

Assuming that the stations are independent, the number of active stations can be modelled by a binomial distribution with parameters (10, 0.1). From the formula for $p(i)$ above, we get

a) $p(1) = \binom{10}{1} 0.1^1 0.9^9 = 0.38$

b) $p(5) = \binom{10}{5} 0.1^5 0.9^5 = 1.49 \times 10^{-3}$

c) $p(10) = \binom{10}{10} 0.1^{10} 0.9^0 = 1 \times 10^{-10}$

This is shown in Figure 3.

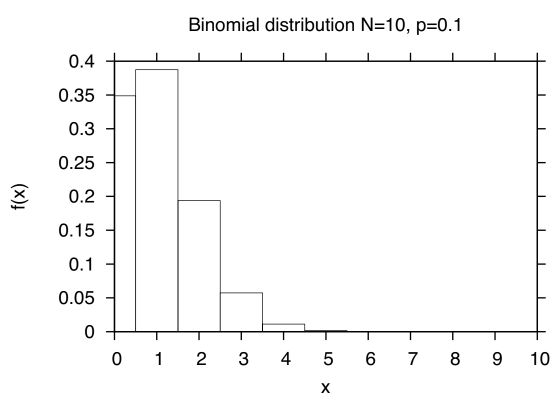


Figure 3. **Example Binomial distribution.**

Note how the probability of one station being active is 0.38, which is actually *greater* than the probability of any single station being active. Note also how rapidly the probability of multiple active stations drops. This is what motivates spatial statistical multiplexing; the provisioning of a link with a capacity smaller than the sum of the demands of the stations.

■

1.5.3 Geometric distribution

Consider a sequence of independent Bernoulli experiments, as before, each of which succeeds with probability p. Unlike earlier, where we wanted to count the number of successes, we want to compute the probability mass function of a random variable X that represents the number of trials before the first success. Such a variable is called a **geometric** random variable and has a probability mass function:

$$p(i) = (1-p)^{i-1}p \qquad \textbf{(EQ 29)}$$

The expected value of a geometrically distributed variable with parameter p is $1/p$.

EXAMPLE 30: GEOMETRIC RANDOM VARIABLE

Consider a link that has a loss probability of 10% and that *packet losses are independent* (although this is rarely true in practice). Suppose that when a packet gets lost this is detected and the packet is retransmitted until it is correctly received. What is the probability that it would be transmitted exactly one, two, and three times?

Solution:

Assuming that the packet transmissions are independent events, we note that the probability of success = p = 0.9. Therefore, $p(1) = 0.1^0 * 0.9 = 0.9$; $p(2) = 0.1^1 * 0.9 = 0.09$; $p(3) = 0.1^2 * 0.9 = 0.009$. Note the rapid decrease in the probability of more than two transmissions, even with a fairly high packet loss rate of 10%. Indeed, the expected number of transmissions is only $1/0.9 = 1.\bar{1}$.

■

1.5.4 Poisson distribution

The **Poisson** distribution is widely encountered in networking situations, usually to model the arrival of packets or new end-to-end connections to a switch or router. A discrete random variable X with the domain {0, 1, 2, 3,...} is said to be a Poisson random variable with parameter λ if, for some $\lambda > 0$:

$$P(X = i) = e^{-\lambda}\left(\frac{\lambda^i}{i!}\right) \qquad \textbf{(EQ 30)}$$

Poisson variables are often used to model the number of events that happen in a fixed time interval. If the events are reasonably rare, then the probability that multiple events occur in a fixed time interval drops off rapidly, due to the $i!$ term in the denominator. The first use of Poisson variables, indeed, was to investigate the number of soldier deaths due to being kicked by a horse in Napoleon's army!

The Poisson distribution (which has only a single parameter λ) can be used to model a binomial distribution with two parameters (n and p) when n is 'large' and p is 'small.' In this case, the Poisson variable's parameter λ corresponds to the product of the two binomial parameters (i.e. $\lambda = n_{Binomial} * p_{Binomial}$). Recall that a binomial distribution arises naturally when we conduct independent trials. The Poisson distribution, therefore, arises when the number of such independent trials is large, and the probability of success of each trial is small. The expected value of a Poisson distributed random variable with parameter λ is also λ.

Consider an endpoint sending a packet on a link. We can model the transmission of a packet by the endpoint in a given time interval as a trial as follows: if the source sends a packet in a particular interval, we will call the trial a success, and if the source does not send a packet, we will call the trial a failure. When the load generated by each source is light, the probability of success of a trial defined in this manner, which is just the packet transmission probability, is small. Therefore, as the number of endpoints grows, and if we can assume the endpoints to be independent, the sum of their loads will be well-modelled by a Poisson random variable. This is heartening, because systems subjected to a Poisson load are mathematically tractable, as we will see in our discussion of queueing theory. Unfortunately, over the last two decades, numerous measurements have shown that actual traffic can be far from Poisson. Therefore, this modelling assumption should be used with care and only as a rough approximation to reality.

EXAMPLE 31: POISSON RANDOM VARIABLE

Consider a link that can receive traffic from one of 1000 independent endpoints. Suppose that each node transmits at a uniform rate of 0.001 packets/second. What is the probability that we see at least one packet on the link during an arbitrary one-second interval?

Solution:

Given that each node transmits packets at the rate of 0.001 packets/second, the probability that a node transmits a packet in any one-second interval is $p_{Binomial}$ = 0.001. Thus, the Poisson parameter λ = 1000*0.001 = 1. The probability that we see at least one packet on the link during any one-second interval is therefore

$1- p(0)$

$= 1-e^{-1}1^0/0!$

$= 1-1/e$

$= 0.64$

That is, there is a 64% chance that, during an arbitrary one-second interval, we will see one or more packets on the link.

■

It turns out that a Poisson random variable is a good approximation to a binomial random variable even if the trials are weakly dependent. Indeed, we do not even require the trials to have equal probabilities, as long as the probability of success of each individual trial is ‘small.’ This is another reason why the Poisson random variable is frequently used to model the behaviour of aggregates.

1.6 Standard continuous distributions

This section presents some standard continuous distributions. Recall from Section 1.3 on page 10 that, unlike discrete random variables, the domain of a continuous random variable is a subset of the real line.

1.6.1 Uniform distribution

A random variable X is said to be uniformly randomly distributed in the domain $[a,b]$ if its density function $f(x) = 1/(b-a)$ when x lies in $[a,b]$ and is 0 otherwise. The expected value of a uniform random variable with parameters a,b is $(a+b)/2$.

1.6.2 Gaussian or Normal distribution

A random variable is **Gaussian** or **normally** distributed with parameters μ and σ^2 if its density is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \textbf{(EQ 31)}$$

We denote a Gaussian random variable X with parameters μ and σ^2 as $X \sim N(\mu,\sigma^2)$, where we read the ‘~’ as “is distributed as.”

The Gaussian distribution can be obtained as the limiting case of the binomial distribution as *n* tends to infinity and *p* is kept constant. That is, if we have a very large number of independent trials, such that the random variable measures the number of trials that succeed, then the random variable is Gaussian. Thus, Gaussian random variables naturally occur when we want to study the statistical properties of aggregates.

The Gaussian distribution is called ‘normal’ because many quantities, such as the heights of people, the slight variations in the size of a manufactured item, and the time taken to complete an activity approximately follow the well-known ‘bell-shaped’ curve[4]. When performing experiments or simulations, it is often the case that the same quantity assumes different values during different trials. For instance, if five students were each measuring the pH of a reagent, it is likely that they would get five slightly different values. In such situations, it is common to assume that these quantities, which are supposed to be the same, are in fact normally distributed about some mean. Generally speaking, if you know that a quantity is supposed to have a certain standard value, but you also know that there can be small variations in this value due to many small and independent random effects, then it is reasonable to assume that the quantity is a Gaussian random variable with its mean centred around the expected value.

The expected value of a Gaussian random variable with parameters μ and σ^2 is μ and its variance is σ^2. In practice, it is often convenient to work with a **standard Gaussian distribution**, that has a zero mean and a variance of 1. It is possible to convert a Gaussian random variable *X* with parameters μ and σ^2 to a Gaussian random variable *Y* with parameters 0,1 by choosing $Y = (X - \mu)/\sigma$.

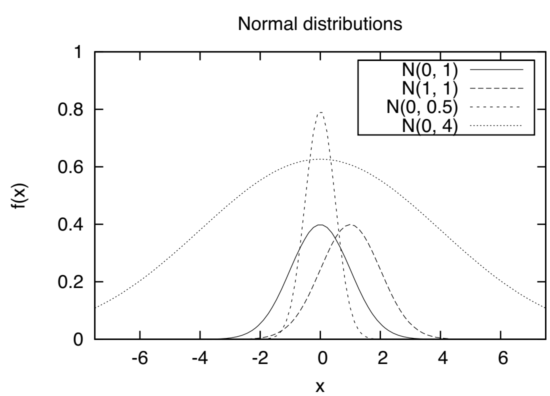


Figure 4. Gaussian distributions for different values of the mean and variance.

The Gaussian distribution is symmetric about the mean and asymptotes to 0 at $+\infty$ and $-\infty$. The σ^2 parameter controls the width of the central ‘bell’: the larger this parameter, the wider the bell, and the lower the maximum value of the density function. The probability that a Gaussian random variable *X* lies between $\mu - \sigma$ and $\mu + \sigma$ is approximately 68.26%; between $\mu - 2\sigma$ and $\mu + 2\sigma$ is approximately 95.44%; and between $\mu - 3\sigma$ and $\mu + 3\sigma$ is approximately 99.73%.

It is often convenient to use a Gaussian continuous random variable to approximately model a discrete random variable. For example, the number of packets that arrive on a link to a router in a given fixed time interval will follow a discrete distribution. Nevertheless, by modelling it using a continuous Gaussian random variable, we can get quick estimates of its expected extremal values.

EXAMPLE 32: GAUSSIAN APPROXIMATION OF A DISCRETE RANDOM VARIABLE

4. With the obvious caveat that many variables in real life are never negative but the Gaussian distribution extends from $-\infty$ to ∞.

Suppose that the number of packets that arrive on a link to a router in a one-second interval can be modelled accurately by a normal distribution with parameters (20, 4). How many packets can we actually expect to see with at least 99% confidence?

Solution:

The number of packets are distributed (20, 4), so that $\mu = 20$ and $\sigma = 2$. We have more than 99% confidence that the number of packets seen will be $\mu \pm 3\sigma$, i.e., between 14 and 26. That is, if we were to measure packets arrivals over a long period of time, fewer than 1% of the one-second intervals would have packet counts fewer than 14 or more than 26.

■

The MGF of the normal distribution is given by:

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}\,dx$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-\mu-\sigma^2 t)^2}{\sigma^2}}\,dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

where in the last step, we recognize that the integral is the area under a normal curve, which evaluates to $\sigma\sqrt{2\pi}$. Note that the MGF of a normal variable with zero mean and a variance of 1 is therefore:

$$M(t) = e^{\frac{1}{2}t^2} \qquad \text{(EQ 32)}$$

We can use the MGF of a normal distribution to prove some elementary facts about it:

(a) If $X \sim N(\mu,\sigma^2)$ then $a + bX \sim N(a+b\mu,\, b^2\sigma^2)$. This is because the MGF of $a+bX$ is:

$$e^{at}M(bt) = e^{at}e^{\mu bt + \frac{1}{2}\sigma^2(bt)^2}$$

$$= e^{(a+\mu b)t + \frac{1}{2}(\sigma^2 b^2)t^2},$$

which can be seen to be a normally distributed random variable with mean $a + b\mu$ and variance $b^2\sigma^2$.

(b) If $X \sim N(\mu,\sigma^2)$ then $Z = (X-\mu)/\sigma \sim N(0,1)$. This is obtained trivially by substituting for a and b in the expression above. Z is called the **standard normal variable**.

(c) If $X \sim N(\mu_1,\sigma_1^2)$ and $Y \sim N(\mu_2,\sigma_2^2)$ and X and Y are independent, then $X+Y \sim N(\mu_1+\mu_2,\, \sigma_1^2+\sigma_2^2)$. This is because the MGF of their sum is the product of their individual MGFs $= e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$. As a corollary, the sum of any number of independent normal variables is also normally distributed with the mean as the sum of the individual means and the variance as the sum of the individual variances.

1.6.3 Exponential distribution

A random variable X is exponentially distributed with parameter λ, where $\lambda > 0$ if its density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \text{(EQ 33)}$$

Note than when $x = 0$, $f(x) = \lambda$ (see Figure 5). The expected value of such a random variable is $\frac{1}{\lambda}$ and its variance is $\frac{1}{\lambda^2}$.

The exponential distribution is the continuous analogue of the geometric distribution. Recall that the geometric distribution measures the number of trials until the first success. Correspondingly, the exponential distribution arises when we are trying to measure the duration of time before some event happens (i.e. achieves success). For instance, it is used to model the time between two consecutive packet arrivals on a link.

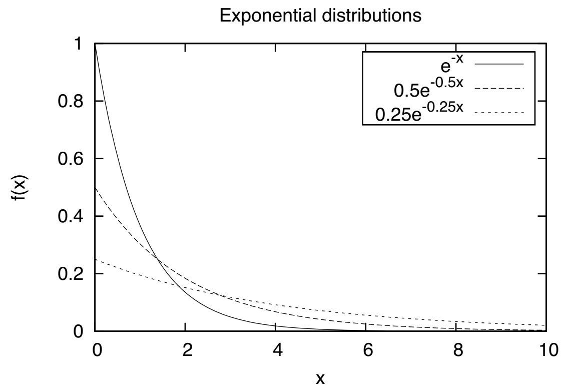


Figure 5. Exponentially distributed random variables with $\lambda = \{1, 0.5, 0.25\}$.

The cumulative density function of the exponential distribution, $F(X)$, is given by:

$$F(X) = p(X \leq x) = 1 - e^{-\lambda x} \qquad \text{(EQ 34)}$$

EXAMPLE 33: EXPONENTIAL RANDOM VARIABLE

Suppose that measurements show that the average length of a phone call is three minutes. Assuming that the length of a call is an exponential random variable, what is the probability that a call lasts more than six minutes?

Solution:

Clearly, the λ parameter for this distribution is 1/3. Therefore, the probability that a call lasts more than six minutes is 1- $F(6) = 1 - e^{-6/3} = 1 - e^{-2} = 13.5\%$

■

An important property of the exponential distribution is that, like the geometric distribution, it is **memoryless** and, in fact, it is the *only* memoryless continuous distribution. Intuitively, this means that the expected remaining time until the occurrence of an event with an exponentially distributed waiting time is *independent* of the time at which the observation is made. More precisely, $P(X > s+t \mid X>s) = P(X>t)$ for all s, t. From a geometric perspective, if we truncate the distribution to the left of any point on the positive X axis, then rescale the remaining distribution so that the area under the curve is 1, we will obtain the original distribution. The following examples illustrate this useful property.

EXAMPLE 34: MEMORYLESSNESS 1

Suppose the time taken by a teller at a bank is an exponentially distributed random variable with an expected value of one minute. When you arrive at the bank, the teller is already serving a customer. If you join the queue now, you can expect to wait one minute before being served. However, suppose you decide to run an errand and return to the bank. If the same customer is still being served (i.e. the condition $X>s$), if you join the queue now, the expected waiting time for you to be served would *still* be 1 minute!

■

EXAMPLE 35: MEMORYLESSNESS 2

Suppose that a switch has two parallel links to another switch and packets can be routed on either link. Consider a packet *A* that arrives when both links are already in service. Therefore, the packet will be sent on the first link that becomes free. Suppose this is link 1. Now, assuming that link service times are exponentially distributed, which packet is likely to finish transmission first: packet *A* on link 1 or the packet continuing service on link 2?

Solution:

Because of the memorylessness of the exponential distribution, the expected remaining service time on link 2 at the time that *A* starts transmission on link 1 is exactly the same as the expected service time for *A*, so we expect both to finish transmission at the same time. Of course, we are assuming we don't know the service time for *A*. If a packet's service time is proportional to its length, and we know *A*'s length, then we no longer have an expectation for its service time: we know it precisely, and this equality no longer holds.

■

1.6.4 Power law distribution

A random variable described by its minimum value x_{min} and a scale parameter $\alpha > 1$ is said to obey the power law distribution if its density function is given by:

$$f(x) = \frac{(\alpha - 1)}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha} \quad \text{(EQ 35)}$$

Typically, this function needs to be normalized for a given set of parameters to ensure that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Note that $f(x)$ decreases rapidly with x. However, the decline is not as rapid as with an exponential distribution (see Figure 6). This is why a power-law distribution is also called a 'heavy-tailed' distribution. When plotted on a log-log scale, the graph of $f(x)$ vs. x shows a linear relationship with a slope of $-\alpha$, which is often used to quickly identify a potential power-law distribution in a data set.

Intuitively, if we have objects distributed according to an exponential or power law, then there are a few 'elephants' that occur frequently and are common and many 'mice' that are relatively uncommon. The elephants are responsible for most of the probability mass. From an engineering perspective, whenever we see such a distribution, it makes sense to build a system that deals well with the elephants, even at the expense of ignoring the mice. Two rules of thumb that reflect this are the *90/10 rule* (90% of the output is derived from 10% of the input) and the dictum '*optimize for the common case*.'

When $\alpha < 2$, the expected value of the random variable is infinite. A system described by such a random variable is unstable (i.e. its value is unbounded). On the other hand when $\alpha > 2$, the tail probabilities fall rapidly enough that a power-law random variable can usually be well-approximated by an exponential random variable.

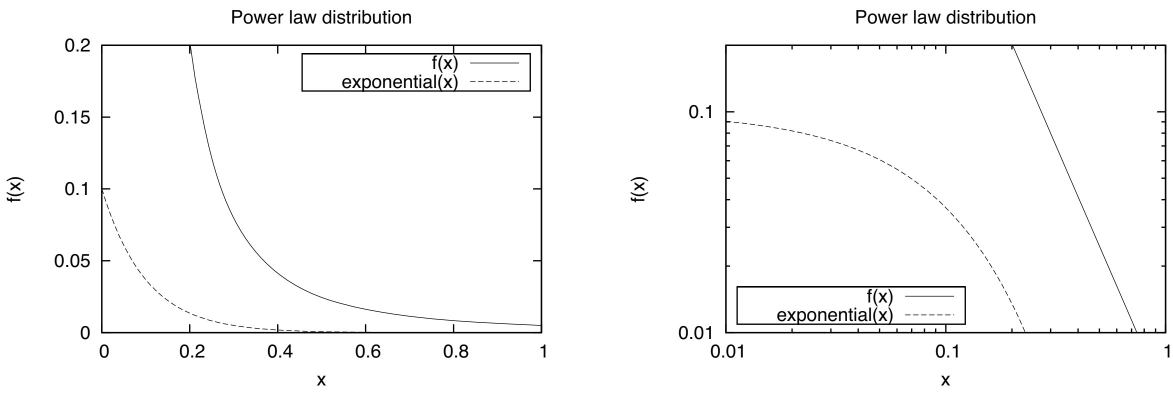


Figure 6. A typical power law distribution with parameters $x_{min} = 0.1$ and $\alpha = 2.3$ compared to an exponential distribution using a linear-linear (left) and a log-log (right) scale.

A widely-studied example of power-law distribution is the random variable that describes the number of users who visit one of a collection of websites on the Internet on any given day. Traces of website accesses almost always show that all but a microscopic fraction of websites get fewer than one visitor a day: traffic is mostly garnered by a handful of well-known websites.

1.7 Useful theorems

This section discusses some useful theorem: Markov's and Chebyshev's inequality allow us to bound the amount of mass in a the tail of a distribution knowing nothing more than its expected value (Markov) and variance (Chebyshev). Chernoff's bound allows us to bound both the lower and upper tails of distributions arising from independent trials. The law of large numbers allows us to relate real-world measurements with the expectation of a random variable. Finally, the central limit theorem shows why so many real-world random variables are normally distributed.

1.7.1 Markov's inequality

If X is a *non-negative* random variable with mean μ, then for any constant $a > 0$

$$p(X \geq a) \leq \frac{\mu}{a} \qquad \text{(EQ 36)}$$

Thus, we can bound the probability mass to the right of any constant a by a value proportional to the expected value of X and inversely proportional to a (Figure 7). Markov's inequality requires knowledge only of the mean of the distribution. Note that this inequality is trivial if $a < \mu$ (why?). Note also that the Markov inequality does not apply to some standard distributions, such as the normal distribution, because they are not always non-negative.

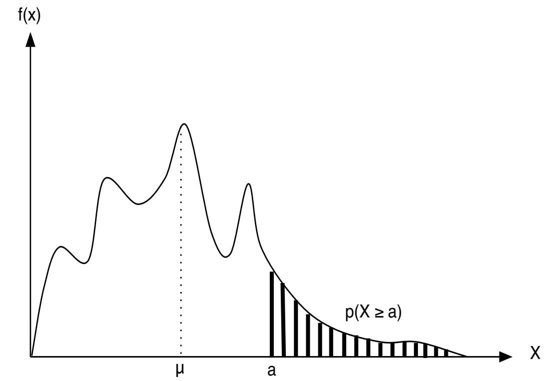


Figure 7. **Markov's inequality**

EXAMPLE 36: MARKOV INEQUALITY

Use the Markov inequality to bound the probability mass to the right of the value 0.75 of a uniform (0,1) distribution.

Solution:

The mean of this distribution is 0.5, so $p(X \geq 0.75) \leq \frac{0.5}{0.75} = 0.66$. The actual probability mass is only 0.25, so the Markov bound is quite loose. This is typical of a Markov bound.

■

1.7.2 Chebyshev's inequality

If X is a random variable with a finite mean μ and variance σ^2, then for any constant $a > 0$

$$p(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2} \quad \textbf{(EQ 37)}$$

Chebyshev's inequality bounds the 'tails' of a distribution on both sides of the mean, given the variance. Roughly, the further away we get from the mean (the larger a is), the less mass there is in the tail (because the right hand size decreases by a factor quadratic in a).

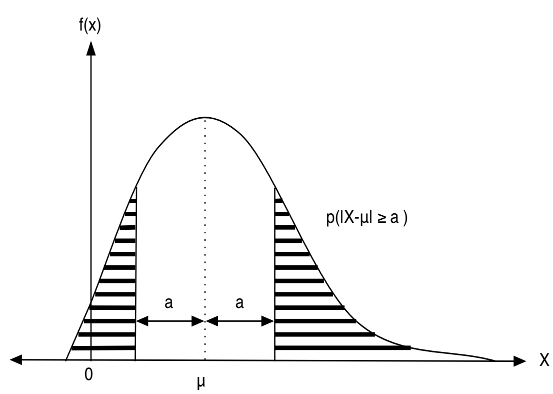


Figure 8. **Chebyshev's inequality**

EXAMPLE 37: CHEBYSHEV BOUND

Use the Chebyshev bound to compute the probability that a standard normal random variable has a value greater than 3.

Solution:

For a standard normal variable, $\mu = 0$ and $\sigma = 1$. We have $a = 3$. So, $p(|X| \geq 3) \leq \frac{1}{9}$, so that $p(X > 3) \leq \frac{1}{18}$ or about 5.5%. Compare this to the tight bound of 0.135% (Section 1.6.2 on page 21).

■

1.7.3 Chernoff bound

Let the random variable X_i denote the outcome of the ith iteration of a process, with $X_i = 1$ denoting success and $X_i = 0$ denoting failure. Assume that the probability of success of each iteration is independent of the others (this is critical!). Denote the probability of success of the ith trial by $p(X_i = 1) = p_i$. Let X be the number of successful trials in a run of n trials. Clearly, $X = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} p_i$. Let $E[X] = \mu$ be the expected value of X (the expected number of successes). Then we can state two Chernoff bounds that tell us the probability that there are 'too few' or 'too many' successes.

The **lower bound** is given by:

$$p(X < (1-\delta)\mu) < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}, \qquad 0 < \delta \leq 1 \qquad \textbf{(EQ 38)}$$

This is somewhat hard to compute. A weaker but more tractable bound is:

$$p(X < (1-\delta)\mu) < e^{\frac{-\mu\delta^2}{2}}, \qquad 0 < \delta \leq 1 \qquad \textbf{(EQ 39)}$$

Note that both equations bound the area under the density distribution of X between $-\infty$ and $(1-\delta)\mu$. The second form makes it clear that the probability of too few successes declines quadratically with δ.

The **upper bound** is given by:

$$p(X > (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}, \qquad \delta > 0 \qquad \textbf{(EQ 40)}$$

A weaker but more tractable bound is:

$$\begin{aligned} p(X > (1+\delta)\mu) &< e^{\frac{-\mu\delta^2}{4}} && \text{if } \delta < 2e-1 \\ p(X > (1+\delta)\mu) &< 2^{-\delta\mu} && \text{if } \delta > 2e-1 \end{aligned} \qquad \textbf{(EQ 41)}$$

EXAMPLE 38: CHERNOFF BOUND

Use the Chernoff bound to compute the probability that a packet source that suffers from independent packet losses, where the probability of each loss is 0.1, suffers from more than 4 packet losses when transmitting 10 packets.

Solution:

We define a 'successful' event to be a packet loss, with the probability of success being $p_i = 0.1 \ \forall i$. We have $E[X] = (10)(0.1) = 1 = \mu$. Also, we want to compute $p(X > 4) = p(X > (1+3)\mu)$ so that $\delta = 3$. So,

$$p(X > 4) < \left(\frac{e^3}{(1+3)^{1+3}}\right)^1 = \frac{e^3}{256} = 0.078 \ .$$

As with all bounds, this is looser than the exact value computed from the binomial theorem, given by

$$(1 - p(X = 0) + p(X = 1) + p(X = 2) + p(X = 3) + p(X = 4))$$

$$= 1 - \binom{10}{0}(0.9)^{10} - \binom{10}{1}(0.1)(0.9)^9 - \binom{10}{2}(0.1)^2(0.9)^8 - \binom{10}{3}(0.1)^3(0.9)^7$$

$= 0.0033.$

■

1.7.4 Strong law of large numbers

The law of large numbers relates the **sample mean—**the average of a set of observations of a random variable—with the **population** or **true mean**, which is its expected value. The **strong** law of large numbers, the better-known variant, states that if $X_1, X_2, \ldots, X_n$ are n independent, identically distributed random variables with the same expected value μ, then:

$$P\left(\lim_{n \to \infty}(X_1 + X_2 + \ldots + X_n)/n = \mu\right) = 1 \qquad \text{(EQ 42)}$$

No matter how X is distributed, by computing an average over a sufficiently large number of observations, this average can be made to be as close to the true mean as we wish. This is the basis of a variety of statistical techniques for hypothesis testing, as described in Chapter 2.

We illustrate this law in Figure 9, which shows the average of 1,2,3,..., 500 successive values of a random variable drawn from a uniform distribution in the range [0, 1]. The expected value of this random variable is 0.5, and the average converges to this expected value as the sample size increases.

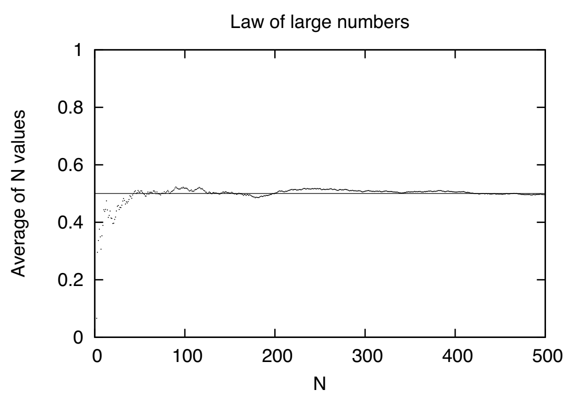


Figure 9. **Strong law of large numbers. As *N* increases, the average value of sample of *N* random values converges to the expected value of the distribution.**

1.7.5 Central limit theorem

The central limit theorem deals with the sum of a *large* number of *independent* random variables that are arbitrarily distributed. The theorem states that no matter how each random variable is distributed, as long as its contribution to the total is 'small,' the sum is well-described by a Gaussian random variable.

More precisely, let $X_1, X_2, \ldots, X_n$ be n independent, identically distributed random variables, each with a finite mean μ and variance σ^2. Then, the distribution of the normalized sum given by $\frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard (0,1) normal as $n \to \infty$. The central limit theorem is the reason why the Gaussian distribution is the limit of the binomial distribution.

In practice, the central limit theorem allows us to model aggregates by a Gaussian random variable if the size of the aggregate is large and the elements of the aggregate are independent.

The Gaussian distribution plays a central role in statistics because of the central limit theorem. Consider a set of measurements of a physical system. Each measurement can be modelled as an independent random variable whose mean and variance are those of the population. From the central limit theorem, their sum, and therefore their mean (which is just the normalized sum) is approximately normally distributed. As we will study in Chapter 2, this allows us to infer the population mean from the sample mean, which forms the foundation of statistical confidence. We now prove the central limit theorem using MGFs.

The proof proceeds in three stages. First, we compute the MGF of the sum of n random variables in terms of the MGFs of each of the random variables. Second, we find a simple expression for the MGF of a random variable when the variance is large (a situation we expect when adding together many independent random variables). Finally, we plug in this simple expression back into the MGF of the sum to obtain the desired result.

Consider a random variable $Y = X_1 + X_2 + \ldots + X_n$, the sum of n *independent* random variables X_i. Let μ_i and σ_i denote the mean and standard deviation of X_i and let μ and σ denote the mean and standard deviation of Y. Because the X_is are independent,

$$\mu = \sum \mu_i \ ; \ \sigma^2 = \sum \sigma_i^2 \tag{EQ 43}$$

Define the random variable W_i to be (X_i -μ_i): it represents the distance of an instance of the random variable X_i from its mean. By definition, the rth moment of W_i about the origin is the rth moment of X_i about its mean. Also, because the X_i are independent, so are the W_i. Denote the MGF of X_i by $M_i(t)$ and the MGF of W_i by $N_i(t)$.

Note that $Y - \mu = X_1 + X_2 + \ldots + X_n - \sum \mu_i = \sum (X_i - \mu_i) = \sum W_i$. So the MGF of $Y - \mu$ is the product of the MGFs of the $W_i = \prod_{i=1}^{n} N_i(t)$. Therefore, the MGF of $(Y - \mu)/\sigma$ denoted $N^*(t)$ is given by:

$$N^*(t) = \prod_{i=1}^{n} N_i\left(\frac{t}{\sigma}\right) \tag{EQ 44}$$

Consider the MGF $N_i(t/\sigma)$, which is given by $E\left(e^{\frac{W_i t}{\sigma}}\right)$. Expanding the exponential, we find that

$$N_i\left(\frac{t}{\sigma}\right) = E\left(e^{\frac{W_i t}{\sigma}}\right) = 1 + E(W_i)\frac{t}{\sigma} + \frac{E(W_i^2)}{2!}\left(\frac{t}{\sigma}\right)^2 + \frac{E(W_i^3)}{3!}\left(\frac{t}{\sigma}\right)^3 + \ldots \tag{EQ 45}$$

Now, $E(W_i) = E(X_i\text{-}\mu_i) = E(X_i)\text{-}\mu_i = \mu_i\text{-}\mu_i = 0$, so we can ignore the second term in the expansion. Recall that σ is the standard deviation of the sum of n random variables. When n is large, then σ is also large, which means that, to first order, we can ignore terms that have σ^3 and higher powers of σ in the denominator in Equation 45. Therefore, for large n, we can write:

$$N^i\left(\frac{t}{\sigma}\right) \approx \left(1 + \frac{E(W_i^2)}{2!}\left(\frac{t}{\sigma}\right)^2\right) = 1 + \frac{\sigma_i^2}{2}\left(\frac{t}{\sigma}\right)^2 \tag{EQ 46}$$

where we have used the fact that $E(W_i^2) = E(X_i - \mu)^2$, which is the variance of $X_i = \sigma_i^2$

Returning to the expression in Equation 44, we find that

$$\log N^*(t) = \log\left(\prod_{i=1}^{n} N_i\left(\frac{t}{\sigma}\right)\right) = \sum_{i=1}^{n} \log\left(N_i\left(\frac{t}{\sigma}\right)\right) \approx \sum_{i=1}^{n} \log\left(1 + \frac{\sigma_i^2}{2}\left(\frac{t}{\sigma}\right)^2\right) \tag{EQ 47}$$

It is easily shown by the Taylor series expansion that when h is small (so that h^2 and higher powers of h can be ignored) $\log(1+h)$ can be approximated by h. So, when n is large, and σ is large, we can further approximate

$$\sum_{i=1}^{n} \log\left(1 + \frac{\sigma_i^2}{2}\left(\frac{t}{\sigma}\right)^2\right) \approx \sum_{i=1}^{n} \frac{\sigma_i^2}{2}\left(\frac{t}{\sigma}\right)^2 = \frac{1}{2}\left(\frac{t}{\sigma}\right)^2 \sum_{i=1}^{n} \sigma_i^2 = \frac{1}{2}t^2 \tag{EQ 48}$$

where, for the last simplification, we used Equation 43. Thus, $\log N^*(t)$ is approximately $1/2\ t^2$, which means that

$$N^*(t) \approx e^{\frac{t^2}{2}} \tag{EQ 49}$$

But this is just the MGF of a standard normal variable with zero mean and a variance of 1 (Equation 32). Therefore, $(Y - \mu)/\sigma$ is a standard normal variable, which means that $Y \sim N(\mu, \sigma^2)$. We have therefore shown that the sum of a large number of independent random variables is distributed as a normal variable whose mean is the sum of the individual means and whose variance is the sum of the individual variances (Equation 43), as desired.

1.8 Jointly distributed random variables

So far, we have considered distributions of one random variable. We now consider situations where we want to study the distribution of two random variables simultaneously.

EXAMPLE 39: JOINT PROBABILITY DISTRIBUTION

Consider the two events: ‘rain today’ and ‘rain tomorrow.’ Let the random variable X be 0 if it does not rain today and 1 if it does. Similarly, let the random variable Y be 0 if it does not rain tomorrow and 1 if it does. There are four possible values for the random variables X and Y considered together: 00, 01, 10, and 11, corresponding to four joint events. We can associate probabilities with these events with the usual restrictions that these probabilities lie in [0,1] and that their sum be 1. For instance, consider the following distribution:

$p(00) = 0.2$;

$p(01) = 0.4$;

$p(10) = 0.3$;

$p(11) = 0.1$

where the ‘00’ is now interpreted as shorthand for X=0 AND Y=0, and so on. This defines the **joint probability** distribution of X and Y which is denoted $p_{XY}(xy)$ or sometimes $p(X,Y)$. Given this joint distribution, we can extract the distribution of X alone, which is the probability of X=0 and of X=1, as follows: $p(X=0) = p(00) + p(01) = 0.1 + 0.4 = 0.6$. Similarly, $p(X=1) = 0.3 + 0.1 = 0.4$. As expected $p(X=0) + p(X=1) = 1$. Similarly, note that $p(Y=0) = 0.5$ and $p(Y=1) = 0.5$.

■

We call the distribution of X alone as the **marginal** distribution of X and denote it p_X. Similarly, the marginal distribution of Y is denoted p_Y. Generalizing from the example above, we see that to obtain the marginal distribution of X, we should set X to each value in its domain and then sum over *all possible values of Y*. Similarly, to obtain the marginal distribution of Y, we set Y to each value in its domain and sum over all possible values of X.

An important special case of a joint distribution is when the two variables X and Y are **independent**. Then, $p_{XY}(xy) = p(X=x \text{ AND } Y=y) = p(X=x) * p(Y=y) = p_X(x)p_Y(y)$. That is, each entry in the joint distribution is obtained simply as the product of the marginal distributions corresponding to that value. We sometimes denote this as $= p_X(x)p_Y(y)$.

EXAMPLE 40: INDEPENDENCE

In Example 39, $p_{XY}(00) = 0.2$, $p_X(0) = 0.6$ and $p_Y(0) = 0.5$, so X and Y are *not* independent: we *cannot* decompose the joint distribution into the product of the marginal distributions.

■

Given the joint distribution, we define the **conditional probability mass function of X,** denoted by $p_{X|Y}(x|y)$ by $p(X=x \mid Y=y) = p(X=x \text{ AND } Y=y)/p(Y=y) = \dfrac{p_{XY(xy)}}{p_Y(y)}$.

EXAMPLE 41: CONDITIONAL PROBABILITY MASS FUNCTION

Continuing with Example 21, suppose we wanted to compute the probability that it will rain tomorrow, given that it rained today. This is $p_{Y|X}(1|1) = p_{XY}(11)/p_X(1) = 0.1/0.4 = 0.25$. Thus, knowing that it rained today makes it more probable that it will rain tomorrow.

■

We can generalize the notion of joint probability in several ways. We outline these generalizations next. Note that the concepts we have developed for the simple case above continue to hold for these generalizations.

1. Instead of having only two values, 0 and 1, X and Y could assume any number of finite discrete values. In this case, if there are n values of X and m values of Y, we would need to specify, for the joint distribution, a total of nm values. If X and Y are independent, however, we only need to specify $n+m$ values to completely specify the joint distribution.
2. We can generalize this further and allow X and Y to be continuous random variables. Then, the joint probability distribution $p_{XY}(xy)$ is implicitly defined by:

$$p(a \le X \le a + \alpha, b \le Y \le b + \beta) = \int_b^{(b+\beta)} \int_a^{(a+\alpha)} p_{XY}(xy)\,dx\,dy \quad \text{(EQ 50)}$$

   Intuitively, this is the probability that a randomly chosen two-dimensional vector will be in the vicinity of (a,b).
3. As a further generalization, consider the joint distribution of n random variables, $X_1, X_2, \ldots, X_n$, where each variable is either discrete or continuous. If they are all discrete, then we need to define the probability of each possible choice of each value of X_i. This grows exponentially with the number of random variables and with the size of each domain of each random variable. Thus, it is impractical to completely specify the joint probability distribution for a large number of variables. Instead, we exploit pairwise independence between the variables, using the construct of a Bayesian network, which is described next.

1.8.1 Bayesian networks

Bayes' rule allows us to compute the degree to which one of a set of mutually exclusive prior events contribute to a posterior condition. Suppose the posterior condition was itself a prior to yet another posterior and so on. We could then imagine tracing this chain of conditional causation back from the final condition to the initial causes. This, in essence, is a Bayesian network. We will study one of the simplest forms of a Bayesian network next.

A Bayesian network is a directed acyclic graph whose vertices represent random variables and whose edges represent conditional causation between these random variables: there is an edge from a random variable E_i, called the 'parent' or 'cause', to every random variable E_j whose outcome depends on it, called its 'children' or 'effects.' If there is no edge between E_i and E_j, they are independent. Each node in the Bayesian network stores the conditional probability distribution $p(E_i|\text{parents}(E_i))$ also called its **local distribution**. Note that if the node has no parents, then its distribution is unconditionally known. The network allows us to compute the joint probability $p(E_1E_2...E_n)$ as:

$$p(E_1E_2\ldots E_n) = \prod_i p(E_i|parents(E_i)) \qquad \text{(EQ 51)}$$

That is, the joint distribution is simply the product of the local distributions. This greatly reduces the amount of information required to describe the joint probability distribution of the random variables. Choosing the Bayesian graph is a non-trivial problem and one that we will not discuss further. An overview can be found in the text by Russel and Norvig cited in Section 1.9 on page 35.

Note that, because the Bayesian network encodes the full joint distribution, in principle, we can extract any probability we want from it. Usually we want to compute something much simpler. A Bayesian network allows us to compute probabilities of interest without having to compute the entire joint distribution, as the next example demonstrates.

EXAMPLE 42: BAYESIAN NETWORK

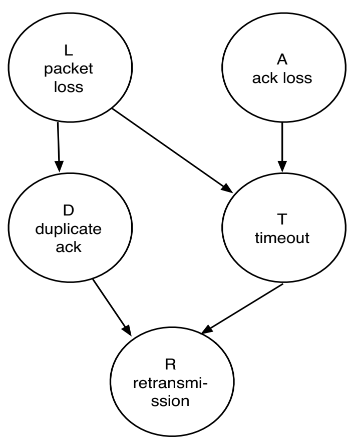


Figure 10. **A Bayesian network to represent TCP retransmissions**

Consider the Bayesian network in Figure 10. Each circle shows a discrete random variable that can assume only two values, true or false. Each random variable is associated with an underlying event in the appropriate sample space as shown in the figure. The network shows that if *L*, the random variable representing packet loss event has the value true (the cause), this may lead to a timeout event at the TCP transmitter (effect), so that the random variable representing this, i.e., *T*, has a higher probability of having the value true. Similarly, the random variable denoting the loss of an acknowledgement packet may also increase the probability that *T* assumes the value 'true.' The node marked *T* therefore stores the probabilty that it assumes the value true conditional on the parents assuming the set of values {(true, true), (true, false), (false, true), (false, false)}.

The network also represents the fact that a packet loss event affects the likelihood of a duplicate acknowledgment event. However, packet and ack loss events are mutually exclusive, as are duplicate acks and timeouts. Finally, if there is either a duplicate ack or a timeout at the transmitter, it will surely retransmit a packet.

The joint distribution of the random variables (L, A, D, T, R) would assign a probability to every possible combination of the variables, such as p(*packet loss AND no ack loss AND no duplicate ack AND timeout AND no retransmission*). In practice, we rarely need the joint distribution. Instead, we may only be interested in computing the following probability: p(*packet loss* | *retransmission*) = $p(L|R)$. That is, we observe the event that the transmitter has retransmitted a packet. What is the probability that the event packet loss occurred, i.e. what is $p(L|R)$?

For notational simplicity, let $p(R=\text{'true'}) = p(R) = r$, $p(L=\text{'true'}) = p(L) = l$, $p(T=\text{'true'}) = p(T) = t$, $p(A=\text{'true'}) = p(A) = a$ and $p(D=\text{'true'}) = p(D) = d$. From the network, it is clear that we can write $p(R)$ as $p(R|T)t + p(R|D)d$. Similarly, $t = p(T|L)l + p(T|A)a$ and $d = p(D|L)l$. Therefore,

$$p(R) = r = p(R|T)(p(T|L)l + p(T|A)a) + p(R|D)p(D|L)l$$

If we know a and l and the conditional probabilities stored at each node, we can therefore compute r.

From the definition of conditional probabilities:

$$p(L|R) = \frac{p(LR)}{r} \qquad \textbf{(EQ 52)}$$

We have already seen how to compute the denominator. To compute the numerator, we sum across all possibilities for L and R as follows:

$$p(LR) = p(LRTD) + p(LRT\bar{D}) + p(LR\bar{T}D) + p(LR\bar{T}\bar{D})$$

where the overbar represents the probability that the random variable assumes the value 'false.' However, note that T and D are mutually exclusive, so

$$p(TD) = 0$$

$$p(T\bar{D}) = p(T)$$

$$p(\bar{T}D) = p(D)$$

Thus,

$$p(LR) = p(LRT) + p(LRD) + p(LR\bar{T}\bar{D})$$

The last term is 0 because we do not have a retransmission unless there is either a timeout or a duplicate ack. Thus,

$$p(LR) = P(LRT) + P(LRD).$$

Replacing this in Equation 52, we get:

$$p(PLR) = \frac{p(LRT) + p(LRD)}{p(R|T)(p(T|L)l + p(T|A)a) + p(R|D)p(D|L)l}$$

All these variables can be computed by observations over sufficiently long durations of time. For instance, to compute $p(LRT)$, we can compute the ratio of all retransmissions where there was both a packet loss and timeout event to the number of transmissions. Similarly, to compute $p(R|T)$, we can compute the ratio of the number of times a retransmission happens due to a timeout to the number of times a timeout happens. This allows us to compute $p(L|R)$ in practice.

■

1.9 Further Reading

There are a number of excellent introductory texts on probability that treat this subject in more detail, such as S. Ross, A First Course in Probability, 7th edition, Prentice Hall, 2006. A more sophisticated treatment is the classic text: W. Feller, An Introduction to Probabilty Theory and its Applications, 3rd edition, John Wiley, 1968. Bayesian analysis is described in the standard textbook on Artificial Intelligence: S. Russell and P. Norvig, Artificial Intelligence: A Modern Approach, 3rd edition, Pearson, 2010.

1.10 Exercises

1 Sample space

In the IEEE 802.11 protocol, the congestion window (CW) parameter is used as follows: initially, a terminal waits for a random time period (called *backoff*) chosen in the range $[1, 2^{CW}]$ before sending a packet. If an acknowledgement for the packet is not received in time, then CW is doubled, and the process is repeated, until CW reaches the value CWMAX. The initial value of CW is CWMIN. What is the sample space for (a) the value of CW? (b) the value of the backoff?

2 Interpretations of probability

Consider the statement: given the conditions right now, the probability of a snowstorm tomorrow morning is 25%. How would you interpret this statement from the perspective of an objective, frequentist, and subjective interpretation of probability (assuming these are possible)?

3 Conditional probability

Consider a device that samples packets on a link. (a) Suppose that measurements show that 20% of packets are UDP, and that 10% of all packets are UDP packets with a packet size of 100 bytes.What is the conditional probability that a UDP packet has size 100 bytes? (b) Suppose 50% of packets were UDP, and 50% of UDP packets were 100 bytes long. What fraction of all packets are 100 byte UDP packets?

4 Conditional probability again

Continuing with Ex. 3: How does the knowledge of the protocol type change the sample space of possible packet lengths? In other words, what is the sample space before and after you know the protocol type of a packet?

5 Bayes' rule

For Exercise 3(a), what additional information do you need to compute P(UDP|100)? Setting that value to x, express P(UDP|100) in terms of x.

6 Cumulative distribution function

(a) Suppose discrete random variable D take values $\{1, 2, 3,...,i,...\}$ with probability $1/2^i$. What is its CDF?

(b) Suppose continuous random variable C is uniform in the range $[x_1, x_2]$. What is its CDF?

7 Expectations

Compute the expectations of the D and C in Exercise 6.

8 Variance

Prove that $V[aX] = a^2V[X]$.

9 Moments

Prove that $M_\mu^3 = M_0^3 - 3M_0^2M_0^1 + 2(M_0^1)^3$

10 MGFs

Prove that the MGF of a uniform random variable, expressed in terms of its series expansion is

$$E(e^{tx}) = \int_0^1 \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots\right) dx = \frac{1}{t}[e^t - 1] .$$

11 MGFs

Prove that the r^{th} moment of the uniform distribution about the origin is $1/(r+1)$.

12 MGF of a sum of two variables

Use MGFs to find the variance of the sum of two independent uniform standard random variables.

13 MGF of a normal distribution

Prove that if $X \sim N(\mu,\sigma^2)$ then $(X-\mu)/\sigma \sim N(0,1)$.

14 Bernoulli distribution

A hotel has 20 guest rooms. Assuming outgoing calls are independent and that a guest room makes 10 minutes worth of outgoing calls during the busiest hour of the day, what is the probability that 5 calls are simultaneously active during the busiest hour? What is the probability of 15 simultaneous calls?

15 Geometric distribution

Consider a link that has a packet loss rate of 10%. Suppose that every packet transmission has to be acknowledged. Compute the expected number of data transmissions for a successful packet+ack transfer.

16 Poisson distribution

Consider a binomially distributed random variable X with parameters n=10, p=0.1. (a) Compute the value of $P(X=8)$ using both the binomial distribution and the Poisson approximation. (b) Repeat for n=100, p=0.1

17 Gaussian distribution

Prove that if X is Gaussian with parameters (μ, σ^2), then the random variable $Y=aX + b$, where a and b are constants, is also Gaussian, with parameters $(a\mu + b, (a\sigma)^2)$.

18 Exponential distribution

Suppose that customers arrive to a bank with an exponentially distributed inter-arrival time with mean 5 minutes. A customer walks into the bank at 3pm. What is the probability that the next customer arrives no sooner than 3:15?

19 Exponential distribution

It is late August and you are watching the Perseid meteor shower. You are told that the time between meteors is exponentially distributed with a mean of 200 seconds. At 10:05 pm, you see a meteor, after which you head to the kitchen for a bowl of icecream, returning outside at 10:08pm. How long do you expect to wait to see the next meteor?

20 Power law

Consider a power-law distribution with $x_{min} = 1$ and $\alpha = 2$ and an exponential distribution with $\lambda = 2$. Fill in the following table:

x	$f_{power\_law}(x)$	$f_{exponential}(x)$
1		
5		
10		
50		
100		

It should now be obvious why a power-law distribution is called ‘heavy-tailed’!

21 Markov’s inequality

Consider a random variable X that exponentially distributed with parameter $\lambda = 2$. What is the probability that $X > 10$ using (a) the exponential distribution (b) Markov’s inequality.

22 Joint probability distribution

Consider the following probability mass function defined jointly over the random variables, X, Y, and Z:

$p(000) = 0.05$; $p(001) = 0.05$; $p(010) = 0.1$; $p(011)=0.3$; $p(100) = 0.05$; $p(101) = 0.05$; $p(110) = 0.1$; $p(111)=0.3$. (a) Write down p_X, p_Y, p_Z, p_{XY}, p_{XZ}, p_{YZ}. (b) Are X and Y, X and Z, or Y and Z independent? What is the probability that X=0 given that Z=1.

CHAPTER 2 Statistics

This chapter reviews basic statistical concepts and techniques. We start by considering a critical problem in statistics: that of choosing a representative sample. We then discuss statistical techniques to deal with some situations that frequently arise in carrying out research in computer networking: describing data parsimoniously, inferring the parameters of a population from a sample, comparing outcomes, and inferring correlation or independence of variables. We conclude with some approaches to dealing with large data sets and a description of common mistakes in statistical analysis and how to avoid them.

2.1 Sampling a population

The universe of individuals under study constitutes a **population** that can be characterized by its inherent **parameters** such as its range, minimum, maximum, mean, or variance. In many practical situations the population is infinite, so we have to estimate its parameters by studying a carefully chosen subset or **sample**. The parameters of a sample, such as its range, mean, and variance, are called its **statistics**. In standard notation, population parameters are denoted using the Greek alphabet and sample statistics are represented using the Roman alphabet. For example, the population mean and variance parameters are denoted μ and σ^2 respectively and the corresponding sample mean and variance statistics are denoted m (or $\bar{x}$) and s^2 respectively.

When choosing a sample, it is important to carefully identify the underlying population, as the next example illustrates.

EXAMPLE 1:CHOICE OF POPULATION

Suppose that you capture a trace of all UDP packets sent on a link from your campus router to your university's Internet Service Provider from 6am to 9pm on Monday, November 17, 2008. What is the underlying population? There are many choices:

- The population of UDP packets sent from your campus router to your university's Internet Service provider from *12:00:01 am to 11:59:59 pm* on November 17, 2008.
- The population of UDP packets sent from your campus router to your university's Internet Service provider from 12:00:01 am to 11:59:59 pm *on Mondays*.
- The population of UDP packets sent from your campus router to your university's Internet Service provider from 12:00:01 am to 11:59:59 pm *on days that are not holidays.*
- The population of UDP packets sent from your campus router to your university's Internet Service provider from 12:00:01 am to 11:59:59 pm *on a typical day.*
- The population of UDP packets sent *from a typical university's campus router to a typical university's* Internet Service provider from 12:00:01 am to 11:59:59 pm on a typical day.

- The population of UDP packets sent from a *typical access router to a typical ISP router* from 12:00:01 am to 11:59:59 pm on a typical day.
- ...
- The population of all UDP packets sent on the Internet in 2008.
- The population of all UDP packets sent since 1969 (the year the Internet was created).

Each population in this list is a superset of the previous population. As you go down the list, therefore, conclusions regarding the population that you draw from your sample are more general. Unfortunately, these conclusions are also less valid. For instance, it is hard to believe that a single day's sample on a single link is representative of all UDP packets sent on the Internet in 2008!

The difficulty when setting up a measurement study is determining a sample that is representative of the population under study. Conversely, given a sample, you are faced with determining the population that is represented by the sample. This population lies in the spectrum between the most specific population—which is the sample itself—where your conclusions are certainly true and the most general population, about which usually no valid conclusions can be drawn. Unfortunately, the only guide to making this judgement is experience and even experts may disagree with any decision you make.

■

2.1.1 Types of sampling

As Example 1 shows, collecting a sample before identifying the corresponding population puts the metaphorical cart in front of the horse. Instead, one should first identify a population to study and only then choose samples that are **representative** of that population. By representative, we mean a sample chosen such that every member of the population is equally likely to be a member of the sample. In contrast, if the sample is chosen so that some members of the population are more likely to be in the sample than others, then the sample is **biased** and the conclusions drawn from it may be inaccurate. Of course, representativeness is in the eye of the beholder. Nevertheless, explicitly stating the population and then the sampling technique will aid in identifying and removing otherwise hidden biases.

Here are some standard sampling techniques:

- **Random**: In random or **proportional** sampling, an unbiased decision rule is used to select elements of the sample from the population. An example of such a rule is: 'Choose an element of the population with probability 0.05.' In doing Monte Carlo simulations, varying seed values in random number generators randomly perturbs simulation trajectories so that one can argue that the results of the simulation are randomly selected from the space of all possible simulation trajectories.
- **Stratified random**: In this approach, the population is first categorized into groups of elements that are expected to differ in some significant way. Then, each group is randomly sampled to create an overall sample of the population. For example, one could first categorize packets on a link according to their transport protocol (TCP, UDP, or 'other'), then sample each category separately in proportion to their ratio in the population.
- **Systematic**: This approach is similar to random sampling but sometimes simpler to carry out. We assume that the population can be enumerated in some random fashion (i.e., with no discernible pattern). Then, the systematic sampling rule is to select every *k*th element of this random enumeration. For instance, if we expected packet arrivals to a switch to be in no particular order with respect to their destination port, then the destination port of every 100th arriving packet would constitute a systematic sample.
- **Cluster**: Cluster sampling, like stratified sampling, is appropriate when the population naturally partitions itself into distinct groups. As with stratified sampling, the population is divided into groups and each group is separately sampled. Grouping may reflect geography or an element type. However, unlike stratified sampling, with cluster sampling the identity of the cluster is preserved, and statistics are computed individually for each cluster. In contrast to stratified sampling, where the grouping attempts to increase precision, with cluster sampling, the goal is to reduce the cost of creating the sample. Cluster sampling may be done hierarchically, with each level of the hierarchy or **stage** further refining the grouping.
- **Purposive**: Here, the idea is to sample only elements that meet a specific definition of the population. For example, suppose we wanted to study all IP packets that are 40 bytes long (corresponding to a zero data payload). Then, we could set up a packet filter that captured only these packets, constituting a purposive sample.

- **Convenience**: A convenience sample involves studying the elements of the population that happen to be conveniently available. For example, you may examine call traces from a cooperative cell phone operator to estimate mean call durations. Although it may not be possible to claim that call durations on that provider are representative of all cellular calls (because the duration is influenced by pricing policies of each operator), this may be all that is available and, on balance, is probably better than not having any data at all.

2.1.2 Scales

Gathering a sample requires measuring some physical quantity along a scale. Not all quantities correspond to values along a real line. We distinguish between four types of scales:

1. **Nominal**: A nominal (or **categorical**) scale corresponds to categories. Quantities arranged in a nominal scale cannot be mutually compared. For example, the transport-protocol type of a packet (i.e., UDP, TCP, other) constitutes a nominal scale.
2. **Ordinal**: An ordinal scale defines an ordering but distances along the ordinal scale are meaningless. A typical ordinal scale is the **Likert** scale, where 0 corresponds to ‘strongly disagree,’ 1 to ‘disagree,’ 2 to ‘neutral,’ 3 to ‘agree,’ and 4 to ‘strongly agree.’ A similar scale, with the scale ranging from ‘poor’ to ‘excellent’ is often used to compute the Mean Opinion Score (MOS) of a set of consumers of audio or video content to rank the quality of the content.
3. **Interval**: An interval scale defines an ordering where distances between index values are meaningful but there is no absolute zero value. That is, the values are invariant to an affine scaling (multiplication by a constant followed by addition of another constant). A good example is vonNeumann-Morgenstern utilities.
4. **Ratio**: A ratio scale is an interval scale that also has a well-defined zero element, so that all indices are unique. Quantities such as packet length and inter-arrival time are measured on ratio scales.

It is important to keep track of the type of scale corresponding to each measured quantity. A typical mistake is to assume that an ordinal scale can be treated as an interval or ratio scale.

2.1.3 Outliers

A common problem when collecting data is to find that some data elements are significantly out of line compared with the rest. These outliers can arise due to extreme variations in normal operating conditions or due to errors such as a failure in the measuring instrument or software test harness, or overflowing measurement counters

Although ignoring outliers is common practice, there are two reasons for treating them with care. First, the presence of an outlier is often indicative of poor data collection practices. Examining the root cause of an outlier often reveals problems with the underlying measurement setup. Fixing these problems usually not only eliminates outliers but also results in the collection of statistically valid data. Second, outliers indicate the presence of unusual or unsuspected complexity in the operation of a system. Explaining outliers can reveal to deeper appreciation of the underlying system. Therefore, when collecting samples, it is imperative to pay special attention to outliers, making certain that these are truly statistical aberrations before dismissing them or removing them from the data set.

2.2 Describing a sample parsimoniously

After gathering a sample, the next step is to describe it parsimoniously, that is, with a few well-chosen statistics. These statistics can be thought to constitute a **model** of the sample: each data item in the sample can be viewed as arising partly from this model and partly from an unmodelled error term, that is:

$$Data = Model + Error$$

A good model accounts for each element of the sample while minimizing the error. Naturally, the greater the number of parameters in a model, the better it fits the sample: the best model of a sample is the sample itself. However, a model with a hundred or a hundred million parameters provides no insight. Our goal is to describe as much of the data as possible with the *fewest* number of parameters. We consider some standard descriptors of sample data in this section.

2.2.1 Tables

The simplest technique to represent data is by tabulation. Let the *i*th sample value be denoted x_i and let $n(x)$ denote the number of occurrences of the value x in a sample. Then, a table is defined as the set of tuples $(x, n(x))$.

2.2.2 Bar graphs, histograms, and cumulative histograms

Bar graphs and histograms graphically represent the number of occurrences of sample values (i.e., $n(x)$) as a function of x. When x is measured on a nominal or ordinal scale, histograms and bar graphs both consist of a set of bars or rectangles of height proportional to $n(x)$ for each value of x. When x is measured on an interval or a ratio scale, the scale is first divided into contiguous ranges called **bins**. Bins may differ in width, which is also called the **bin size**. If all bins are the same size then histograms and bar graphs are identical. However, if bin sizes differ, in a histogram the height of the bar is inversely proportional to the bin size;in a bar graph, however, the height is unchanged. In a bar graph only the height of the bar (rectangle) is significant, whereas for a histogram the area of the rectangle is significant. A histogram is, therefore, a quantized (or approximate) probability density function[1] of the underlying population, becoming identical to it in the limit as the number of bins goes to infinity.

EXAMPLE 2: BAR GRAPHS AND HISTOGRAMS

Consider the following set of observations in a sample:

Data value	Frequency
1	5
2	7
3	2
7	2
10	1

TABLE 1. Sample data for Example 2

Note that the number of samples is quite sparse after the value 3. We have many choices of representation. For example, we can treat the data value as being measured on an ordinal scale and show the frequency of each data value found in the sample. This results in the bar graph/histogram show in Figure 1.

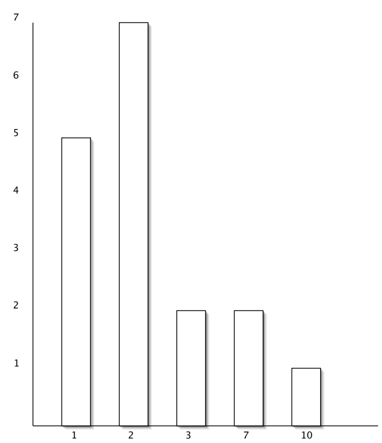


FIGURE 1. Bar graph of the sample with data values on an ordinal scale.

1. Probability density functions are described in more detail in Section 1.3.1 on page 11.

We could also treat the data values as being on an interval scale, with bins [0.5,1.5), [1.5, 2.5), [2.5,4.5), [4.5, 10.5] where the bin limits are chosen so that there can be no ambiguity as to which bin a value falls into. Note that the bins are not equally sized. This results in the bar graph and histogram shown in Figure 2.

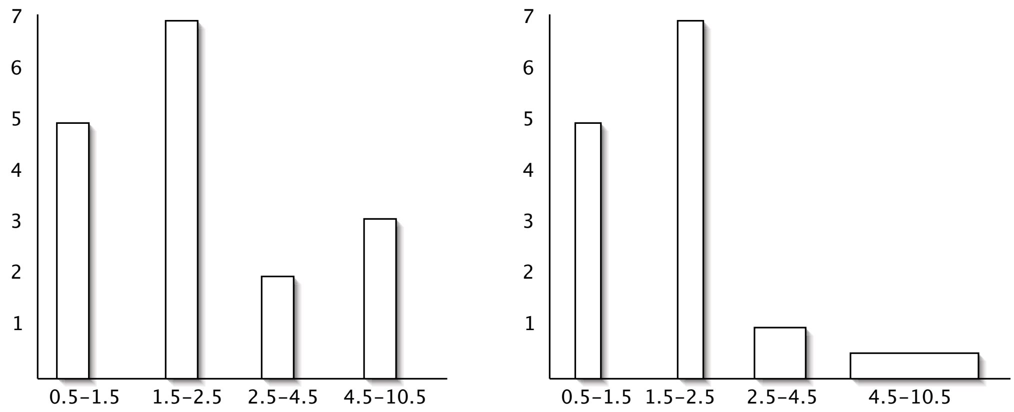


FIGURE 2. Bar graph (left) and histogram (right) for the sample with data values on an interval scale

■

Choosing a bin size for variables measured on a interval or ratio scales requires choosing appropriate bin sizes. Unfortunately, this can usually only be accomplished by trial and error. If the bin sizes are too small, then many bins are likely to be empty and the histogram is visually too wide. In contrast, if the bin sizes are too large, then all the data values may cluster into a handful of bins, hiding finer-scale structure. Several heuristics for bin sizing are known for the case of equally sized bins. For example, **Scott's choice** of bin width is $width = \frac{3.5s}{\sqrt[3]{n}}$ where s is the standard deviation of the sample (i.e., the square root if its variance - see Section 2.2.5 on page 46).

The cumulative histogram of a sample is a histogram where the value represented by the mth bin is the count of sample data values up to and including those in the mth bin, i.e., $\sum_{i=1}^{m} n(x_i)$. This can be viewed as the quantized version (or approximation) of the cumulative density function[2] of the underlying population.

2.2.3 The sample mean

The sample mean $\bar{x}$ of a sample with n elements is defined as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_i x_i \qquad \textbf{(EQ 1)}$$

Alternatively, given a sample in tabular form, we can sum over the different possible values of x:

$$\bar{x} = \frac{1}{n}\sum_x n(x)x \qquad \textbf{(EQ 2)}$$

Adopting a frequentist approach to probability, where $P(x)$, the probability of a value x, is defined as the limiting value of $n(x)/n$, we see that:

2. Cumulative density functions are described in Section 1.3.2 on page 12).

$$\lim_{n \to \infty} \bar{x} = \lim_{n \to \infty} \frac{1}{n} \sum_x n(x)x = \sum_x xP(x) = \mu \qquad \textbf{(EQ 3)}$$

This shows that as the sample size becomes very large, its mean is the expected value of a data item (which is the Strong Law of Large Numbers described in Section 1.7.4 on page 29).

The sample mean $\bar{x}$ is a random variable whose value depends on the values taken on by the elements in the sample. It turns out that the expected value of this random variable, that is, E($\bar{x}$) is also μ (this is not the same as the limiting value of the sample mean). To see this, we start with the definition of the sample mean: $\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$, so that $n\bar{x} = (x_1 + x_2 + \ldots + x_n)$. Taking expectations of both sides,

$$E(n\bar{x}) = E(x_1 + x_2 + \ldots + x_n) \qquad \textbf{(EQ 4)}$$

From the sum rule of expectations, we can rewrite this as:

$$E(n\bar{x}) = E(x_1) + E(x_2) + \ldots + E(x_n) = n\mu \qquad \textbf{(EQ 5)}$$

Therefore,

$$E(\bar{x}) = E\left(\frac{n\bar{x}}{n}\right) = \frac{E(n\bar{x})}{n} = \mu \qquad \textbf{(EQ 6)}$$

as stated.

The mean of a sample is a good representative of the sample for two reasons. First, for a finite sample size, the sample mean is an **estimator** of the population mean. An estimator is **unbiased** if its expected value is the corresponding population parameter. From Equation 3, the sample mean is an unbiased estimator of the population mean. If a population is normally distributed, then the sample mean is also the most **efficient unbiased estimator** of the population mean, in that it has the least variance of all unbiased estimators of the population mean. Second, it can be easily shown that the mean value of a sample is the value of x^* that minimizes $\sum_{i=1}^{n} (x_i - x^*)^2$ (i.e., the sum of squared deviations from x^*, which can be interpreted as errors in choosing x^* as a representative of the sample). In this sense, the mean is the 'central' value of a sample.

The mean is therefore the most widely used first-order descriptor of a population. Nevertheless, when using the mean of a sample as its representative, the following issues must be kept in mind:

- The mean of a sample may significantly differ from the true population mean. Consider the means of m samples, where the ith sample mean is derived from a sample with n_m data items (Figure 3). These means are likely to differ in value and can be thought of as themselves being data items in a sample with m elements, drawn from a population of sample means. The distribution of this population of sample means is called the **sampling distribution of the mean**. If this distribution has a large variance, then the mean of any particular sample may be far from representative of the population mean. Therefore, the mean of a sample, especially when the sample size is small, should be treated only as an approximation to the truth. We will examine the topic of statistical significance of the mean of a sample in Section 2.3 on page 48.

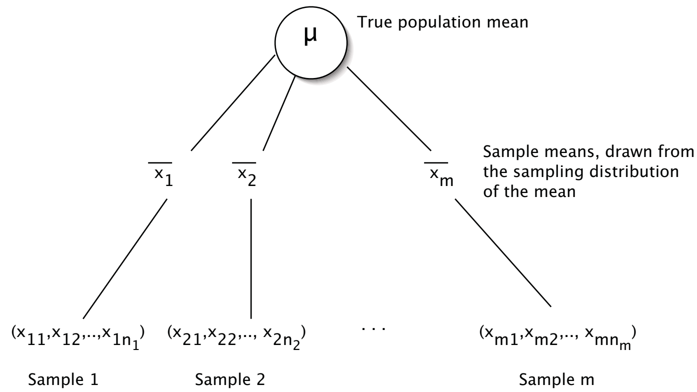


FIGURE 3. Sampling distribution of the mean

- The mean of a sample can be greatly influenced by outliers. Imagine a system where most packet interarrival times are small but there is one very large gap between packet bursts, corresponding to a very large interarrival time. Unless this outlier is excluded, this single value can bias the mean interarrival time.

EXAMPLE 3: EFFECT OF OUTLIERS ON THE MEAN

Consider the following sample:

Data value	Frequency
1	5
2	7
3	2
7	2
1000	1

TABLE 2. A sample with an outlier

The sample mean excluding the last value is (1*5 + 2*7 + 3*2 + 7*2)/(5+7+2+2) = 2.43. The sample mean including the last value is 61.1. It is hard to argue that the mean is representative of this sample, due to the influence of the outlier.

■

- The mean of a multi-modal distribution usually does not represent a sample very well. Consider a link where interarrival times are either small (< 1ms) or large (> 10s). This will result in a bi-modal distribution of sample values. The mean of any sample is therefore not a good representative of a given sample. In such cases, it is best to cluster the sample and compute the means for each cluster separately.

The variance of the sample mean (that is, the variance of the sampling distribution of the mean) can be computed as follows. Recall that

$$n\bar{x} = (x_1 + x_2 + \ldots + x_n)$$

Taking the variance of both sides,

$$V(n\bar{x}) = V(x_1 + x_2 + \ldots + x_n) \quad \text{(EQ 7)}$$

Now, the variance of a sum of a set of independent random variables is the sum of their variances. If we assume that each data value in a sample is independent (an assumption that may not always hold true), then

$$V(n\bar{x}) = V(x_1) + V(x_2) + \ldots + V(x_n) = n\sigma^2 \qquad \text{(EQ 8)}$$

Therefore,

$$V(\bar{x}) = V\left(\frac{n\bar{x}}{n}\right) = \frac{V(n\bar{x})}{n^2} = \frac{\sigma^2}{n} \qquad \text{(EQ 9)}$$

Therefore, the variance of the sample mean is $1/n$ of the variance of the population variance. As the size of a sample increases, the sample mean (whose expected value is μ) has a smaller and smaller variance, and the mean of each sample will be tightly clustered around μ.

2.2.4 The sample median

The median value of a sample is the value such that 50% of the samples are larger than this value. For a sample with an odd number of elements, it is the middle element after sorting. For a sample with an even number of elements, it is the mean of the two middle elements after sorting.

The median is a better representative of the mean for samples that contain outliers, in that it is relatively insensitive to outliers. It is also an unbiased estimator of the population mean. However, it can be shown that if the underlying distribution is normal, then the asymptotic variance of the median of a sample is 1.57 times larger than the asymptotic variance of the sample mean. Hence, if the underlying distribution is normal, the same accuracy in estimating the population mean can be obtained by collecting 100 observations and computing their mean or by collecting 157 samples and computing their median. If the underlying distribution is unimodal and sharper-peaked than normal (also called **leptokurtic**), then the median is a more efficient estimator than the mean, because, in such situations, the variance of the mean is higher than the variance of the median due to the presence of outliers.

2.2.5 Measures of variability

Unlike the mean or the median, which seek to represent the 'central tendency' of a sample, we now consider ways of representing the degree to which the data values in a sample differ from each other. These are also called 'measures of variability.'

The simplest measure of variability is the **range**, which is the difference between the largest and smallest value. The range is susceptible to outliers and therefore not reliable. A better measure is to sort the data values and then determine the data values that lie at *q*% and 1-*q*%. The difference between the two values is the range of values in which the central 1-2*q*% of the sample lies. This conveys nearly the same information as the range but with less sensitivity to outliers. A typical value of *q* is 25, in which case this measure is also called the **inter-quartile range**. In the context of delay bounds and service-level agreements, a typical value of *q* is 5 (so that the span is 5%-95%).

EXAMPLE 4: INTER-QUARTILE RANGE

Consider the sample in Table 2. There are 17 data values, so the 25th percentile index is the fourth one, and the 75th percentile index is the 13th one. The fourth value in sorted order is 1 and the 13th value is 3. Hence, the inter-quartile range is 2.

■

Although ranges convey some information, they do not tell us what fraction of the data values are clustered around the sample mean. This information can be represented by the sample variance m_2 which is defined as:

$$m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{x}^{n}(x - \bar{x})^2 n(x) \qquad \text{(EQ 10)}$$

Clearly, the variance increases if the sample values are distant from the mean (so that the mean is a poor representative of the sample) and is zero if all the data values are exactly clustered at the mean (in which case the mean perfectly represents the sample). The positive square root of the variance is called the **standard deviation**. Unlike the variance, the standard deviation has the same units as the data values in the sample.

A simple technique to compute the variance of a sample is to maintain a running total of three quantities n, $\sum_i x_i$, and $\sum_i x_i^2$. Then, the variance can be computed as:

$$m_2 = \frac{1}{n}\left(\sum_i x_i^2 - \frac{\left(\sum_i x_i\right)^2}{n}\right) \quad \textbf{(EQ 11)}$$

In the same way that the sample mean estimates the population mean, the sample variance is an estimator for the population variance, i.e. $E(X-\mu)^2$. However, the sample variance is *not* an unbiased estimator of the population variance—it is slightly smaller—with $E(m_2) = (n-1)\sigma^2/n$.

To prove this, recall that each element x_i in a sample can be thought of as being a random variable whose distribution is identical to that of the population, with an expected value of μ and a variance of σ^2. That is, $E(x_i) = \mu$ and $V(x_i) = E((x_i-\mu)^2) = \sigma^2$. Now, by definition of m_2:

$$m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right) \quad \textbf{(EQ 12)}$$

where the second step can be verified by expansion. Taking expectations on both sides, we find that:

$$E(m_2) = \frac{1}{n}E\left(\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right) = \frac{1}{n}\left(E\left(\sum_{i=1}^{n}(x_i-\mu)^2\right) - nE((\bar{x}-\mu)^2)\right) \quad \textbf{(EQ 13)}$$

Now, $E\left(\sum_{i=1}^{n}(x_i-\mu)^2\right) = \sum_{i=1}^{n}E(x_i-\mu)^2 = \sum_{i=1}^{n}\sigma^2 = n\sigma^2$. Also, because $E(\bar{x}) = \mu$, $E((\bar{x}-\mu)^2) = V(\bar{x}) = \frac{\sigma^2}{n}$. Substituting these into Equation 13, we find that

$$E(m_2) = \frac{1}{n}(n\sigma^2 - \sigma^2) = \frac{(n-1)}{n}\sigma^2 \quad \textbf{(EQ 14)}$$

as stated. Therefore, to obtain an unbiased estimate of the population variance σ^2, we should multiply m_2 by $n/(n-1)$, so that the unbiased estimator of the population variance is given by:

$$E[\sigma^2] = \frac{1}{n-1}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right) \quad \textbf{(EQ 15)}$$

2.3 Inferring population parameters from sample parameters

Thus far we have focused on statistics that describe a sample in various ways. A sample, however, is usually only a subset of the population. Given the statistics of a sample, what can we infer about the corresponding population parameters? If the sample is small or if the population is intrinsically highly variable, then there is not much we can say about the population. However, if the sample is large, there is reason to hope that the sample statistics are a good approximation to the population parameters. We now quantify this intuition.

Our point of departure is the central limit theorem, which states that the sum of the n independent random variables, for large n, is approximately normally distributed (see Section 1.7.5 on page 30). Suppose we collect a set of m samples, each with n elements, from some population. (In the rest of the discussion we will assume that n is large enough that the central limit theorem applies.) If the elements of each sample are independently and randomly selected from the population, we can treat the sum of the elements of each sample as the sum of n independent and identically distributed random variables $X_1, X_{2,...}, X_n$. That is, the first element of the sample is the value assumed by the random variable X_1, the second element is the value assumed by the random variable X_2, and so on. From the central limit theorem, the sum of these random variables is normally distributed. The mean of each sample is the sum divided by a constant, so the mean of each sample is also normally distributed. This fact allows us to determine a range of values where, with high confidence, the population mean can be expected to lie.

To make this more concrete, refer to Figure 3 and consider sample 1. The mean of this sample is $\overline{x_1} = \frac{1}{n}\sum_i x_{1i}$. Similarly $\overline{x_2} = \frac{1}{n}\sum_i x_{2i}$, and, in general, $\overline{x_k} = \frac{1}{n}\sum_i x_{ki}$. Define the random variable $\overline{X}$ as taking on the values $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}$. The distribution of $\overline{X}$ is called the **sampling distribution of the mean**. From the central limit theorem, $\overline{X}$ is approximately normally distributed. Moreover, if the elements are drawn from a population with mean μ and variance σ^2, we have already seen that $E(\overline{X}) = \mu$ (Equation 6) and $V(\overline{X}) = \sigma^2/n$ (Equation 9). These are, therefore, the parameters of the corresponding normal distribution, i.e., $\overline{X} \sim N(\mu, \sigma^2/n)$. Of course, we do not know the *true* values of μ and σ^2.

If we know σ^2, we can estimate a range of values in which μ will lie, with high probability, as follows. For any normally distributed random variable $Y \sim (\mu_Y, \sigma_Y^2)$, we know that 95% of the probability mass lies within 1.96 standard deviations of its mean, and 99% of the probability mass lies within 2.576 standard deviations of its mean. So, for any value y:

$$P(\mu_Y - 1.96\ \sigma_Y < y < \mu_Y + 1.96\ \sigma_Y) = 0.95 \quad \text{(EQ 16)}$$

The left and right endpoints of this range are called the **critical values** at the 95% confidence level: an observation will lie beyond the critical value, assuming that the true mean is μ_Y, in less than 5% (or 1%) of observed samples. This can be rewritten as:

$$P(|\mu_Y - y| < 1.96\ \sigma_Y) = 0.95 \quad \text{(EQ 17)}$$

Therefore, from symmetry of the absolute value:

$$P(y - 1.96\ \sigma_Y < \mu_Y < y + 1.96\ \sigma_Y) = 0.95 \quad \text{(EQ 18)}$$

In other words, given any value y drawn from a normal distribution whose mean is μ_Y, we can estimate a range of values where μ_Y must lie with high probability (i.e. 95% or 99%). This is called the **confidence interval** for μ_Y.

We just saw that $\overline{X} \sim N(\mu, \sigma^2/n)$. Therefore, given the sample mean $\overline{x}$:

$$P\left(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95 \quad \text{(EQ 19)}$$

and

$$P\left(\bar{x} - 2.576\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.576\frac{\sigma}{\sqrt{n}}\right) = 0.99 \quad \textbf{(EQ 20)}$$

Assuming we knew the sample mean and σ^2, this allows us to compute the range of values where the population mean will lie with 95% or 99% confidence.

Note that a confidence interval is constructed from the observations in such a way that there is a known probability, such as 95% or 99%, of it containing the population parameter of interest. It is not the population parameter that is the random variable–the interval itself is the random variable.

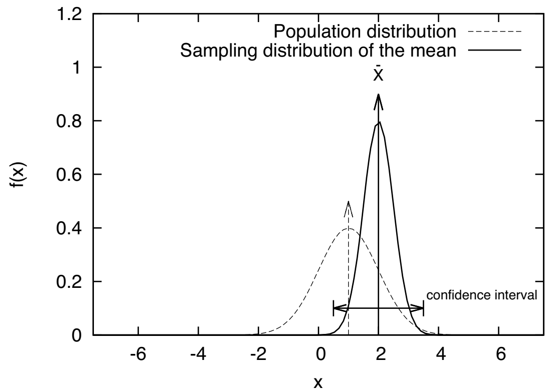


FIGURE 4. Population and sample mean distributions

The situation is graphically illustrated in Figure 4. Here, we assume that the population is normally distributed with mean 1. The variance of the sampling distribution (i.e. of $\overline{X}$) is σ^2/n, so it has a narrower spread than the population (with the spread decreasing as we increase the number of elements in the sample). A randomly chosen sample happens to have a mean of 2.0. This mean is the value assumed by a random variable $\overline{X}$ whose distribution is the sampling distribution of the mean and whose expected value from this single sample is centered at 2 (if we were to take more samples, the means of these samples would converge towards the true mean of the sampling distribution. The double headed arrow around $\bar{x}$ in the figure indicates a confidence interval in which the population mean μ must lie with high probability.

In almost all practical situations, we do not know σ^2. All is not lost, however. Recall that an unbiased estimator for σ^2 is $\frac{1}{n-1}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$ (Equation 15). Therefore, assuming that this estimator is of good quality (in practice, this is true when $n > \sim 20$), $\overline{X} \sim N\left(\mu, \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)\right)$. Therefore, when n is sufficiently large, we can still compute the confidence interval in which the population mean lies with high probability.

EXAMPLE 5: CONFIDENCE INTERVALS

Consider the data values in Table 1 on page 42. What is the 95% and 99% confidence interval in which the population mean lies?

Solution:

We will ignore the fact that n =17 < 20, so that the central limit theorem is not likely to apply. Pressing on, we find the sample mean is 2.88. We compute $\sum_{i=1}^{n}(x_i-\bar{x})^2$ as 107.76. Therefore, the variance of the sampling distribution of the mean is estimated as 107.76/(17*16) = 0.396 and the standard deviation of this distribution is estimated as its square root, i.e., 0.63. Using the value of +/- 1.96σ for the 95% confidence interval and +/- 2.576σ for the 99% confidence interval, the 95% confidence interval is [2.88-1.96*0.63, 2.88+1.96*0.63] = [1.65, 4.11] and the 99% confidence interval is [1.26, 4.5].

■

Because $\bar{x}$ is normally distributed with mean μ and variance σ^2/n, $\frac{(\bar{x}-\mu)}{\left(\frac{\sigma}{\sqrt{n}}\right)}$ is a $N(0,1)$ variable, also called the **standard Z variable**. In practice, when $n > 20$, we can substitute $m_2(n/(n-1))$ as an estimate for σ^2 when computing the standard Z variable.

So far, we have assumed that n is large, so that the central limit theorem applies. In particular, we have made the simplifying assumption that the estimated variance of the sampling distribution of the mean is identical to the actual variance of the sampling distribution. When n is small, this can lead to underestimating the variance of this distribution. To correct for this, we have to re-examine the distribution of the normally distributed standard random variable $\frac{(\bar{x}-\mu)}{\left(\frac{\sigma}{\sqrt{n}}\right)}$, which we actually estimate as the random variable $\frac{(\bar{x}-\mu)}{\sqrt{\frac{\sum_i (x_i-\bar{x})^2}{n(n-1)}}}$. The latter variable is *not* normally distributed. Instead, it is called the **standard t variable** that is distributed according to the ***t*** distribution with n-1 **degrees of freedom** (a parameter of the distribution). The salient feature of the t distribution is that, unlike the normal distribution, its shape varies with the degrees of freedom, with its shape for $n > 20$ becoming nearly identical to the normal distribution.

How does this affect the computation of confidence intervals? Given a sample, we proceed to compute the estimate of the mean as $\bar{x}$ as before. However, to compute the, say, 95% confidence interval, we need to change our procedure slightly. We have to find the range of values such that the probability mass under the t distribution (not the normal distribution) centered at that mean and with variance $\frac{\sum_i (x_i-\bar{x})^2}{n(n-1)}$ is 0.95. Given the degrees of freedom (which is simply n-1), we can look this up in a standard t table. Then, we can state with 95% confidence that the population mean lies in this range.

EXAMPLE 6: CONFIDENCE INTERVALS FOR SMALL SAMPLES

Continuing with the sample in Table 1 on page 42, we will now use the t distribution to compute confidence intervals. The unbiased estimate of the population standard deviation is 0.63. n=17, so this corresponds to a t distribution with 16 degrees of freedom. We find from the standard t table that a (0,1) t variable reaches the 0.025 probability level at 2.12, so that there is 0.05 probability mass beyond 2.12 times the standard deviation on both sides of the mean. Therefore, the 95% confidence interval is [2.88-(2.12*0.63), 2.88+(2.12*0.63)] = [1.54, 4.22]. Compare this to the 95% confidence interval of [1.65, 4.11] obtained using the normal distribution in Example 5. Similarly, the t distribution reaches the 0.005 probability level at 2.921, leading to the 99% confidence interval of [2.88-(2.921*0.63), 2.88+(2.921 *0.63)] = [1.03, 4.72] compared to [1.26, 4.5].

■

So far, we have focussed on estimating the population mean and variance and have computed the range of values in which the mean is expected to lie with high probability. These are obtained by studying the sampling distribution of the mean. We can obtain corresponding confidence intervals for the population variance by studying the **sampling distribution of the var-**

iance. It can be shown that if the population is normally distributed, this sampling distribution is the χ^2 distribution (discussed in Section 2.4.7 on page 58). However, this confidence interval is rarely derived in practice, and so we will omit the details of this result.

2.4 Testing hypotheses about outcomes of experiments

We often need to study the outcomes of experiments conducted to measure the performance of computer systems. We typically would either like to assert that a metric associated with the system has a certain value (such as having a mean value of 1.5 units), or that a new heuristic or algorithm improves the performance of the system. Here, we study techniques for making statistically valid assertions about the outcomes of experiments where we compare at most two values: we will study outcomes involving more than two experiments in Section 2.6 on page 67.

2.4.1 Hypothesis testing

Assertions about outcomes of an experiment can usually be re-formulated in terms of testing a **hypothesis**: a speculative claim about the outcome of an experiment. The goal of an experiment is to either show that the hypothesis is unlikely to be true (i.e., we can reject the hypothesis), or to show that the experiment is consistent with the hypothesis (i.e., the hypothesis need not be rejected).

This last statement bears some analysis. Suppose we are asked to check whether a coin is biased. We will start with the tentative hypothesis that the coin is unbiased, that is, P(heads) = P(tails) = 0.5. Then we toss the coin three times. Suppose we get three heads in a row. What does this say about our hypothesis? Conditional on the hypothesis being true, we have a probability of 0.5*0.5*0.5 = 12.5% that we obtain the observed outcome. This is not too unlikely, so perhaps the three heads in a row were simply due to chance. At this point, all we can state is that the experimental outcome is consistent with the hypothesis.

Now, suppose we flip the coin 10 times and see that it comes up heads nine times. If our hypothesis were true, then the probability of getting nine heads in 10 coin flips is given by the binomial distribution as $\binom{10}{1} 0.5^9 0.5^1 = 10*0.5^{10} = 10/1024 < 1\%$. Thus, if the hypothesis were true, this outcome is fairly unlikely (setting the bar for ‘unlikeliness’ at 1%). This is typically stated as: “we reject the hypothesis at the 1% confidence level.”

The probability of an outcome conditional on a hypothesis being true is called its **p-value**. If the outcome of an experiment has a *p*-value less than 1% (or 5%), then we would interpret the experiment as grounds for rejecting a hypothesis at the 1% (or 5%) level.

It is important to realize that the non-rejection of a hypothesis does not mean that the hypothesis is valid. For example, instead of starting with the hypothesis that the coin was unbiased, we could have made the hypothesis that the coin was biased, with P(heads) = 0.9. If we toss the coin three times and get three heads, the probability of that event, assuming the hypothesis were true, would be 0.9*0.9*0.9 = 0.73. So we cannot reject the hypothesis that the coin is biased. Indeed, with such a small number of experiments, *we cannot invalidate an infinite number of mutually incompatible hypotheses*!

We are therefore led to two inescapable conclusions. First, even the most careful experiment may lead to an incorrect conclusion due to random errors. Such errors may result in rejection of a hypothesis even though it ought not be rejected, or in non-rejection, when it should. Second, an experiment cannot result in the acceptance of a hypothesis, but only in its rejection or non-rejection (which is not the same as acceptance). We deal with each conclusion in turn.

2.4.2 Errors in hypothesis testing

Testing a hypothesis can never be entirely accurate. Random fluctuations in the outcomes of experiments may lead to non-rejection of a hypothesis when it should be rejected and rejecting it when it should not. We now discuss these errors in hypothesis testing.

Consider two universes in each of which a particular hypothesis is either valid or invalid. In each universe, we can expect one of two results from hypothesis testing: “The hypothesis is not rejected,” or “The hypothesis is rejected.” The four possible outcomes of testing are represented in the following table:

	Outcome of the experiment	
State of the universe	*Reject hypothesis*	*Do not reject hypothesis*
Hypothesis is invalid	Good outcome C_{00}	Bad outcome C_{01} False negative or Type II error
Hypothesis is valid	Bad outcome C_{10} False positive or Type I error	Good outcome C_{11}

- If the hypothesis is invalid and is rejected, then we have a good outcome. The probability of this event is denoted C_{00}.
- If the hypothesis is invalid but is not rejected, then we have a bad outcome. The probability of this event is denoted C_{01}.
- If the hypothesis is valid and is not rejected, then we have a good outcome. The probability of this event is denoted C_{11}.
- If the hypothesis is valid but is rejected, then we have a bad outcome. The probability of this event is denoted C_{10}.

We can use the C_{ij}s to define the following quantities:

Term	**Definition**	**Meaning**
Concordance	$C_{11}+C_{00}$	The probability of an accurate prediction
Error rate	$C_{10}+C_{01}$	The probability of an inaccurate prediction
Sensitivity	$C_{11}/(C_{11}+C_{01})$	Ability to predict correctly conditional on the hypothesis actually being valid
Specificity	$C_{00}/(C_{10}+C_{00})$	Ability to eliminate a false hypothesis conditional on the hypothesis actually being invalid.

These apply to all types of hypothesis testing. Our goal is to design experiments that maximize good outcomes while minimizing bad outcomes. In certain cases, we may trade off a higher sensitivity for a higher error rate, or a higher specificity for a lower concordance. A common rule is to limit Type I errors to 5%. That is, if the hypothesis is valid, we should not mistakenly reject it more than 5% of the time. Note that at the 5% confidence level, we might reject the hypothesis one time in twenty just by chance.

2.4.3 Formulating a hypothesis

We now return to the problem that an experiment can only result in rejection or non-rejection of a hypothesis. Therefore, we may end up not rejecting an invalid hypothesis. What guidelines should we use in choosing a hypothesis?

The standard technique is to formulate a **null hypothesis** that we believe is sufficient to explain the data unless statistical evidence strongly indicates otherwise. The null hypothesis should be formulated conservatively, that is, preserving the *status quo*, where this is applicable. A good way to think about this is in terms of a criminal trial. The judicial system starts with the presumption of innocence. It is up to the prosecution to prove that the defendant is guilty. If the prosecution cannot prove beyond reasonable doubt that the defendant is guilty, then the defendant is released. No doubt, this will let some guilty parties go unpunished. But it is preferable to the alternative, where the defendant is assumed guilty and must prove innocence.

In formulating a null hypothesis, it is necessary to be precise. In the words of Sir R.A. Fisher, the inventor of this approach, the null hypothesis should be “free from vagueness and ambiguity.” Otherwise, it may be impossible to reject it, making our effort fruitless. Moreover, a hypothesis should be about a population parameter, not a sample (unless the sample includes the entire population).

EXAMPLE 7: FORMULATING A NULL HYPOTHESIS

Consider a router that can execute either scheduling algorithm A or scheduling algorithm B. Suppose our goal is to show that scheduling algorithm A is superior to scheduling algorithm B for some metric. An acceptable conservative null hypothesis would be “Scheduling algorithm A and scheduling algorithm B have identical performance.” Given this assumption, we would expect that the performance metrics for both scheduling algorithms to be roughly the same (i.e., this is our expectation regarding the state of the world). If our experiments show this to be the case, for example, if the sample means of the performance metrics for both scheduling algorithms were nearly identical, then we would conclude that we do not have sufficient evidence to prove that scheduling algorithm B improved the system, a conservative and scientifically valid decision. In contrast, if the sample mean for algorithm A were much higher than the sample mean for algorithm B (we will quantify this shortly), then the experiment would be inconsistent with our null hypothesis, and we would reject it, giving credence to the belief that scheduling algorithm A was indeed better[3].

If we were to invert the null hypothesis, for example stating it as “Scheduling algorithm A is better than scheduling algorithm B.” then by being unable to reject the null hypothesis, we may come to an unwarranted conclusion. In any case, this hypothesis is imprecise, in that we did not quantify *how* much better algorithm A is supposed to be better than scheduling algorithm B, so ought to be deprecated on those grounds alone.

■

We represent the null hypothesis using the symbol H_0. Alternatives to the null hypothesis are usually labelled H_1, H_2, and so on. The steps in hypothesis testing depend on whether the outcome of an experiment is being compared with a fixed quantity, or whether it being compared with the outcome of another experiment. In the next sub-section, we will consider outcomes that are compared with a fixed quantity, deferring comparison of outcomes of two experiments to 2.4.5 on page 54.

Hypotheses can be ‘two-tailed’ or ‘one-tailed.’ We reject a two-tailed hypothesis if the sample statistic significantly differs from the conjectured corresponding population parameter in absolute value. For example, suppose we hypothesize that the population mean μ is 0. If observations indicate that $|\bar{x}| > a$, where a is the *critical value,* then we reject the hypothesis. In contrast, we reject a one-tailed hypothesis if the sample statistic significantly differs from the corresponding population parameter in a pre-specified direction (i.e., is smaller than or larger than the conjectured population parameter), where experimental conditions allow us to rule out deviations in the other direction. An example of a one-tailed hypothesis is $\mu < 0$. If $\bar{x} > a$, where a, again, is the critical value, we can reject this hypothesis. Note that we do not consider the ‘other tail,’ that is, the possibility that $\bar{x} < a$ (why?).

2.4.4 Comparing an outcome with a fixed quantity

To develop intuition, let us start with a simple example. Suppose that physical considerations lead us to expect that the population mean of a population under study is 0. Assume that there is no particular *status quo* that we are trying to maintain. Therefore, a reasonable null hypothesis is:

$$H_0\text{: the population mean is } 0$$

To test this hypothesis, we need to sample the population multiple times and compute the corresponding sample means. If he number of samples is large, the central limit theorem implies that each sample mean will be drawn from a normal distribution. We can use this fact to compute its confidence interval (say at the 95% level) using the techniques in Section 2.3 on page 48. We then check if 0 lies within this interval. One of two cases arise:

- 0 lies in the 95% (99%) confidence interval of the sample mean. In this case, we cannot reject the null hypothesis. This is usually incorrectly interpreted to mean that with 95% (99%) confidence the population mean is indeed 0. Of course, all have have shown is that, conditional on the population mean being 0, the outcome of this experiment has a likelihood greater than 95% (99%) (i.e., it is consistent with the null hypothesis).
- 0 does not lie in the 95% (99%) confidence interval of the sample mean. In this case, we reject the null hypothesis. This is usually interpreted to mean that, with high confidence, the population mean is not 0. Again, all we have shown is that, conditional on the mean being 0, the outcome we saw was rather unlikely, so we have good reason to be suspicious of the null hypothesis.

3. In the Bayesian formulation of hypothesis testing, we view the experiment as updating a prior expectation on the state of the world, thus refining our model for the state of the world. Here, we are presenting the classical statistical view on hypothesis testing.

This example is easily generalized. Suppose we want to establish that the population mean is μ_0. We compute the sample mean $\bar{x}$ as before. Then, we test the hypothesis:

$$H_0: (\bar{x} - \mu_0) = 0$$

which can be tested as described above, with identical conclusions being drawn about the results[4].

EXAMPLE 8: TESTING FOR A ZERO MEAN

Returning to Example 10, note that the 99% confidence interval for the mean of the sample data, using the *t* test, was [1.03, 4.72]. Therefore, we can state with 99% confidence (with the caveats stated above) that the mean of the underlying population is not 0.

■

2.4.5 Comparing outcomes from two experiments

Suppose we want to test the hypothesis that two samples are drawn from different populations. To fix ideas, consider the situation where we are comparing two systems—a system currently in use and a system that incorporates a new algorithm—on the basis of a particular performance metric. We assume that we can collect performance metrics from each system multiple times to obtain two samples. If the systems do not differ to a statistically significant degree, then both samples would be drawn from the same underlying population, with, for example, the same population mean, and therefore would have similar statistics, such as the sample mean. However, if the statistics are significantly different, then we infer that the two samples are likely to be drawn from different populations and the new algorithm does indeed affect the performance of the system.

The null hypothesis is the statement:

$$H_0\text{: the two systems are identical}$$

We reject H_0 if it is sufficiently unlikely that the two samples are drawn from the same population because this will result in the conservative position that the new algorithm does not improve the performance of the system.

Suppose we collect n sets of samples from the first system, labelled A, to get sample means $a_1, a_2,...,a_n$ and collect m sets of samples from the second system, labelled B, to get sample means $b_1, b_2,...,b_m$. Let the means of these means be denoted $\bar{a}$ and $\bar{b}$ with corresponding variances $m_2(a)$ and $m_2(b)$.

If $n=m$, then we define an auxiliary random variable C=A-B, which takes values $c_i = a_i - b_i$. Then, we redefine the hypothesis as:

$$H_0\text{: the population mean of } C \text{ is zero}$$

This can be easily tested using the approach described in 2.4.4 on page 53.

EXAMPLE 9: COMPARING TWO SAMPLES

4. Advanced readers may note that in this section, we have switched from the Fisher to the Neyman-Pearson approach to hypothesis testing. This hybrid approach is widely used in modern scientific practice.

Suppose that you are using simulations to study the effect of buffer size at some network queue on packet loss rate. You would like to see if increasing the buffer size from 5 packets to 100 packets has a significant effect on loss rate. To do so, suppose that you run 10 simulations for each buffer size, resulting in loss rates shown below:

Loss rate with 5 buffers	1.20%	2.30%	1.90%	2.40%	3.00%	1.80%	2.10%	3.20%	4.50%	2.20%
Loss rate with 100 buffers	0.10%	0.60%	1.10%	0.80%	1.20%	0.30%	0.70%	1.90%	0.20%	1.20%

Does the buffer size have a significant influence on the packet loss rate?

Solution:

Note that each loss rate measurement in each simulation is itself a sample mean. Therefore, these loss rates can be assumed to be distributed approximately normally. Denoting by a_i the loss rate with a buffer of size 5 packets and by b_i the loss rate with a buffer size of 100 packets, we define the auxiliary variable $c_i = a_i - b_i$ that takes values (1.2-0.1), (2.3-0.6),..., (2.2-1.2), so that c is given by:

c = {1.1, 1.7, 0.8, 1.6, 1.8, 1.5, 1.4, 1.3, 4.3, 1.0}

We compute the sample mean as 1.65 and sample variance m_2 as 0.87, so that the unbiased estimator for the population variance is given by $(n/n\text{-}1)m_2$, = 0.97. The variance of the sample mean is given by m_2/n = 0.097, corresponding to a standard deviation of 0.31. Because the number of values is smaller than 20, we use the t distribution with 9 degrees of freedom to compute the confidence interval at the 95% level as 1.65±0.70. This interval does not include 0. Thus we conclude that the change in the buffer size does significantly affect the loss rate.

■

If $n \neq m$, the situation is somewhat more complex. We first use $m_2(a)$ to compute the confidence interval for A's performance metric around its sample mean $\bar{a}$ and similarly use $m_2(b)$ to compute the confidence interval for B's performance metric around its sample mean $\bar{b}$ (using the normal or t distribution, as appropriate). Now, one of the following two cases hold:

- The confidence intervals do not overlap. Recall that with 95% (or 99%) confidence, A's and B's population means lie within the computed confidence intervals. If the null hypothesis were true and the population means coincided, then it must be the case that either A's population mean or B's population mean lies outside of its computed confidence interval. However, this has a probability lower than 5% (1%). Therefore, we reject the hypothesis.
- The confidence intervals overlap. In this case, there is some chance that the samples are drawn from the same population. The next steps depend on whether we can make one of two assumptions: (1) the sample variances are the same or (2) n and m are both large.

(1) *Sample variances are the same*. In this case, we define the auxiliary variable s by:

$$s^2 = \frac{\sum_{i=1}^{n}(a_i - \bar{a})^2 + \sum_{i=1}^{m}(b_i - \bar{b})^2}{m + n - 2} \qquad \text{(EQ 21)}$$

Then, it can be shown that if the two samples are drawn from the same population, the variable c defined by:

$$c = \frac{\bar{a} - \bar{b}}{s\sqrt{\frac{1}{m} + \frac{1}{n}}} \qquad \text{(EQ 22)}$$

is a standard t variable (i.e., with zero mean and unit variance) with m+n-2 degrees of freedom. Therefore, we can use a t test to determine whether c has a zero mean, using the approach in Section 2.4.4 on page 53.

EXAMPLE 10: COMPARING TWO SAMPLES WHEN SAMPLE SIZES DIFFER

Continuing with Example 9, assume that we have additional data points for the simulation runs with 5 buffers, as shown below. Can we still claim that the buffer size plays a role in determining the loss rate?

Loss rate with 5 buffers	0.20%	0.30%	0.90%	1.40%	1.00%	0.80%	1.10%	0.20%	1.50%	0.20%	0.50%	1.20%	0.70%	1.30%	0.90%
Loss rate with 100 buffers	0.10%	0.60%	1.10%	0.80%	1.20%	0.30%	0.70%	1.90%	0.20%	1.20%					

Solution:

Here, m = 15 and n = 10, so we cannot use the approach of Example 9. Instead, we will first compute the mean and confidence intervals of both samples to see if the intervals overlap. It is easily found that, at the 95% level, using a t distribution with 14 and 9 degrees of freedom respectively, the confidence intervals are 0.81±0.25 and 0.81±0.40 which overlap. However, the sample variances are not the same, and there is a good chance that the population variances also differ. Nevertheless, for the purpose of this example, we will make the assumption that the population variances are the same. Therefore, we compute s^2 using Equation 21 as 2.89, so that s = 1.7. We then use Equation 22 to compute the standard t variate c as 0.0033/ $(1.7(1/15 + 1/10)^{1/2})$ = 0.0048. Since this has unit variance, it is easy to see using the t test with 23 degrees of freedom that 0 lies in the confidence interval for c, which implies that, with this data set, buffer size has no statistically significant effect on packet loss rate.

■

(2) *Sample variances differ, but* m *and* n *are both large.* In this case, it can be shown that the variable c defined by:

$$c = \frac{\bar{a} - \bar{b}}{\sqrt{\left(\frac{\sum_{i=1}^{n}(a_i - \bar{a})^2}{m(m-1)}\right) + \left(\frac{\sum_{i=1}^{m}(b_i - \bar{b})^2}{n(n-1)}\right)}} \qquad \text{(EQ 23)}$$

is a standard normal variable (i.e., with a zero mean and unit variance). Therefore, we can use a standard normal test to determine whether c has a zero mean, using the approach discussed in Section 2.4.4 on page 53.

If neither assumption can be made, then it is difficult to draw meaningful comparisons, other than by using **non-parametric tests**, such as the Mann-Whitney U test, which is beyond the scope of this text.

2.4.6 Testing hypotheses regarding quantities measured on ordinal scales

So far, we have tested hypotheses where a variable takes on real values. We now consider the case where a variable takes on nominal (categorical) values (such as ‘UDP’ or ‘TCP’) or ordinal values (such as ‘bad’, ‘satisfactory’, and ‘good’) (these terms are defined in Section 2.1.2 on page 41). In such cases, hypothesis testing using the techniques described above is meaningless because a sample cannot be described by a mean, nor can we define real confidence intervals about the mean. Instead, for such variables, hypotheses are of the form:

H_0*: the observed values are drawn from the expected distribution*

Then, we use a statistical test, such as the *Pearson chi-squared test* (described below), to reject or not reject the hypothesis.

EXAMPLE 11: HYPOTHESIS FORMULATION WITH NOMINAL SCALES

Suppose that you want to check if the distribution of packet types on a link from your campus to the Internet is similar to that reported in the literature. For instance, suppose that 42% of the bytes originating at the University of Waterloo (UW) during the measurement period can be attributed to P2P applications. Suppose you measure 100 GB of traffic and find that 38 GB can be attributed to P2P applications. Then, a reasonable null hypothesis would be:

H_0: the observed traffic on the campus Internet access link is similar to that at UW

■

How should we test hypotheses of this form? A clue comes from the following thought experiment. Suppose we have a possibly biased coin and that we want to determine whether it is biased or not. The null hypothesis is:

$$H_0: P(heads) = P(tails) = 0.5$$

We assume that we can toss the coin as many times as we want and that the outcome of each toss is independent. Let T denote the outcome 'Tails' and H denote the outcome 'Heads.' We will represent a set of outcomes such as 'Nine heads and one tails' by the notation TH^9. As we saw earlier, if a coin is unbiased, this outcome has the probability $\binom{10}{1}0.5^9 0.5^1$. Any outcome from n coin tosses—such as a Heads, represented by H^aT^{n-a}–can be viewed as one sample drawn at random from the set of all possible outcomes when tossing a coin n times. A little thought indicates that the probability of this outcome, given that the probability of heads is p and of tails is $q=1-p$, is given by the binomial distribution $\binom{n}{a}p^a q^{n-a}$, which is also the ath term of the expansion of the expression $(p + q)^n$. As $n \to \infty$, the binomial distribution tends to the normal distribution, so that the probability of each outcome is approximated by the normal distribution.

Now, consider an experiment where each individual outcome is independent of the others, and where an outcome results in one of k ordinal values, $o_1, o_2, \ldots, o_k$. Let the expected probability of the ith outcome be p_i, so that the expected count for the ith outcome, $e_i=np_i$. Suppose we run the experiment n times, and the ith outcome occurs n_i times with $\sum_i n_i = n$. We can represent any particular outcome by $o_1^{n_1} o_2^{n_2} \ldots o_k^{n_k}$ and this outcome can be viewed as one sample drawn at random from the set of all possible outcomes. The probability of such an outcome is given by the **multinomial** distribution as:

$$P\left(o_1^{a_1} o_2^{a_2} \ldots o_k^{a_k}\right) = \binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n-\sum_i^{k-1} n_i}{n_k} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \quad \text{(EQ 24)}$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \quad \text{(EQ 25)}$$

This outcome is one of the terms from the expansion of $(p_1+p_2+\ldots+p_k)^n$. As with the binomial distribution, we can use the multinomial distribution to test if any particular outcome, conditional on a null hypothesis on the p_is being true, is 'too unlikely,' indicating that the null hypothesis should be rejected.

In many cases, using the multinomial distribution for testing the validity of a hypothesis can be cumbersome. Instead, we use a standard mathematical result that the variable $X_i = \frac{n_i - e_i}{\sqrt{e_i}}$, for values of $e_i > 5$, closely approximates a standard normal variable with zero mean and unit variance. But we immediately run into a snag: the n_i are not independent. For example, if $n_3 = n$, then all the other n_i must be zero. Therefore, the X_i are also not independent. However, it can be proved that this set of k *dependent* variables X_i can be mapped to a set of k-1 *independent* standard normal variables while keeping the sums of squares of the variables constant. By definition, the sum of squares of k-1 independent standard normal variables follows the χ^2 (also written chi-squared and pronounced kai-squared) distribution with k-1 degrees of freedom. Therefore, if the null hypothesis is true (that is, the observed quantities are drawn from the distribution specified implicitly by the expected values) the variable

$$X = \sum_{i=1}^{k} \frac{(n_i - e_i)^2}{e_i} \quad \textbf{(EQ 26)}$$

is a χ^2 variable with k-1 degrees of freedom. Standard statistical tables tabulate P($X > a$) where X is a χ^2 variable with k degrees of freedom. We can use this table to compute the degree to which a set of observations corresponds to a set of expected values for these observations. This test is the *Pearson* χ^2 *test*.

EXAMPLE 12: (CHI-SQUARED TEST)

We use the Pearson χ^2 test to test if the observation in Example 11 results in rejection of the null hypothesis. Denote P2P traffic by ordinal 1 and non-P2P traffic by ordinal 2. Then, e_1 = 42, e_2 = 58, n_1 = 38, n_2 = 62. Therefore, $X = (38\text{-}42)^2/42 + (62\text{-}58)^2/58 = 0.65$. From the χ^2 table with 1 degree of freedom, we see that $P(X > 3.84)= 0.05$, so that any value greater than 3.84 occurs with probability less than 95% and is ‘unlikely.’ Since 0.65 < 3.84, the observation is not unlikely, which means that we cannot reject the null hypothesis.

In contrast, suppose the observation was n_1 = 72, n_2 = 28. Then, $X = (72\text{-}42)^2/42 + (28\text{-}58)^2/58 = 36.9$. Since 36.9 > 3.84, such an observation would suggest that we should reject the null hypothesis at the 5% level.

■

2.4.7 Fitting a distribution

When testing a hypothesis using a chi-square test we need to compute the expected distribution of sample values. These expected values may come from prior studies, as in the example above, or from physical considerations. In many cases, however, the expected values can be derived by assuming that the observations arise from a standard distribution (such as the Poisson, exponential, or normal distributions) and then choosing the parameters of the distribution to best match the observed values. This is called ‘fitting’ a distribution to the observations. A general technique for fitting a distribution is called the **method of maximum likelihood** and we discuss it next.

Suppose that random variables $X_1, X_2,...,X_n$ have a known joint density function $f_\theta(x_1,x_2,...,x_n)$ where θ denotes the unknown parameters of the distribution, such as its mean and variance. Given the observation $X_i=x_i$, where i=1,2,...,n, we would like to compute the **maximum likelihood estimate** (*mle*) of θ, that is, the value of θ that makes the observed data the ‘most likely.’ Intuitively, conditional on the observations being what they are, we would like to work backwards to find the value of θ that made these observations likely: we then assume that we observed what we did because the parameters were what they were.

Assuming that the X_is are independent and identically distributed according to $f_\theta(.)$, the joint probability that the observation is $(x_1,x_2,...,x_n)$ is simply the product of the individual probabilities $\prod_{i=1}^{n} f_\theta(X_i)$. Note that the distribution function is parametrized by θ. We make this explicit by defining *likelihood*(θ) as

$$likelihood(\theta|x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} f_\theta(X_i) \quad \textbf{(EQ 27)}$$

We find the mle by maximizing *likelihood*(θ) with respect to θ. In practice, it is more convenient to maximize the natural logarithm of *likelihood*(.) denoted l(.), defined by

$$l(\theta|x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} \log(f_\theta(X_i)) \quad \text{(EQ 28)}$$

For example, suppose that we want to fit a Poisson distribution with parameter λ to an observation (x_1,x_2,...,x_n). Recall that for a Poisson distribution, $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$. If the X are independent and identically distributed (i.i.d.) Poisson variables, their joint probability is the product of their individual distributions, so that

$$l(\lambda) = \sum_{i=1}^{n} (X_i \log\lambda - \lambda - \log X_i!)$$
$$l(\lambda) = \log\lambda \sum_{i=1}^{n} X_i - n\lambda - \sum_{i=1}^{n} \log X_i! \quad \text{(EQ 29)}$$

We maximize $l(.)$ by differentiating it with respect to λ and setting the derivative to 0:

$$\frac{dl}{d\lambda} = \frac{1}{\lambda}\sum_{i=1}^{n} X_i - n = 0 \quad \text{(EQ 30)}$$

which yields the satisfying result:

$$\lambda = \bar{X} \quad \text{(EQ 31)}$$

Thus, we have found that the mean of a set of observations is the value that maximizes the probability that we obtain that particular set of observations, conditional on the observations being independent and identically distributed Poisson variables.

Proceeding along similar lines, it is possible to show that the maximum likelihood estimators for a set of i.i.d normal variables is

$$\mu = \bar{X}$$
$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2} \quad \text{(EQ 32)}$$

Note the mle for the standard deviation is not a consistent estimator - to get a consistent estimator, we need to divide by n-1, rather than n, as discussed in Section 2.2.5 on page 46. Maximum likelihood estimators for other distributions can be found in standard texts on mathematical statistics.

It is possible to obtain confidence intervals for maximum likelihood estimators by considering the sampling distribution of the estimated parameters. This is discussed in greater depth in more advanced texts.

Note that if we use the sample itself to estimate p parameter values of the population, then we reduce the number of degrees of freedom in the sample by p. Recall that a sample that has n counts (ordinal types), has $n-1$ degrees of freedom. If, in addition, p parameters are estimated to compute the expected counts, then the degrees of freedom when conducting a chi-squared test is $n-1-p$.

EXAMPLE 13: FITTING A POISSON DISTRIBUTION

In an experiment, a researcher counted the number of packets arriving to a switch in each 1ms time period. The table below shows the count of the number of time periods with a certain number of packet arrivals. For instance, there were 146 time periods that had 6 arrivals. The researcher expects the packet arrival process to be a Poisson process. Find the best Poisson fit for the sample. Use this to compute the expected count for each number of arrivals. What is the chi-squared variable value for this data set? Determine whether the Poisson distribution adequately describes the data.

Number of packet arrivals	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Count	18	28	56	105	126	146	164	165	120	103	73	54	23	16	9	5

Solution:

The total number of time periods is 18+28+...+5 = 1211. The total number of arrivals is (18*1)+(28*2)+...+(5*16) = 8935. Therefore, the mean number of packets arriving in 1ms is 8935/1211 = 7.38. This is the best estimate for the mean of a fitted Poisson distribution. We use this to generate the probability of a certain number of arrivals in each 1ms time period. This probability multiplied by the total number of time periods is the expected count for that number of arrivals, and this is shown below. For instance, we compute P(1) = 0.0046 and 0.0046*1211 = 6.

Number of packet arrivals	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Count	18	28	56	105	126	146	164	165	120	103	73	54	23	16	9	5
Expected count	6	21	51	93	138	170	179	165	135	100	67	41	23	12	6	3

Although, at first glance, the fit appears to be good, it is best to compute the chi-squared value. This is computed as $(18-6)^2/6) + (28-21)^2/21 + ... + (5-3)^2/3 = 48.5$. Since we estimated one parameter from the sample, the degrees of freedom = 16-1-1 = 14. From the chi-squared table, with 14 degrees of freedom, at the 95% confidence level, the critical value is 23.68. Therefore, we reject the hypothesis that the sample is well-described by a Poisson distribution at this confidence level. That is, we have 95% confidence that this sample was not drawn from a Poisson population. The critical value at the 99.9% level for 14 degrees of freedom is 36.12. So, we can be even more confident, and state that with 99.9% confidence, the sample is not drawn from a Poisson population.

At first glance, this is a surprising result, because the fit appears quite good. The reason why the test fails is clear when we examine the $(O-E)^2/E$ values. The largest value is 27.6, which is for 1 packet arrival, where we expected a count of 6 but got 18. Because the denominator here is small (6), the contribution of this sample value to the chi-squared variable is disproportionate. If we were to ignore this value as an outlier and computed the fit only for 2-16 packet arrivals, then the revised estimate of the distribution mean is 7.47, and the revised chi-squared variable is 19.98 (see Exercise 12). This does meet the goodness-of-fit criterion with 13 degrees of freedom even at the 95% confidence level. In cases like these, it is worthwhile looking into why there was a deviation from the Poisson process: it could be a systematic error in the experiment, or perhaps due to a heretofore unknown phenomenon.

■

2.4.8 Power

Recall that when we test a hypothesis, we determine the probability of obtaining an observed outcome conditional on the null hypothesis being true. If the outcome is less probable than the significance level such as 95% or 99%, then we reject the null hypothesis. Of course, the hypothesis could still be true. Nevertheless, we reduce the Type I error, that of rejecting a hypothesis when it is in fact true, to a value below the significance level.

We now discuss a related concept: the power of a test. The **power** of a statistical test is the probability that the test will reject a null hypothesis when it is in fact false. If the power is low, then we may not reject a null hypothesis even when it is false, a Type II error. Thus, the greater the power, the lower the chance of making a Type II error. Usually, the only way to increase the power of a test is to increase its significance level (which makes a Type I error more likely).

The practical difficulty in computing the power of a test is that we don't know the ground truth. So, it becomes impossible to compute the probability that we will reject the null hypothesis conditional on the ground truth being different from the null hypothesis. For instance, suppose that the ground truth differs infinitesimally from the null hypothesis. Then, the probability that we reject the null hypothesis (which is false) is essentially the same as the significance level (why?). In contrast, suppose that the ground truth is far from the null hypothesis. Then, the sample mean is likely to be near the ground truth and we are likely to reject the null hypothesis, increasing the power of the test. But we have no way of knowing which of these situations hold. Therefore, we can only precisely compute the power of a test in the context of an alternative hypothesis about the state of the world. Unfortunately, in many cases, this is impossible to determine. Therefore, despite its intuitive merit, the power of a test is rarely computed.

2.5 Independence and dependence: regression, and correlation

Thus far, we have primarily studied single variables in isolation. In this section, we study data sets with two variables. In this situation, some questions immediately crop up: Are the variables independent of each other? If not, are pairs of variables correlated with each other? Do some of the variables depend linearly or non-linearly on the others? Can the variability in one of the variables be explained as being due to variability in another variable? These are the types of questions that we will study in this section.

2.5.1 Independence

Consider a data set where each element can be simultaneously placed into more than one category. For example, we could characterize an IP packet both by its size and its type. Are these variables independent of each other? Given the size, can we say anything about the type and vice versa? If knowing the value of one variable does not give us any additional information about the other, then the variables are **independent**. We now describe a test to determine whether we can confidently reject the hypothesis that two variables are independent.

In testing for independence, it is useful to represent the data set in the form of a **contingency table**. For a sample that has two variables that take one of m and n ordinal values respectively, the contingency table has $m \times n$ cells, with each cell containing the count of the number of sample elements that simultaneously fall into both corresponding categories.

Given the contingency table, we can use the Pearson chi-squared test to test whether two variables are independent as follows. We use the sample to estimate the population parameters, and, from these estimates, assuming independence of the variables, we compute the expected numbers of sample values that will fall into each cell of the contingency table. We can then compare the actual counts in each cell with these expected values to compute the chi-squared statistic. If this statistic is larger than the critical value, then we can, with high confidence, reject the hypothesis that the variables are independent.

In computing the chi-squared statistic, it is important to correctly compute the degrees of freedom. Recall that for a variable that falls into one of k classes, the number of degrees of freedom is k-1. It can be shown that the number of degrees of freedom is further reduced by each parameter that is estimated from the sample (instead of being known *a priori*) as the next example shows.

EXAMPLE 14: TESTING FOR INDEPENDENCE

Suppose that in a packet trace with a million packets, you obtain the following contingency table:

Packet size	TCP	UDP	Other	Row sum
40	12412	15465	300	**28177**
100	85646	12561	15613	**113820**
150	9846	68463	4561	**82870**
512	4865	45646	23168	**73679**
1024	48651	95965	48913	**193529**
1200	98419	59678	48964	**207061**
1450	156461	48916	51952	**257329**

1500	16516	24943	2076	**43535**
Column sum	**432816**	**371637**	**195547**	**1000000**

Is the packet size independent of the packet type?

Solution:

We do not know the actual frequency of each packet type in the population (of all IP packets), so we will estimate the population frequencies from this sample using the column sums as follows:

P(TCP) = 432816/1000000 = 0.433

P(UDP) = 371637/1000000 = 0.372

P(Other) = 195547/1000000 = 0.195

Similarly, we compute the probability of each packet size from the row sums as follows:

P(40) = 28177/100000 = 0.028

P(100) = 113820/1000000 = 0.114

...

P(1500) = 43535/1000000 = 0.043

If these probabilities were independent, then each cell could be computed as follows:

Count of TCP AND 40 = P(TCP) * P(40) * 1000000 = 0.433*0.028*1000000 = 12195.

...

Count of Other and 1500 = P(Other) * P(1500) * 1000000 =0.195* 0.043*1000000 = 8513.

We therefore have both the observed and expected values for each cell. We compute the chi-squared statistic as the sum of squares of the variable (observed value - expected value)^2/(expected value). This value turns out to be 254,326. Here *k* is 3*8 = 24. Moreover, we have estimated nine parameters from the data (we get two probabilities ‘for free’ since probabilities sum to 1). Therefore, the degrees of freedom still left is 24-1-9 = 14. Looking up the chi-square table with 14 degrees of freedom, we find that the critical value for the 0.001 confidence level to be 36.12. Since the statistic far exceeds this value, we can be more than 99.9% confident that packet type and packet size are *not* independent in the population from which this trace was drawn.

■

Note that, given the large sample size, even a tiny deviation from the expected values will lead to the null hypothesis of independence being rejected. We discuss this further in Section 2.9.5 on page 73.

2.5.2 Regression

When two random variables are not independent, it is sometimes the case that one variable depends on—or can be thought to depend on—the other, in that the value of the second variable is approximately known if the value of the first is known. Let the independent variable X take on specific values such as x, and let the dependent variable be Y. Then, the **regression** of Y on x is a graph that shows $E(Y|x)$ as a function of x. Note that this graph is only defined when both X and Y are defined on interval or ratio scales (why?).

In the simplest case, we model Y as varying linearly with x. In this case, we define a *best-fit line* that minimizes the sum of squared deviations of observed from the estimated values of y. If our observations are of the form $\{(x_i, y_i)\}$, then the model for the regression line is:

$$y = a + bx \tag{EQ 33}$$

and therefore we seek to minimize

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \tag{EQ 34}$$

To find a and b we set the partial derivative of S with respect to a and b to zero. This gives us:

$$a = \bar{y} - b\bar{x}$$
$$b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \tag{EQ 35}$$

Substituting for a in Equation 33 we see that the point $(\bar{x}, \bar{y})$ satisfies the regression equation, so that the regression line always passes through this point, which is also called the *centroid* of the sample. We interpret a as the Y intercept of the best-fit line. b is the mean change in Y with a unit increase in X.

When Y does not depend on X linearly, it is sometimes possible to transform Y so that the dependency is more nearly linear, as the next example demonstrates.

EXAMPLE 15: COMPUTING A LINEAR REGRESSION AFTER TRANSFORMATION

Consider the following data set, which shows the packet loss rate for a given buffer size, where three simulations were run for each buffer size setting. Compute the linear regression of the loss rate on the buffer size.

Buffer size	10 packets	20 packets	50 packets	100 packets	200 packets	500 packets
Run 1	30.20	10.20	5.20	1.10	0.20	0.01
Run 2	27.40	11.30	6.37	1.70	0.23	0.01
Run 3	29.10	9.80	5.82	1.30	0.17	0.01

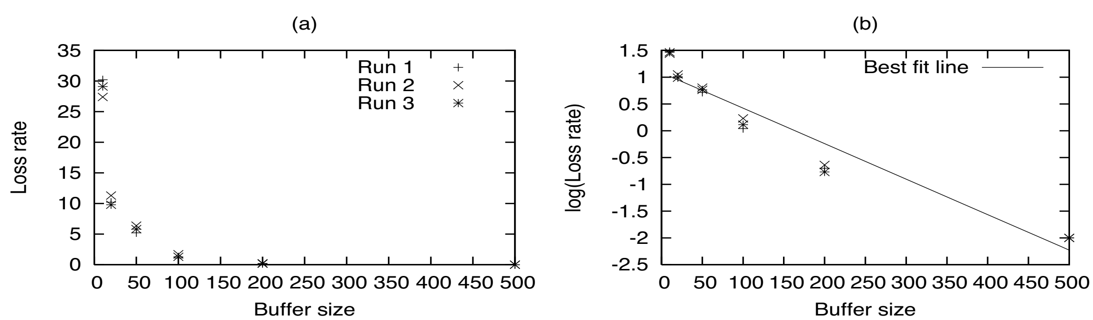


FIGURE 5. **Scatter plot of loss rate vs. buffer size**

It is instructive to look at the scatter plot of the data, shown in Figure 5(a). It is immediately obvious that the relationship between the loss rate and the buffer size is far from linear. In such cases, it is necessary to transform the data values to extract a more linear relationship. Figure 5(b) is a scatter plot which plots the logarithm of the loss rate with respect to the buffer size. It is clear that the relationship is far more linear than before. We compute the best-fit line, using Equation 35, as y = 1.0568 - 0.0066 x which is the regression of Y on x. This best fit line is also shown in the figure.

■

The best-fit line, in addition to minimizing the sum of squared errors, has several useful properties. It is the maximum likelihood estimator for the population parameters a and b. Moreover, it is possible to construct confidence intervals for the values of a and b by means of the t distribution.

Note that it is always possible to compute a linear regression of one variable on another, even if the two variables are not linearly related. Therefore, it is always a good idea to use a statistical test for linearity (or the correlation coefficient, described next), to validate that the relationship is reasonably linear before computing the best-fit line.

We now briefly discuss three extensions to simple linear regression:

- The **least-squares best-fit** approach discussed above assumes that the degree of variation in the dependent variable is more or less the same, regardless of the value of the independent variable. In some cases, the greater (or smaller) the value of the independent variable, the greater (or smaller) the variance in the dependent variable. For example, in the preceding example, we see a greater variation in log(loss rate) for smaller values of the buffer size. Such samples are said to be **heteroscedastic**, and if the departure from uniform variability is significant, it is necessary to resort to advanced techniques to compute the regression.
- In some cases, the relationship between the dependent and independent variable is non-linear even after transformation of the dependent values. In such cases, it may become necessary to perform **non-linear** regression.
- Finally, we have considered a dependent variable that depends on only a single independent variable. In general, the dependency may extend to multiple independent variables. This is the subject of **multiple regression**.

These three topics are treated in greater depth in more advanced texts on statistics.

2.5.3 Correlation

When computing a regression, we can use physical considerations to clearly identify independent and dependent variables. In some cases, however, the outcomes of an experiment can be thought of as being mutually dependent on each other. This dependency is captured in the statistical concept of *correlation*. Moreover, as we will see later, even if one variable depends on the other, the correlation coefficient allows us to determine the degree to which variations in the dependent variable can be explained as a consequence of variations in the independent variable.

EXAMPLE 16: CORRELATED VARIABLES

Suppose we transfer a small file over a cellular modem ten times, each time measuring the *round-trip delay* (from a 'ping' done just before transferring the file) and the *throughput* achieved (by dividing the file size by the transfer time). The round-trip delay may be large because the network interface card may have a low capacity, so that even a small ping packet experiences significant delays. On the other hand, the file transfer throughput may be low because the path delay is large. So, it is not clear which variable ought to be the dependent variable and which variable ought to be the independent variable. Suppose that the measured round-trip delays and throughputs are as shown below:

Throughput (kbps)	46	65	53	38	61	89	59	60	73
Round-trip delay (ms)	940	790	910	1020	540	340	810	720	830

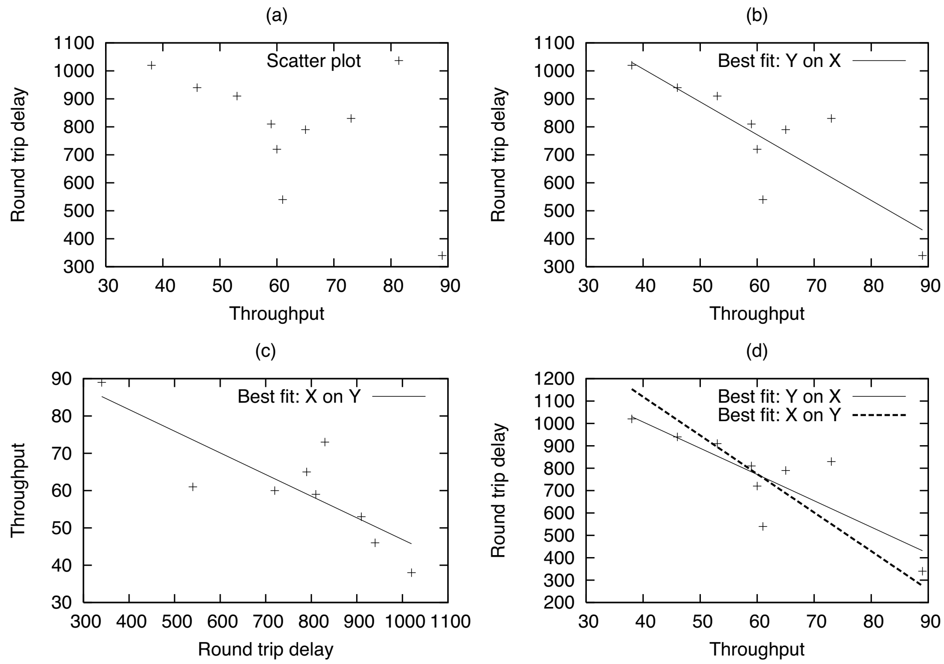


FIGURE 6. Regression and correlation

Figure 6(a) shows the scatter plot of the two variables. There appears to be an approximately linear decline in the round-trip delay with an increase in throughput. We arbitrarily choose throughput to be the independent variable and do a regression of round-trip delay on it, as shown in Figure 6(b). We see that the best-fit line has a negative slope, as expected.

There is no reason why we could not have chosen the round-trip delay to be the independent variable and have done a similar regression. This is shown in Figure 6(c). Again, we see that as the round-trip delay increases, the throughput decreases, indicating a negative relationship. We also see the best-fit line with a negative slope.

Note that the two regression lines are not the same! In one case, we are trying to minimize the sum of the squared errors in round-trip delay, and in the other, the sum of squared errors in throughput. So, the best fit lines will, in general, not be the same. This is shown in Figure 6(d) where we show both best-fit lines (one drawn with transposed axes).

■

The reason why two regression lines do not coincide in general is best understood by doing a thought experiment. Suppose two outcomes of an experiment, say *X* and *Y*, are completely independent. Then, $E(XY) = E(X)E(Y)$, by definition of independence. In the context of a single sample, we rewrite this as:

$$E(XY) = \frac{\sum x_i y_i}{n} = E(X)E(Y) = \frac{\sum x_i}{n}\frac{\sum y_i}{n} = \bar{x}\bar{y} \quad \textbf{(EQ 36)}$$

Recall that $b = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$. We expand the numerator as $\sum x_i y_i - \bar{x}\sum y_i - \bar{y}\sum x_i + n\bar{x}\bar{y}$. Rewriting $\sum y_i$ as $n\bar{y}$ and $\sum x_i$ as $n\bar{x}$, and using Equation 36, we get

$$b = \frac{\sum x_i y_i - \bar{x}\sum y_i - \bar{y}\sum x_i + n\bar{x}\bar{y}}{\sum (x_i - \bar{x})^2} = \frac{n\bar{x}\bar{y} - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}}{\sum (x_i - \bar{x})^2} = 0 \quad \textbf{(EQ 37)}$$

so that the regression line has zero slope, i.e., is parallel to the X axis. Symmetrically, the regression of X on Y will be parallel to the Y axis. Therefore, the two regression lines meet at right angles when the outcomes are independent. Recalling the we can interpret b as the expected increment in Y with a unit change in X, $b = 0$ implies that a unit change in X does not change Y (in expectation), which is consistent with independence.

On the other hand, if one outcome is perfectly linearly related to the other, then $Y = tX$. Clearly, $\bar{y} = t\bar{x}$, so that

$b = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \dfrac{\sum (x_i - \bar{x})(tx_i - t\bar{x})}{\sum (x_i - \bar{x})^2} = t$. Denoting the regression of X on Y by $x = a' + b'y$, the expression for b' is given by $\dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (y_i - \bar{y})^2} = \dfrac{\sum (x_i - \bar{x})(tx_i - t\bar{x})}{\sum (tx_i - t\bar{x})^2} = \dfrac{1}{t}$. With transposed axes, this line exactly overlaps the best fit line for the regression of Y on X. In other words, when there is exact linear dependence between the variables, the best fit regression lines meet at zero degrees. Thus, we can use the angle between the regression lines as an indication of the degree of linear dependence between the variables.

In practice, the standard measure of dependence, or **correlation**, is the square root of the product bb', denoted r, also called **Pearson's correlation coefficient**, and is given by

$$r = \sqrt{\frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}\frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (y_i - \bar{y})^2}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left(\sum (x_i - \bar{x})^2\right)\left(\sum (y_i - \bar{y})^2\right)}} \quad \textbf{(EQ 38)}$$

When the slopes are perpendicular, $r = 0$, and when the slopes are inverses of each other, so that the regression lines overlap, then $r = 1$. Moreover, when X and Y are perfectly negatively correlated, so that $Y = -tX$, $r = -1$. Therefore, we interpret r as the degree of correlation between two variables, ranging from -1 to +1, with its sign indicating direction of correlation (positive or negative), and its magnitude indicating the degree of correlation.

EXAMPLE 17: CORRELATION COEFFICIENT

Compute the correlation coefficient for the variables in Example 16.

Solution:

We compute the mean throughput as 54.4 kbps and the mean delay as 690 ms. Substituting these values into Equation 38, we find that $r = -0.56$. This indicates a negative correlation, but it is not particularly linear.

■

There are many interpretations of the correlation coefficient[5]. One particularly insightful interpretation is based on the sum of squares minimized in a linear regression: $S^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$. Substituting for a and b, it is easily shown (see Exercise 14) that

5. See Joseph Lee Rodgers and W. Alan Nicewander, "Thirteen Ways to Look at the Correlation Coefficient," *The American Statistician*, Vol. 42, No. 1 (Feb., 1988), pp. 59-66.

$$S^2 = (1 - r^2)\sum (y_i - \bar{y})^2 \quad \text{(EQ 39)}$$

That is, r^2 is the degree to which a regression is able to reduce the sum of squared errors, which we interpret as the degree to which the independent variable explains variations in the dependent variable. When we have perfect linear dependency between Y and X, then the degree of correlation is 1 in absolute value, and the regression line is perfectly aligned with the data, so that it has zero error.

In computing a correlation coefficient, it is important to remember that it only captures linear dependence. A coefficent of zero does not mean that the variables are independent: they could well be non-linearly dependent. For example, if $y^2 = 1 - x^2$, then for every value of X, there are two equal and opposite values of Y, so that the best fit regression line is the X axis, which leads to a correlation coefficient of 0. But, of course, Y is not independent of X! Therefore, it is important to be cautious in drawing conclusions regarding independence when using the correlation coefficient. For drawing such conclusions, it is best to use the chi-square goodness-of-fit test described earlier.

Like any statistic, the correlation coefficient r can have an error due to random fluctuations in a sample. It can be shown that if X and Y are jointly normally distributed, then the variable $z = 0.5\log\left(\frac{1+r}{1-r}\right)$ is approximately normally distributed with a mean of $0.5\ \log\left(\frac{1+\rho}{1-\rho}\right)$ and a variance of $1/(n\text{-}3)$. This can be used to find the confidence interval around r in which we can expect to find ρ.

A specific form of correlation that is relevant in the analysis of time series is **autocorrelation**. Consider a series of values of a random variable that are indexed by discrete time, i.e., $X_1, X_2,...,X_n$. Then, the autocorrelation of this series with lag l is the correlation coefficient between the random variable X_i and X_{i-l}. If this coefficient is large (close to 1) for a certain value of l, then we can infer that the series has variation on the time scale of l. This is often much easier to compute than a full scale harmonic analysis by means of a Fourier transform.

Finally, it is important to recognize that correlation is not the same as causality. We must not interpret a correlation coefficient close to 1 or -1 to infer causality. For example, it may be the case that packet losses on a wireless network are positively correlated with mean frame size. One cannot infer that larger frame sizes are more likely to be dropped. It could be the case, for example, that the network is heavily loaded when it is subjected to video traffic, which uses large frames. The increase in the loss rate could be due to the load, rather than the frame size. Yet, the correlation between these two quantities would be strong.

To go from correlation to causation, it is necessary to determine the physical causes that lead to causation. Otherwise, the unwary researcher may be led to unsupportable and erroneous conclusions.

2.6 Comparing multiple outcomes simultaneously: analysis of variance

In the discussion so far, we have focused on comparing the outcomes of experiments corresponding to at most two choices of experimental controls, or ‘treatments’ (Section 2.4.5 on page 54) and determining dependence and independence between two variables (Section 2.5 on page 61). Suppose we wanted to compare outcomes of multiple treatments simultaneously. For example, Example 9 and Example 10 compared the packet loss rate at a queue with 5 buffers with the loss rate at a queue with 100 buffers. Instead, we may want to compare the loss rates with 5, 100, 200, 500, and 1000 buffers with each other. How should we proceed?

Theoretically, we could perform a set of pairwise comparisons, where each comparison used a normal or t test. For example, we could test the hypothesis that the loss rates with 5 and 100 buffers were identical, the hypothesis that the loss rates with 5 and 200 buffers were identical, and so on. This approach, however, leads to a subtle problem. Recall that when we reject a hypothesis, we are subject to a Type I error, that is, rejecting a hypothesis that is true. If we perform many pairwise comparisons, although the probability of Type I error for any one test is guaranteed to be below 5% (or 1%), the overall probability of making at least one Type I error can be greater than 5% (or 1%)! To see this, think of flipping a coin ten times and looking for ten heads. This has a probability of about 1/1024 = 0.1%. But if we were to flip 1024 coins, chances are good that we would get at least one run of ten heads. Arguing along similar lines, it is easy to see that, as the number of comparisons increases, the overall possibility of a Type I error increases. What is needed, therefore, is a way to perform a single test that avoids numerous pairwise comparisons. This is achieved by the technique of 'Analysis of Variance' or ANOVA.

ANOVA is a complex topic with considerable depth. We will only discuss the simplest case of the 'one-way layout' with fixed effects. Multi-way testing and random effects are discussed in greater depth in advanced statistical texts.

2.6.1 One-way layout

In the analysis of a one-way layout, we group observations according to their corresponding treatment. For instance, we group repeated measurements of the packet loss rate for a given buffer size, say 5 buffers. The key idea in ANOVA is that if none of the treatments–such as the buffer size– affect the observed variable–such as the loss rate–then all the observations would be drawn from the same population. Therefore, the sample mean computed for observations corresponding to each treatment should not be too far from the sample mean computed across all the observations. Moreover, the estimate of population variance computed from each group separately should not differ too much from the variance estimated from the entire sample. If we do find a significant difference between statistics computed from each group separately and the sample as a whole, we reject the null hypothesis. That is, we conclude that, with high probability, the treatments affect the observed outcomes. By itself, that is all that basic ANOVA can tell us. Further testing is necessary to determine which treatments affect the outcome and which do not.

We now make this more precise. Suppose we can divide the observations into I groups of J samples each (we assume that all groups have the same number of samples: this is usually not a problem because the treatments are under the control of the experimenter). We denote the jth observation of the ith treatment by the random variable Y_{ij}. We model this observation as the sum of an underlying population mean μ, the true effect of the ith treatment α_i, and a random fluctuation ε_{ij}:

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \text{(EQ 40)}$$

These errors are assumed to be independent and normally distributed with zero mean and a variance of σ^2. For convenience, we normalize the α_is so that $\sum_{i=1}^{I} \alpha_i = 0$. Note that the expected outcome for the ith treatment is $E(Y_{ij}) = \mu + \alpha_i$.

The null hypothesis is that the treatments have no effect on the outcome. If the null hypothesis holds, then the expected value of each group of observations would be μ, so that $\forall i, \alpha_i = 0$. Moreover, the population variance would be σ^2.

Let the mean of the ith group of observations be denoted $\overline{Y_{i.}}$ and the mean of all the observations be denoted $\overline{Y_{..}}$. We denote the sum of squared deviations from the mean *within* each sample by

$$SSW = \sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij} - \overline{Y_{i.}})^2 \qquad \text{(EQ 41)}$$

$\frac{SSW}{I(J-1)}$ is an unbiased estimator of the population variance σ^2 because it sums I unbiased estimators, each given by $\frac{1}{J-1}\sum_{j=1}^{J}(Y_{ij} - \overline{Y_{i.}})^2$. Similarly, we denote the sum of squared deviations from the mean *between* samples by

$$SSB = J\sum_{i=1}^{I}(\overline{Y_{i.}} - \overline{Y_{..}})^2 \qquad \text{(EQ 42)}$$

SSB/(I-1) is also an unbiased estimator of the population variance σ^2 because $\frac{1}{I-1}\sum_{i=1}^{I}(\overline{Y_{i.}}-\overline{Y_{..}})^2$ is an unbiased estimator of $\frac{\sigma^2}{J}$. So, the ratio $\frac{SSB/(I-1)}{SSW/I(J-1)}$ should be 1 if the null hypothesis holds.

It can be shown that *SSB/(I-1)* is a χ^2 variable with *I*-1 degrees of freedom and that *SSW/I(J-1)* is a χ^2 variable with *I*(*J*-1) degrees of freedom. The ratio of two χ^2 variables with *m* and *n* degrees of freedom follows a distribution called the ***F*** distribution with (*m*,*n*) degrees of freedom. Therefore, the variable $\frac{SSB/(I-1)}{SSW/I(J-1)}$ follows the *F* distribution with (*I-1, I*(*J-1*)) degrees of freedom, and has an expected value of 1 if the null hypothesis is true.

To test the null hypothesis, we compute the value of $\frac{SSB/(I-1)}{SSW/I(J-1)}$ and compare it with the critical value of an *F* variable with (*I-1, I*(*J-1*)) degrees of freedom. If the computed value exceeds the critical value, then the null hypothesis is rejected. Intuitively, this would happen if *SSB* is 'too large', that is, there is significant variation in the sums of squares between treatments, which is what we expect when the treatment does have an effect on the observed outcome.

EXAMPLE 18: SINGLE FACTOR ANOVA

Continuing with Example 9, assume that we have additional data for larger buffer sizes, as shown below. Can we still claim that the buffer size plays a role in determining the loss rate?

Loss rate with 5 buffers	1.20%	1.30%	0.90%	1.40%	1.00%	1.80%	1.10%	1.20%	1.50%	1.20%
Loss rate with 100 buffers	0.10%	0.60%	1.10%	0.80%	1.20%	0.30%	0.70%	1.90%	0.20%	1.20%
Loss rate with 200 buffers	0.50%	0.45%	0.35%	0.60%	0.75%	0.25%	0.55%	0.15%	0.35%	0.40%
Loss rate with 500 buffers	0.10%	0.05%	0.03%	0.08%	0.07%	0.02%	0.10%	0.05%	0.13%	0.04%
Loss rate with 1000 buffers	0.01%	0.02%	0.01%	0.00%	0.01%	0.01%	0.00%	0.02%	0.01%	0.00%

Here, I = 5 and J = 10. We compute $\overline{Y_{5.}}$ = 1.26%, $\overline{Y_{100}}$= 0.81%, $\overline{Y_{200.}}$ = 0.44%, $\overline{Y_{500.}}$ = 0.07%, and $\overline{Y_{1000.}}$ = 0.01%. This allows us to compute $SSW = 5.13*10^{-5}$ and $SSB = 1.11*10^{-3}$. The *F* statistic is therefore $(1.11*10^{-3}/ 4)/(5.13*10^{-5}/45)$ = 242.36. Looking up the *F* table we find that even with only (4, 40) degrees of freedom, the critical *F* value at the 1% confidence level is 3.83. The computed statistic far exceeds this value. Therefore, the null hypothesis is rejected.

■

The *F* test is somewhat anticlimactic: it only indicates that a treatment has an effect on the outcome, but it does not quantify the degree of effect. Nor does it identify whether any one treatment is responsible for the failure of the test. These questions can be resolved by *post-hoc* analysis. For example, to quantify the degree of effect, we can compute the regression of the observed effect as a function of the treatment. To identify the treatment that is responsible for the failure of the test, we can re-run the *F* test eliminating one treatment at a time. If the *F* test does not reject the null hypothesis with a particular treatment removed, then we can hypothesize that this treatment has a significant effect on the outcome, testing this hypothesis with a two-variable test.

If these approaches do not work, two more advanced techniques to perform multi-way comparisons are **Tukey's method** and the **Bonferroni method** (see Section 2.10 on page 73 for texts where these methods are discussed).

2.6.2 Multi-way layouts

It is relatively straightforward to extend the one-way layout to two or more treatments that are simultaneously applied. For instance, we may want to study the joint effect of buffer size and cross traffic workload on the loss rate. The details of this so-called *two-way layout* are beyond the scope of this text. We will merely point out that in the context of such designs, we not only have to determine the effect of a treatment on the outcome, but also deal with the possibility that only certain combinations of treatment levels affect the outcome (for instance, the combination of small buffer size and heavy cross traffic). Such *interaction effects* greatly complicate the analysis of multi-way layouts.

2.7 Design of experiments

The statistically rigorous design of experiments is a complex topic. Our goal here will be to give an intuitive understanding of its essentials. Details can be found in more advanced texts devoted to the topic.

The goal of an experiment is, in the words of Sir R.A. Fisher, "...to give the facts a chance of disproving the null hypothesis." The first step in designing an experiment is to formulate a precise hypothesis that can be rejected (or not) on the basis of its results. Many experimental studies in the field of computer systems fail to meet even this obvious requirement! The careful choice of a null hypothesis cannot be over-emphasized.

Note that our analysis of hypothesis testing assumes that the elements of each sample are independently and randomly selected from the population, so that we can treat the sum of the elements of each sample as the sum of n independent and identically distributed random variables. Therefore, in conducting an experiment, it is necessary to ensure that each observation is as nearly independent of the others as possible. Moreover, observations should be made so that each member of the population has an equal chance of being represented. If observations come from sampling a population, then care should be taken that no obvious bias be introduced in the sampling process.

A second consideration in the design of experiments is that enough data be collected so that the hypothesis can be conclusively rejected if necessary. To take a trivial example, it is impossible to reject the hypothesis that a coin is biased from a single coin flip. We can increase the **sensitivity** of an experiment either by collecting more observations within each sample (**enlargement**), or by collecting more samples (**repetition**).

Third, it is necessary to ensure that the experimental conditions be kept as constant as possible ('**controlled**') so that the underlying population does not change when making observations. Otherwise, it is impossible to determine the population whose parameters are being estimated by the statistics of the sample. For example, in a wireless network that is subject to random external interference, packet loss rates are determined not only by the signal strength of the transmitter, but also the signal strength of the interferer. If an external interferer is not controlled, a study that tries to relate a MAC data rate selection algorithm to the packet loss rate, for example, may draw incorrect conclusions.

Finally, when studying the effect of more than one treatment on the outcome of an experiment, we need to take into account the fact that the treatments may not be independent of each other. If treatments were orthogonal, we would simply need to change one treatment at a time, which reduces the analysis of the experiment to analysing a set of one-way layouts. If they are not, then we need to design a set of experiments that explores all combinations of treatments. For example, if both buffer size and cross traffic workload intensity can affect the packet loss rate at a router, and these treatments were non-orthogonal, we need to take into account the so-called interaction effects (see Section 2.6.2 on page 70). A trivial solution to take interactions into account is to perform a **full factorial** design, where we set up the cross product of every possible level of each treatment. For example, if we could choose between five different buffer sizes and three workload levels, we would need to conduct 15 experiments. In many cases, a full factorial design is impossible. If so, there is a considerable body of work on the design of **fractional factorial** experiments, where we may change two or more treatment levels at the same time, using statistical analysis to identify the effect of each individual treatment. These schemes can be fairly complex, in that they need to consider all possible two-way, three-way,..., n-way interactions that may affect the observed outcomes. Specifically, the designs must deal with the problem of **aliasing**, that is, not being able to make out the difference between alternative combinations of treatment levels that have the same effect on the output. If certain combinations of levels can be safely ignored, or are practically unimportant, then we can greatly reduce the number of experiments without affecting the quality of the conclusions.

2.8 Dealing with large data sets

In the statistical analysis of modern computer systems, all too often, the problem is not the lack of experimental data, but its surfeit. With the extensive logging of system components and the growing number of components in a system, the analyst is often confronted with the daunting task of extracting comprehensible and statistically valid results from large volumes of data. In view of this, the practice of statistical analysis—long focused on the extraction of statistically valid results from a handful of experiments—changes its character. This section discusses a pragmatic approach to the analysis of large data sets based on the author's own experiences over the last two decades[6].

Unlike the classical notion of careful experimental design in order to test a hypothesis, the situation in contemporary systems evaluation is to focus, at least to begin with, on **data exploration**. We typically have access to a large compendium of logs and traces, and the questions we would like to answer typically fall into the following broad categories:

- How can the data help us identify the cause of poor overall performance?
- What is the relative performance of alternative implementations of one component of the system?
- Are there implicit rules that describe the data?

In answering these questions, the following procedure has proved useful.

Step 1: Extract a small sample from the entire data set and carefully read through it. Even a quick glance at the data will often point out salient characteristics that can be used to speed up subsequent analysis. Moreover, this allows a researcher to spot potential problems (certain variables not being logged, or having clearly erroneous values). Proceeding with a complex analysis in the presence of such defects only wastes time.

Step 2: Attempt to visualize the *entire* data set. For example, if every sample could be represented by a point, the entire data set could be represented by a pixellated bitmap. The human eye is quick to find non-obvious patterns, but only if presented with the entire data set. If the data set is too large, it may help to sub-sample it (taking every fifth, tenth, or hundredth sample) before visualization. Again, skipping this step will result in missing patterns that may otherwise be only revealed with considerable effort.

Step 3: Look for outliers. The presence of outliers usually indicates a deeper problem, usually either with data collection or data representation (for example, due to underflow or overflow). Usually, the removal of outliers results in the discovery of problems in the logging or tracing software, and the entire data set may have to be collected again. Even if part of the data set can be sanitized to correct for errors, it is prudent to collect the data set again.

Step 4: Formulate a preliminary null hypothesis. Choose this hypothesis with care, being conservative in your selection, so that the non-rejection of the hypothesis does not lead you to a risky conclusion.

Step 5: Use the data set to attempt to reject the hypothesis, using the techniques described earlier in this chapter.

Step 6: Frame and test more sophisticated hypotheses. Often preliminary results reveal insights into the structure of the problem whose further analysis will require the collection of additional data. The problem here is that if data is collected at different times, it is hard to control extraneous influences. The workload may have changed in the interim, or some system components may have been upgraded. Therefore, it is prudent to discard the entire prior data set to minimize the effects of uncontrolled variables. Step 6 may be repeated multiple times until the initial problem has been satisfactorily answered.

6. An alternative view from the perspective of computer system performance evaluation can be found in R. Jain, The Art of Computer Systems Performance Analysis, John Wiley, 2001.

Step 7: Present and interpret the results of the analysis using appropriate graphics. An excellent source for presentation guidelines is (E. Tufte, The Visual Display of Quantitative Information, 2e, Graphics Press, 2001).

When dealing with very large data sets, where visualization is impossible, techniques derived from **data mining** and **machine learning** are often useful. We briefly outline two elementary techniques for data clustering.

The goal of a data clustering algorithm is to find hidden patterns in the data, in this case, the fact that the data can be grouped into 'clusters,' where each cluster represents 'closely' related observations. For example, in a trace of packets observed at a router interface, clusters may represent packets that fall into a certain range of lengths. A clustering algorithm automatically finds clusters in the data set that for our example, would correspond to a set of disjoint ranges of packet lengths.

A clustering algorithm takes as input a distance metric that quantifies the concept of a distance between two observations. Distances can be simple metrics, such as packet lengths, or may be more complex, such as the number of edits (that is, insertions and deletions) that need to be made to a string-valued observation to transform it into another string-valued observation. Observations within a cluster will be closer, according to the specified distance metric, than observations placed in different clusters.

In **agglomerative clustering**, we start with each observation in its own cluster. We then merge the two closest observations into a single cluster and repeat the process until the entire data set is in a single cluster. Note that to carry out repeated mergings, we need to define the distance between a point and a cluster and between two clusters. The distance between a point and a cluster can be defined either as the distance from that point to the closest point in the cluster or as the average of all the distances from that point to all the points in the cluster. Similarly, the distance between clusters can be defined to be the closest distance between their points or the distance between their centroids. In either case, we compute a tree such that links higher up in the tree have longer distance metrics. We can therefore truncate the tree at any point and treat the forest so created as the desired set of clusters. This approach usually does not scale beyond about 10,000 observation types on a single server; distributed computation techniques allow the processing of larger data sets.

k-means clustering is a technique to cluster data into *k* classes. The earliest and most widely-used algorithm for *k*-means clustering is Lloyd's algorithm. In this algorithm, we start with a set of *k* empty containers. We partition the observations into *k* sets, either randomly, or on the basis of a subsample, allocating one set to each container. We then compute the centroid of each container (this is the point that minimizes the sum of distances from all points in the set to itself). Now, each point is re-allocated to the container with the closest centroid. This may result in the container's centroid moving to a different point. We therefore re-compute the centroid for each container, re-allocating points as before. This process iterates until convergence, when no points move from one cluster to another. In most practical cases, the algorithm is found to converge after a few iterations to a globally optimal clustering. However, convergence may result in a local optimum. Several variants of this algorithm with better convergence properties are described in texts on machine learning and data mining.

2.9 Common mistakes in statistical analysis

We now present some common problems in statistical analysis, especially in the context of computer systems.

2.9.1 What is the population?

A question commonly left unanswered in statistical analyses is a precise statement of the underlying population. As we saw in Section 2.1 on page 39, the same sample can correspond to multiple underlying populations. It is impossible to interpret the results of a statistical analysis without carefully justifying that the sample is representative of the chosen underlying population.

2.9.2 Lack of confidence intervals in comparing results

Comparing the performance of two systems simply by comparing the mean values of performance metrics is an all-too-common mistake. The fact that one mean is greater than another is not statistically meaningful and may lead to erroneous conclusions. The simple solution is to always compare confidence intervals, rather than means, as described in Section 2.4.5 on page 54.

2.9.3 Not stating the null hypothesis

Although the process of research necessitates a certain degree of adjustment of hypotheses, a common problem is to carry out a statistical analysis without stating the null hypothesis. Recall that we can only reject or not reject the null hypothesis from observational data. Therefore, it is necessary to carefully formulate and clearly state the null hypothesis.

2.9.4 Too small a sample

If the sample size is too small, then the confidence interval associated with the sample is large, so that even a null hypothesis that is actually false will not be rejected. By computing the confidence interval around the mean during exploratory analysis, it is possible to detect this situation, and collect larger samples for populations with greater inherent variance.

2.9.5 Too large a sample

If the sample size is too large, then a sample that deviates even slightly from the null hypothesis will cause the null hypothesis to be rejected. This is because the confidence interval around the sample mean varies as $\frac{1}{\sqrt{n}}$. Therefore, when interpreting a test that rejects the null hypothesis, it is important to take the *effect size* into account, which is the (subjective) degree to which the rejection of the null hypothesis accurately reflects reality. For instance, suppose we hypothesize that the population mean was 0, and we found from a very large sample that the confidence interval was 0.005 ± 0.0001. This rejects the null hypothesis. However, in the context of the problem, perhaps the value 0.005 is indistinguishable from zero, and therefore has a small ‘effect.’ In this case, we would still not reject the null hypothesis.

2.9.6 Not controlling all variables when collecting observations

The effect of controlling variables in running an experiment is to get a firm grasp on the nature of the underlying population. If the population being sampled changes during the course of the experiment, then the collected sample is meaningless. For example, suppose you are observing the mean delay from a campus router to a particular data centre. Suppose that during data collection, your ISP were to change their Tier 1 provider. Then, the observations made subsequent to the change would be likely to reflect a new population. It is necessary, therefore, during preliminary data analysis, to ensure that such uncontrollable effects have not corrupted the data set.

2.9.7 Converting ordinal to interval scales

Ordinal scales where each ordinal is numbered, such as the Likert scale (where 1 may represent ‘poor’, 2 ‘satisfactory’, 3 ‘good’, 4 ‘outstanding’ and 5 ‘excellent’), are often treated as if they are interval scales. So, if one user were to rate the streaming performance of a video player as 1 and another as 3, then the mean rating is stated to be 2. This is bad practice. It is hard to argue that the gap between ‘poor’ and ‘satisfactory’ is the same as the gap between ‘satisfactory’ and ‘good.’ Yet, that is the assumption being made when ordinal scales such as these are aggregated. In such cases, it is better to ask users to rank an experience on a linear scale from 1 to 5. This converts the ordinal scale to an interval scale and allows aggregation without making unwarranted assumptions.

2.9.8 Ignoring outliers

The presence of outliers should always be a cause for concern. Silently ignoring them, or deleting them from the data set altogether, not only is bad practice, but prevents the analyst from unearthing significant problems in the data collection process. Therefore, they should never be ignored.

2.10 Further reading

This chapter only touches on the elements of mathematical statistics. A delightfully concise summary of the basics of mathematical statistics can be found in M.G. Bulmer, Principles of Statistics, *Oliver and Boyd*, 1965, re-issued by Dover, 1989. Statistical analysis is widely used in the social sciences and agriculture. The classic reference for a plethora of statistical techniques is Snedecor and Cochran, Statistical Methods, 8e, Wiley, 1989. Exploratory data analysis is described from the perspective of a practitioner in G. Myatt, Making Sense of Data: A Practical Guide to Exploratory Data Analysis and Data Mining, Wiley, 2006. Readers who want to learn directly from one of the masters of statistical analysis should refer to R.A. Fisher, Statistical Methods for Research Workers, 1e, Oliver and Boyd, 1925.

2.11 Exercises

1 Means

Prove that the mean of a sample is the value of x^* that minimizes $\sum_{i=1}^{n}(x_i - x^*)^2$

2 Means

Prove Equation 12.

3 Confidence intervals (normal distribution)

Compute the 95% confidence interval for the data values in Table 2 (reproduced below).

Data value	Frequency
1	5
2	7
3	2
7	2
1000	1

4 Confidence intervals (t distribution)

Redo Exercise 8 using the *t* distribution.

5 Hypothesis testing: comparing the mean to a constant

For the sample below, test the null hypothesis that the mean loss rate is 2% at the 95% confidence level.

Loss rate with 5 buffers	1.20%	2.30%	1.90%	2.40%	3.00%	1.80%	2.10%	3.20%	4.50%	2.20%

6 Chi-squared test

In Example 15, what is the value of n_1 beyond which the hypothesis would be rejected?

7 Fitting a distribution and chi-squared test

Continuing with Example 16, consider the data set below. Ignoring the first observation (i.e., (1,18)), find the best Poisson fit for the reduced sample. Use this to compute the expected count for each number of arrivals. What is the chi-squared variate value for this reduced data set? Use this to determine whether the Poisson distribution is indeed a good distribution to describe the reduced data set.

Number of packet arrivals	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Count	18	28	56	105	126	146	164	165	120	103	73	54	23	16	9	5

8 Independence, Regression, and Correlation

A researcher measures the mean uplink bandwidth of 10 desktop computers (in kbps) as well their mean number of peer-to-peer connections over the period of one hour, obtaining the following data set:

Uplink capacity	202	145	194	254	173	94	102	232	183	198
# peers	50	31	47	50	41	21	24	50	41	49

a) If the number of peers were independent of the uplink capacity, what is the expected value of the number of peers for a specific uplink capacity?

b) Can we conclude, using the chi-squared test, that the number of peers is independent of the uplink capacity, at the 95% and 99.9% confidence levels?

c) Compute the regression of the number of peers on the uplink capacity. What is the slope of the best-fit line?

d) What is the correlation coefficient between the two variates? Does this reinforce your conclusions regarding independence or dependence?

e) What portion of the variation in the value of the number of peers can be attributed to the uplink capacity?

9 Correlation coefficient

Prove Equation 39.

10 Single Factor ANOVA

A university is connected to the Internet using three ISPs. To test their relative performance, the IT staff conduct an experiment where they measured the ping times to a well-known website over each of the three providers over a period of ten days. The mean ping time using each ISP on each day is shown below. Use single-factor ANOVA to test the hypothesis that the ISPs are statistically identical.

Day	ISP1	ISP2	ISP3
1	41.2	50.7	41.1
2	34.9	38.5	48.2
3	43.5	56.3	73.2
4	64.2	54.2	48.4
5	64.0	46.4	61.4
6	54.9	58.4	43.2
7	59.3	61.8	63.9
8	73.1	69.4	54.3
9	56.4	66.3	67.4
10	63.8	57.4	58.4

CHAPTER 3 *Linear Algebra*

This chapter presents the essentials of linear algebra. Starting with a basic introduction to vectors and matrices, we study operations on matrices. This motivates the study of a linear combination of vectors that underlies the important concepts of matrix rank and independence. We then use these concepts to study the solution of systems of linear equations. Linear algebra plays a critical role in modelling processes as linear transformations, which we study next. We focus on an intuitive understanding of eigenvalues and eigenvectors. We conclude with a description of stochastic matrices that arise frequently in models of computer networking systems.

3.1 Vectors and matrices

Consider two runs of an experiment where a researcher collects packet traces on an Internet link. Suppose that the first trace contains 312 TCP and 39 UDP packets and that the second trace contains 432 TCP and 21 UDP packets. We can represent these results in the form of these two ordered tuples: [312, 39] and [432, 21]. Here, the positions in the tuple are implicitly associated with the meaning "TCP count" and "UDP count" respectively. We call this representation of an ordered set of **elements** a **vector**.

A vector with *n* elements is said to have *n* **dimensions**. There is a one-to-one mapping from an *n*-dimensional vector with real-valued elements to a point in an *n*-dimensional real space. Returning to our example, the vector [432, 21] corresponds to a point in a two-dimensional real space whose X and Y axes are "TCP count" and "UDP count" respectively and that has a coordinates of (432, 21). If one were to add another measurement to the tuple, say "ICMP count," then we could represent the counts in a packet trace by a vector such as [432, 21, 12] which corresponds to a point in a three-dimensional real space.

Vectors can be represented in one of two ways: as **row-vectors** of the form [312, 12, 88], and as **column-vectors** of the form: $\begin{bmatrix} 312 \\ 12 \\ 88 \end{bmatrix}$. We define the **zero-vector** of *n* dimensions, denoted **0,** as the vector [0 0 0 ... 0]. In this book, vectors are shown in lower case bold italic font and vector elements are shown using italic font.

Returning to our example, we can represent packet counts in both traces simultaneously using an array that looks like this:

$$\begin{bmatrix} 312 & 39 \\ 432 & 21 \end{bmatrix}$$

Such a representation is called a **matrix**. In this book, matrices are shown using upper case bold italic font and matrix elements are shown using lower case italic font.

Unlike a vector, whose elements may be unrelated, elements in the same column of a matrix are usually related to each other; in our example, all elements in the first column are TCP counts. In general, an array with *m* rows and *n* columns is called an $m \times n$ matrix. The element in the *i*th row and *j*th column of a matrix named ***A*** is usually represented by the symbol a_{ij}. In the example above, a_{12} is 39 and a_{21} is 432.

Although vector and matrix representations can be used for arbitrary element types, such as for character strings, in our discussion, we will assume that the elements of a vector or a matrix are members of a mathematical **field**. For completeness, the formal definition of a field is stated below; note that this definition essentially formalizes our intution of the properties of real numbers.

A field *F* is a finite or infinite **set** along with the **operations** of addition (denoted '+') and multiplication (denoted '*') on elements of this set that satisfy the following axioms:

1. **Closure** under addition and multiplication: For *a*, *b* in *F*, if $a+b = c$ and $a*b=d$, then *c* and *d* are also in *F.*
2. **Commutativity** of addition and multiplication: For *a*, *b* in *F*, if $a+b = b+a$ and $a*b=b*a$.
3. **Associativity** of addition and multiplication: For *a*, *b*, *c* in *F*, $(a+b) + c = a + (b+c)$.
4. Existence of distinct additive and multiplicative **identity** elements in the set: There are distinct elements denoted '0' and '1' in *F*, such that for all *a* in *F*, $a+0 = a$ and $a*1 = a$.
5. Existence of additive and multiplicative **inverses**: For every *a* in *F* there is an element *b* also in *F* such that $a+b = 0$. For every *a* in *F* other than '0', there is an element also in *F* such that $a*c = 1$.
6. **Distributivity** of multiplication over addition: For all *a*, *b* and *c* in *F*, the following equality holds: $a *(b+c) = (a*b) + (a*c)$.

3.2 Vector and matrix algebra

This section presents some basic operations on vectors and matrices.

3.2.1 Addition

The **sum of two vectors** of the same dimension is a vector whose elements are the sums of the corresponding elements of each vector. Addition is not defined for vectors with different dimensions.

The **sum of two matrices** with the same number of rows and columns is a matrix whose elements are the sums of the corresponding elements of each matrix. Addition is not defined for matrices with different numbers of rows or columns.

Because the elements of a vector or a matrix are drawn from a field and vector and matrix addition operates element by element, vector and matrix addition inherits the field properties of closure, commutativity, associativity, the existence of an additive inverse and the existence of an additive identity.

3.2.2 Transpose

The **transpose** of a row-vector ***x***—denoted $\boldsymbol{x}^T$—is a column-vector whose *j*th row is the *j*th column of ***x***. The transpose of an $m \times n$ matrix ***A***, denoted $\boldsymbol{A}^T$ is an $n \times m$ matrix whose [*j*, *i*]th element—denoted a_{ji}—is the [*i*, *j*]th of ***A***, a_{ij}.

EXAMPLE 1: TRANSPOSE

The transpose of [1,3,5,1] is $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix}$. The transpose of $\begin{bmatrix} 23 & 2 & 9 \\ 98 & 7 & 89 \\ 34 & 9 & 1 \end{bmatrix}$ is $\begin{bmatrix} 23 & 98 & 34 \\ 2 & 7 & 9 \\ 9 & 89 & 1 \end{bmatrix}$.

■

3.2.3 Multiplication

Multiplying a vector ***x*** by a real number (or **scalar**) *s* results in the multiplication of each element (i.e., **scaling**) of the vector by that real. That is,

$$s[x_1, x_2, \ldots, x_n] = [sx_1, sx_2, \ldots, sx_n] \tag{EQ 1}$$

Similarly, multiplying a matrix ***A*** by a real number (scalar) *s* results in the multiplication of each element of the matrix by that real. That is,

$$s\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} sa_{11} & \ldots & sa_{1n} \\ \ldots & \ldots & \ldots \\ sa_{m1} & \ldots & sa_{mn} \end{bmatrix} \tag{EQ 2}$$

The **product of two vectors** can be defined either in terms of a **dot** product or a **cross** product. The dot product of vector ***x*** with elements x_i and vector ***y*** with elements y_i is defined as the *scalar s* obtained as the sum of the element-by-element product. That is,

$$s = \mathbf{x}.\mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{EQ 3}$$

The dot product is undefined if the two vectors do not have the same dimension.

The **cross** product of two vectors is not relevant to computer networking and will not be discussed further.

Unlike the dot product of two vectors, which is a scalar, the product of two matrices is a matrix whose [*i*, *j*]th element is the dot product of the *i*th *row* of the first matrix and the *j*th *column* of the second matrix. That is, if ***C*** = ***AB***, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{EQ 4}$$

Note that the number of columns in ***A*** (the dimension of each row of ***A***) must equal the number of rows in ***B*** (the dimension of each column in ***B***). Thus, the product of an $m \times n$ matrix by an $n \times o$ matrix results in an $m \times o$ matrix. Therefore, the product of a *n* dimensional row-vector—a matrix of size $1 \times n$—with an $n \times n$ matrix is a row-vector of dimension *n*.

EXAMPLE 1: MATRIX MULTIPLICATION

The product of $\begin{bmatrix} 23 & 2 & 9 \\ 98 & 7 & 89 \\ 34 & 9 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 & -2 \\ 4 & 9 & 8 \\ 3 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 81 & 133 & -21 \\ 491 & 553 & -51 \\ 107 & 251 & 5 \end{bmatrix}$. To obtain c_{11}, for example, we compute 23*2 + 2*4 + 9*3 = 46 + 8 + 27 = 81.

■

Matrix multiplication is associative, that is (***AB***)***C*** = ***A***(***BC***), but it is <u>not</u> commutative. That is, in general,

$$\boldsymbol{AB} \neq \boldsymbol{BA} \tag{EQ 5}$$

This follows trivially from the observation that although ***AB*** may be defined, if the number of columns in ***B*** differs from the number of rows in ***A***, then ***BA*** may not even be defined. Moreover, if ***AB***=***0***, it is not necessary that either ***A*** or ***B*** be the null matrix. This is unlike the case with scalars, where $ab=0$ implies that one of a or b is zero. As a corollary, if $\boldsymbol{AB}=\boldsymbol{AC}$, then $\boldsymbol{B}$ does not necessarily have to be the same as $\boldsymbol{C}$.

3.2.4 Square matrices

A matrix $\boldsymbol{A}$ is **square** if it has the same number of rows and columns. A square matrix with non-zero elements only along the main diagonal is called a **diagonal** matrix.

An $n \times n$ square diagonal matrix $\boldsymbol{I}$ with '1' along the main diagonal and '0' elsewhere has the property that multiplication of any $n \times n$ square matrix $\boldsymbol{A}$ with this matrix does not change $\boldsymbol{A}$. That is, $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$. Hence, $\boldsymbol{I}$ is called the **identity** matrix.

3.2.5 Exponentiation

If a matrix $\boldsymbol{A}$ is square then its product with itself is defined and denoted $\boldsymbol{A}^2$. If $\boldsymbol{A}$ has n rows and columns, so does $\boldsymbol{A}^2$. By induction, all higher powers of $\boldsymbol{A}$ are also defined and also have n rows and columns.

EXAMPLE 2: EXPONENTIATION

Let A be $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then, $A^2 = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 125 & 0 & 0 \\ 0 & 343 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. Finding the higher powers of $\boldsymbol{A}$ in this example is particularly straightforward (why?). Generalizing, we see that if $\boldsymbol{A}$ is a diagonal matrix then the $(i,i)^{th}$ element of the kth power of A is a_{ii}^k.

■

3.2.6 Matrix exponential

If a matrix $\boldsymbol{A}$ is square then all higher powers of $\boldsymbol{A}$ are also defined and also have n rows and columns. Then, the **matrix exponential** denoted $e^{\boldsymbol{A}}$ is defined as the infinite sum:

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \dots \qquad \textbf{(EQ 6)}$$

3.3 Linear combinations, independence, basis, and dimension

This section introduces some important foundational concepts that will be used in later sections.

3.3.1 Linear combinations

Consider a set of k real-valued variables $x_1, x_2,...,x_k$. Suppose we are given a set of k real-valued weights $w_1, w_2,...,w_k$. Then, we can define the weighted sum of these variables as $s = w_1x_1 + w_2x_2 +...+ w_kx_k$. This sum 'mixes' the variables in linear proportion to the weights. Therefore, we call s a **linear combination** of the variables.

We can generalize the notion of linear combination to vectors. Here, each $\boldsymbol{x_i}$ as a vector, so that their linear combination, $\boldsymbol{s}$, is also a vector. Of course, each vector must have the same number of elements. Note that each component of $\boldsymbol{s}$ is a linear combination of the corresponding elements of the underlying vectors.

EXAMPLE 3: LINEAR COMBINATION OF SCALARS

Compute the linear combination s of the scalars 2, 4, 1, 5 with weights 0.1, 0.4, 0.25, 0.25.

Solution:

The linear combination is s = 0.1*2 + 0.4*4 + 0.25*1 + 0.25*5 = 0.2 + 1.6 + 0.25 + 1.25 = 3.3.

■

EXAMPLE 4: LINEAR COMBINATION OF VECTORS

Compute the linear combination of the vectors [2 4 1 5], [3 5 1 2], [5 6 2 1], [9 0 1 3] with weights 0.1, 0.4, 0.25, 0.25.

Solution:

The linear combination is given by 0.1*[2 4 1 5] + 0.4*[3 5 1 2] + 0.25*[5 6 2 1] + 0.25*[9 0 1 3]. Clearly, the first element of $\boldsymbol{s}$ is given by 0.1*2 + 0.4*3 + 0.25*5 + 0.25*9 = 0.2 + 1.2 + 1.25 + 2.25 = 4.9. Similarly, the other elements are 3.9, 1.25 and 2.3, so that $\boldsymbol{s}$ = [4.9 3.9 1.25 2.3].

■

3.3.2 Linear independence

Consider a set of k vectors $\boldsymbol{x}_1, \boldsymbol{x}_2,...,\boldsymbol{x}_k$. Suppose we can express one of the vectors, say $\boldsymbol{x}_i$, as a linear combination of the others. Then, the value of $\boldsymbol{x}_i$ *depends* on the others: if the remaining vectors assume certain values, then the value of $\boldsymbol{x}_i$ is known and cannot be chosen arbitrarily. This means that we have removed some degrees of freedom in assigning arbitrary values to the vectors.

Specifically, suppose we can express $\boldsymbol{x}_i$ as a linear combination of the remaining k-1 vectors using an appropriately chosen set of k-1 weights. Then, we can write

$$\boldsymbol{x}_i = w_1\boldsymbol{x}_1 + ... + w_{i-1}\boldsymbol{x}_{i-1} + w_{i+1}\boldsymbol{x}_{i+1} + ... + w_k\boldsymbol{x}_k \qquad \text{(EQ 7)}$$

Or, transposing terms:

$$w_1\boldsymbol{x}_1 + ... + w_{i-1}\boldsymbol{x}_{i-1} - \boldsymbol{x}_i + w_{i+1}\boldsymbol{x}_{i+1} + ... + w_k\boldsymbol{x}_k = \boldsymbol{0} \qquad \text{(EQ 8)}$$

This motivates the following definition of independence of a set of vectors: we say that a set of vectors is independent if the only set of weights that satisfies Equation 8 is $\boldsymbol{w}=\boldsymbol{0}$.

Note that if a set of vectors is not linearly independent, *any* one of them can be rewritten in terms of the others (why?).

EXAMPLE 5: LINEAR INDEPENDENCE

The three vectors:

$$\boldsymbol{x}_1 = [3\ 0\ 2]$$

$$\boldsymbol{x}_2 = [-3\ 21\ 12]$$

$$\boldsymbol{x}_3 = [21\ -21\ 0]$$

are not linearly independent because $6\boldsymbol{x}_1 - \boldsymbol{x}_2 - \boldsymbol{x}_3 = \boldsymbol{0}$. The first and third vectors are independent because the third element of the first vector cannot be generated by the third vector (why?).

■

In Section 3.4.6 on page 87 we will see that if a set of vectors is linearly independent, then the matrix formed by the vectors is **non-singular**, that is, has a non-zero **determinant**.

3.3.3 Vector spaces, basis, and dimension

Given a set of k vectors, suppose we can identify a subset of r vectors that linearly independent. That is, the remaining vectors can be written as linear combinations of the r vectors. Then, we call these r vectors the **basis** set of this set of vectors. They form the essential core from which we can derive the rest of the vectors. In this sense, the remaining vectors can be thought to be redundant. For instance, in Example 5, the first and third vectors constitute a basis, and the second vector can be generated from the basis set as $\boldsymbol{x}_2 = 6\boldsymbol{x}_1 - \boldsymbol{x}_3$. Note that any linearly independent subset of r vectors can form a valid basis, so the basis set is not unique.

We can now generalize this observation as follows. Suppose we are given a set of r linearly independent vectors. What is the set of vectors that can be generated as linear combinations of this set? Clearly, there are an infinite number of such vectors. We call this infinite set a **vector space** generated by the basis set (note that a 'vector space' is a precisely defined mathematical object - this is only an informal definition of the concept). The number of basis vectors (the cardinality of the basis set) is called the **dimension** of this space.

EXAMPLE 6: BASIS AND DIMENSION

A simple way to guarantee that a set of vectors is linearly independent is to set all but one element in each vector to zero. For instance, the vectors $\boldsymbol{x}_1$ = [1 0 0], $\boldsymbol{x}_2$ = [0 1 0], $\boldsymbol{x}_3$ = [0 0 1] are guaranteed to be linearly independent (why?). Let us find the vector space generated by this basis set. Consider an arbitrary vector $\boldsymbol{x} = [a\ b\ c]$. This vector can be expressed as linear combination of the basis set as $\boldsymbol{x} = a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3$. Therefore, this basis set generates the vector space of all possible vectors with three real-valued elements.

If we think of a vector with three real-valued elements as corresponding to a point in three-dimensional space, where the elements of the vector are its Cartesian coordinates, then the basis vectors generate all possible points in three-dimensional space. It is easy to see that the basis vectors correspond to the three ordinal axes. It should now be clear why we call the generated vectors a 'space', and why the cardinality of the basis set is the dimensionality of this space.

■

3.4 Solving linear equations using matrix algebra

We now turn our attention to an important application of matrix algebra, which is to solve sets of linear equations.

3.4.1 Representation

Systems of linear equations are conveniently represented by matrices. Consider the set of linear equations:

$$\begin{aligned} 3x + 2y + z &= 5 \\ -8x + y + 4z &= -2 \\ 9x + 0.5y + 4z &= 0.9 \end{aligned}$$

We can represent this set of equations by the matrix

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -8 & 1 & 4 & -2 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$$

where the position of a number in the matrix implicitly identifies it either as a coefficient of a variable or a value on the right hand side. This representation can be used for any set of linear equations. If the rightmost column is ***0,*** then the system is said to be **homogeneous**. The submatrix corresponding to the left hand size of the linear equations is called the **coefficient matrix**.

3.4.2 Elementary row operations and Gaussian elimination

Given a set of equations, certain simple operations allow us to generate new equations. For example, multiplying the left- and right-hand sides of any equation by a scalar generates a new equation. Moreover, we can add or subtract the left- and right-hand sides of any pair of equations to also generate new equations.

In our example above, the first two equations are $3x + 2y + z = 5$ and $-8x + y + 4z = -2$. We can multiply the first equation by 3 to get the new equation $9x + 6y + 3z = 15$. We can also add the two equations to get a new equation $(3-8)x + (2+1)y + (1+4)z = (5-2)$, which gives us the equation $-5x + 3y + 5z = 3$.

We can also combine these operations. For example, we could multiply the second equation by 2 and subtract it from the first one like this:

$$(3-(-16))x + (2-2)y + (1-8)z = 5-(-4)$$
$$19x - 7z = 9$$

This results in an equation where the variable y has been eliminated (i.e., does not appear). We can similarly multiply the third equation by 4 and subtract it from the first one to obtain another equation that also eliminates y. We now have two equations in two variables that we can trivially solve to obtain x and z. Putting their values back into any of the three equations allows us to find y.

This approach, in essence, is the well-known technique called **Gaussian elimination**. In this technique, we pick any one variable and use multiplications and additions on the set of equations to eliminate that variable from all but one equation. This transforms a system with n variables and m equations to a system with $n-1$ variables and $m-1$ equations. We can now recurse to obtain, in the end[1], an equation with one variable, which solves the system for that variable. By substituting this value back into the reduced set of equations, we solve the system.

When using a matrix representation of the set of equations, the elementary operations of multiplying an equation by a scalar and of adding two equations correspond to two **row operations**. The first row operation multiplies all the elements of a row by a scalar and the second row operation is the element-by-element addition of two rows. It is easy to see that these are exactly analogous to the operations in the previous paragraphs. The Gaussian technique uses these elementary row operations to manipulate the matrix representation of a set of linear equations so that one row looks like this: [0 0 ... 0 1 0 ... 0 a]. This allows us to read off the value of that variable. We can use this to substitute for this variable in the other equations, so that we are left with a system of equations with one less unknown, and, by recursion, to find the values of all the variables.

EXAMPLE 7: GAUSSIAN ELIMINATION

Use row operations and Gaussian elimination to solve the system given by $\begin{bmatrix} 3 & 2 & 1 & 5 \\ -8 & 1 & 4 & -2 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$.

Solution:

1. Assuming that the equations are self-consistent and have at least one solution. More on this below.

Subtract row 3 from row 2 to obtain $\begin{bmatrix} 3 & 2 & 1 & 5 \\ -17 & 0.5 & 0 & -2.9 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$. Then subtract 0.25 times row 3 from row 1 to obtain

$\begin{bmatrix} 0.75 & 1.875 & 0 & 4.775 \\ -17 & 0.5 & 0 & -2.9 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$. Note that the first two rows represent a pair of equations in two unknowns. Multiply the second row

by 1.875/0.5 = 3.75 and subtract from the first row to obtain $\begin{bmatrix} 64.5 & 0 & 0 & 15.65 \\ -17 & 0.5 & 0 & -2.9 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$. This allows us to read off x as 15.65/

66.525 = 0.2426. Substituting this into row 2, we get -17*0.2426 + 0.5y = -2.9, which we solve to get y = 2.4496. Substituting this into the third row, we get 9*0.2426 + 0.5* 2.4496 + 4z = 0.9, so that z = 0.6271. Checking, 3*0.2426 + 2*2.4484 - 0.6271 = 4.9975, which is within rounding error of 5.

■

In practice, choosing which variable to eliminate first has important consequences. Choosing a variable unwisely may require us to maintain matrix elements to very high degrees of precision, which is costly. There is a considerable body of work on algorithms to carefully choosing the variables to eliminate, which are also called the **pivots**. Standard matrix packages, such as MATLAB, implement these algorithms.

3.4.3 Rank

So far, we have assumed that a set of linear equations always has a consistent solution. This is not always the case. A set of equations has no solution or has an infinite number of solutions if it is either **over-determined** or **under-determined** respectively. A system is over-determined if the same variable assumes inconsistent values. For example, a trivial over-determined system is the set of equations: $x = 1$ and $x = 2$. Gaussian elimination will fail for such systems.

A system is under-determined if it admits more than one answer. A trivial instance of an under-determined system is the system of linear equations: $x + y = 1$ because we can choose an infinite number of values of x and y that satisfy this equation. Gaussian elimination on such a system results in some set of variables expressed as linear combinations of the independent variables. Each assignment of values to the independent variables will result in finding a consistent solution to the system.

Given a system of m linear equations using n variables, the system is under-determined if $m < n$. If m is at least as large as n, the system may or may not be under-determined, depending on whether some equations are 'repeated.' Specifically, we define an equation as being **linearly dependent** on a set of other equations if it can be expressed as a linear combination of the other equations (the vector corresponding to this equation is a linear combination of the vectors corresponding to the other equations). If one equation in a system of linear equations is linearly dependent on the others, then we can reduce the equation to the equation 0 = 0 by a suitable combination of multiplications and additions. Thus, this equation does not give us any additional information and can be removed from the system without changing the solution.

If of m equations in a system, k can be expressed as a linear combination of the other m - k equations, then we really only have m - k equations to work with. This value is called the **rank** of the system, denoted r. If $r < n$, then the system is under-determined. If $r = n$, then there is only one solution to the system. If $r > n$, then the system is over-determined, and therefore inconsistent. Note that the rank of a matrix is the same as the cardinality of the basis set of the corresponding set of row vectors.

EXAMPLE 8: RANK

We have already seen that the system of equations $\begin{bmatrix} 3 & 2 & 1 & 5 \\ -8 & 1 & 4 & -2 \\ 9 & 0.5 & 4 & 0.9 \end{bmatrix}$ has a unique assignment of consistent values to the variables x, y, and z. Therefore, it has a rank of 3.

Consider the system $\begin{bmatrix} 3 & 2 & 1 & 5 \\ -8 & 1 & 4 & -2 \\ 6 & 4 & 2 & 10 \end{bmatrix}$. We see that the third row is just the first row multiplied by 2. Therefore, it adds no additional information to the system and can be removed. The rank of this system is 2 (it is under-determined), and the resultant system has an infinity of solutions.

Now, consider the system $\begin{bmatrix} 3 & 2 & 1 & 5 \\ -8 & 1 & 4 & -2 \\ 9 & 0.5 & 4 & 0.9 \\ 3 & 2 & 1 & 4 \end{bmatrix}$. We know that the first three rows are linearly independent and have a rank of 3. The fourth row is inconsistent with the first row, so the system is over-determined, and has no solution. The resulting system has a rank of 4.

■

Many techniques are known to determine the rank of a system of equations. These are, however, beyond the scope of this discussion. For our purpose, it suffices to attempt Gaussian elimination, and report a system to be over-determined, that is, have a rank of at least *n*, if an inconsistent solution is found, and to be under-determined, that is, with a rank smaller than *n*, if an infinite number of solutions can be found. If the system is under-determined, the rank is the number of equations that do not reduce to the trivial equation 0 = 0.

3.4.4 Determinants

We now turn our attention to the study of a determinant of a matrix. Even the most enthusiastic of mathematicians will admit that the study of determinants is a rather dry topic. Moreover, the determinant of a matrix does not, by itself, have much practical value. Although they can compactly represent the solution of a set of linear equations, actually computing solutions using the determinant is impractical. The real reason to persist in mastering determinants is as a necessary prelude to the deep and elegant area of the eigenvalues of a matrix.

The determinant $D = \det A$ of a two-by-two matrix is a *scalar* defined as follows:

$$D = detA = det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad \text{(EQ 9)}$$

Note the use of vertical lines (instead of brackets) to indicate the determinant of the matrix, rather than the matrix itself. The determinant of a two-by-two matrix is called a determinant of **order** 2.

To describe the determinant of a larger (square) matrix, we will need the concept of **submatrix** corresponding to an element a_{jk}. This is the matrix $\boldsymbol{A}$ from which the j^{th} row and the k^{th} column have been deleted and is denoted $S_{jk}(\boldsymbol{A})$. The determinant of this submatrix, i.e., $det\ S_{jk}(\boldsymbol{A}) = |S_{jk}(\boldsymbol{A})|$ is a scalar called the **minor** of a_{jk} and is denoted M_{jk}.

Note that the submatrix of a matrix has one fewer row and column. The determinant of a *n*-by-*n* matrix has order *n*. Therefore, each of its minors has an order *n*-1.

We now define another auxiliary term, which is the **co-factor** of a_{jk} denoted C_{jk}. This is defined by

$$C_{jk} = (-1)^{j+k} M_{jk} \qquad \text{(EQ 10)}$$

We are now in a position to define the determinant of a matrix. The determinant of a matrix is defined recursively as follows:

$$D = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{k=1}^{n} a_{ki}C_{ki} \qquad \text{(EQ 11)}$$

where i is an arbitrary row or column. It can be shown that D does not change no matter which column or row is chosen for expansion. Moreover, it can be shown (see the Exercises) that the determinant of a matrix does not change if the matrix is transposed. That is, $|\boldsymbol{A}| = |\boldsymbol{A}^T|$.

EXAMPLE 9: DETERMINANTS

Compute the determinant of the matrix $\begin{bmatrix} 2 & 5 & -2 \\ 4 & 9 & 8 \\ 3 & 0 & 1 \end{bmatrix}$.

Solution:

We will compute this by expanding the third row, so that we can ignore the middle co-factor corresponding to the element $a_{32} = 0$. The determinant is given by

$$a_{31}C_{31} + a_{33}C_{33} = 3(-1)^{3+1}M_{31} + 1(-1)^{3+3}M_{33}$$

$$= 3\begin{vmatrix} 5 & -2 \\ 9 & 8 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 4 & 9 \end{vmatrix} = 3(40-(-18)) + 1(18-20) = 174 + (-2) = 172$$

As a check, we expand by the center column to obtain

$$a_{12}C_{12} + a_{22}C_{22} = 5(-1)^{1+2}M_{12} + 1(-1)^{2+2}M_{22}$$

$$= -5\begin{vmatrix} 4 & 8 \\ 3 & 1 \end{vmatrix} + 9\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = -5(4-24) + 9(2-(-6)) = 100 + 72 = 172$$

■

Here are some useful properties of determinants:

- A determinant can be computed by expanding any row or column of a matrix. Therefore, if a matrix has a zero column or row, then its determinant is 0.
- Multiplying every element in a row or a column of a matrix by the constant c results in multiplying its determinant by the same factor.
- Interchanging two rows or columns of matrix $\boldsymbol{A}$ results in a matrix $\boldsymbol{B}$ such that $|\boldsymbol{B}| = -|\boldsymbol{A}|$. Therefore, if a matrix has identical rows or columns, its determinant must be zero, because zero is the only number whose negation leaves it unchanged.
- A square matrix with n rows and columns has rank n if and only it has a non-zero determinant.
- A square matrix has an inverse (is **non-singular**) if and only if has a non-zero determinant.

3.4.5 Cramer's theorem

Computing the determinant of a matrix allows us to (in theory, at least) trivially solve a system of equations. In practice, computing the determinant is more expensive than Gaussian elimination, so **Cramer's theorem**, discussed below is useful mostly to give us insight into the nature of the solution.

Cramer's theorem states that if a system of n linear equations in n variables $\boldsymbol{Ax} = \boldsymbol{b}$ has a non-zero coefficient determinant $D = \det \boldsymbol{A}$, then the system has precisely one solution, given by

$$x_i = D_i/D$$

where D_i is determinant of a matrix obtained by substituting $\boldsymbol{b}$ for the ith column in $\boldsymbol{A}$. Thus, if we know the corresponding determinants, we can directly compute the x_is using this theorem (this is also called **Cramer's rule**).

A system is said to be **homogeneous** if $\boldsymbol{b} = \mathbf{0}$. In this case, each of the D_is is zero (why?), so that each of the x_is is also 0. If the determinant of the coefficient matrix $\boldsymbol{A}$, i.e., D, is 0, and the system is homogeneous, then Cramer's rule assigns each variable the indeterminate quantity 0/0. However, it can be shown that in this case *the system does, in fact, have non-zero solutions*. This important fact is the point of departure for the computation of the eigenvalues of a matrix.

3.4.6 The inverse of a matrix

The inverse of a square matrix $\boldsymbol{A}$ denoted $\boldsymbol{A}^{-1}$ is a matrix such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

EXAMPLE 10: INVERSE

Prove that the inverse of a matrix is unique

Solution:

If A had an inverse B as well as an inverse $\boldsymbol{C}$, then $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{AC}=\boldsymbol{CA}=\boldsymbol{I}$. So, $\boldsymbol{B} = \boldsymbol{BI} = \boldsymbol{B}(\boldsymbol{AC}) = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{IC} = \boldsymbol{C}$.

■

Not all square matrices are invertible: a matrix that does not have an inverse is called a **singular** matrix. All singular matrices have a determinant of zero. If a matrix is not singular, its inverse is given by:

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}[C_{jk}]^T = \frac{1}{|\boldsymbol{A}|}\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \quad \text{(EQ 12)}$$

where C_{jk} is the co-factor of a_{jk}. Note that the co-factor matrix is transposed when compared with $\boldsymbol{A}$. As a special case, the inverse of a two-by-two matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{is} \quad \boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad \text{(EQ 13)}$$

EXAMPLE 11: INVERSE

Compute the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}$.

The determinant of the matrix is 2*2 - 6*3 = -14. We can use Equation 13 to compute the inverse as:

$$\frac{1}{-14}\begin{bmatrix} 2 & -3 \\ -6 & 2 \end{bmatrix}.$$

■

3.5 Linear transformations, eigenvalues and eigenvectors

This section deals with the important problem of linear transformations and their computation using eigenvalues and eigenvectors.

3.5.1 A matrix as a linear transformation

Recall that the product of an $n \times n$ matrix with an n dimensional column vector—a matrix of size $n \times 1$ —is another column vector of dimension n. We can therefore view the matrix as **transforming** the input vector into the output vector.

Note that the k^{th} element of the output column vector is formed by combining all the elements of the input vector using the weights found in the k^{th} row of the matrix. This is just a *linear combination* of the elements of the input vector. A square matrix that represents (the weights corresponding to) a set of such linear combinations and thus is said to represent a **linear transformation** of the input vector.

EXAMPLE 12: MATRIX AS A LINEAR TRANSFORMATION

Consider the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}$ and an input vector $\begin{bmatrix} a \\ b \end{bmatrix}$. The multiplication of this vector with the matrix, i.e., $\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix}$ is the output vector $\begin{bmatrix} 2a + 3b \\ 6a + 2b \end{bmatrix}$. The first element of the output vector can be thought of as combining the input elements a and b with weights 2 and 3 (the first row of the matrix), and the second element of the output vector can be thought of as combining the inputs with weights 6 and 2 (the second row of the matrix).

■

The definition of matrix multiplication allows us to represent the composition of two linear transformations as a matrix product. Specifically, suppose that the matrix $\boldsymbol{A}$ transforms a column vector $\boldsymbol{x}$ to another column vector $\boldsymbol{x}'$ and that the matrix $\boldsymbol{B}$ is now used to transform $\boldsymbol{x}'$ to another vector $\boldsymbol{x}''$. Then, $\boldsymbol{x}''=\boldsymbol{B}\boldsymbol{x}' = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{C}\boldsymbol{x}$, where $\boldsymbol{C}=\boldsymbol{B}\boldsymbol{A}$ is the product of the two transformation matrices. That is, we can represent the composition of the two transformations as the matrix product of the transformation matrices.

EXAMPLE 13: COMPOSITION OF LINEAR TRANSFORMATIONS

We have already seen that the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}$ transforms a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ to the vector $\begin{bmatrix} 2a + 3b \\ 6a + 2b \end{bmatrix}$. Suppose that we apply the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ to this output. The resultant value is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 2a + 3b \\ 6a + 2b \end{bmatrix} = \begin{bmatrix} 4a + 6b \\ 12a + 4b \end{bmatrix}$. Instead, we can compute the product of the two transformation matrices as $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$. Then, applying this product to the initial input gives us

$\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a + 6b \\ 12a + 4b \end{bmatrix}$, as before.

■

3.5.2 The eigenvalue of a matrix

Consider the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and an input vector $\begin{bmatrix} a \\ b \end{bmatrix}$. The multiplication of this vector with the matrix is the output vector $\begin{bmatrix} 2a \\ 2b \end{bmatrix} = 2*\begin{bmatrix} a \\ b \end{bmatrix}$. Therefore, the matrix represents a *doubling* transformation on its input: the result of applying this matrix to any vector is equivalent to multiplying the vector by the scalar 2.

When the result of a matrix multiplication with a *particular* vector is the same as a scalar multiplication with that vector, we call the scalar an **eigenvalue** of the matrix and the corresponding vector an **eigenvector**. More precisely, we define an eigenvalue of a square matrix $\boldsymbol{A}$ to be a scalar λ such that, for *some* non-zero column vector x,

$$\boldsymbol{Ax} = \lambda \boldsymbol{x} \qquad \textbf{(EQ 14)}$$

The magnitude of an eigenvalue indicates the degree to which the matrix operation scales an eigenvector: the larger the magnitude, the greater the scaling effect.

EXAMPLE 14: EIGENVALUES AND EIGENVECTORS

Compute the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$.

Solution:

Let the eigenvector be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then, $\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Expanding the left hand side, we get a system of two equations:

$$4x_1 + 6x_2 = \lambda x_1$$
$$12x_1 + 4x_2 = \lambda x_2$$

Transposing terms, we get

$$(4-\lambda)x_1 + 6x_2 = 0$$
$$12x_1 + (4-\lambda)x_2 = 0$$

We recognize this as a homogenous system of two linear equations. Recall from Section 3.4.5 on page 86 that this system has a non-zero solution if and only if the determinant of the coefficient matrix is zero. That is,

$$\begin{vmatrix} (4-\lambda) & 6 \\ 12 & (4-\lambda) \end{vmatrix} = 0$$

We can expand this to get the quadratic:

$$(4-\lambda)^2 - 72 = 0$$

which we solve to find:

$$(4-\lambda)^2 = 72$$

$$\lambda = 4 \pm \sqrt{72} = 4 \pm 6\sqrt{2}$$

Given these two eigenvalues, we compute the corresponding eigenvectors as follows. To begin with, we substitute the value $\lambda = 4 \pm 6\sqrt{2}$ in Equation to get:

$$(4 - 4 - 6\sqrt{2})x_1 + 6x_2 = 0$$
$$12x_1 + (4 - 4 - 6\sqrt{2})x_2 = 0$$

$$-6\sqrt{2}x_1 + 6x_2 = 0$$
$$12x_1 - 6\sqrt{2}x_2 = 0$$

which both reduce to the equation

$$\sqrt{2}x_1 - x_2 = 0$$

We have two variables and only one equation, which corresponds to an infinite number of solution eigenvectors, parametrized by a single free variable. This is represented as $\begin{bmatrix} x_1 \\ \sqrt{2}x_1 \end{bmatrix}$. For instance, if we set $x_1 = 1$, then one possible eigenvector is $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$. The set of eigenvectors can also be represented as $a\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, where a is an arbitrary scalar.

As a check, note that $\boldsymbol{Ax} = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 + 6\sqrt{2} \\ 12 + 4\sqrt{2} \end{bmatrix}$ and $\lambda \boldsymbol{x} = (4 + 6\sqrt{7}) * \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 + 6\sqrt{2} \\ 12 + 4\sqrt{2} \end{bmatrix}$.

We interpret this geometrically as follows. Suppose we represent the Cartesian coordinates of a point (x, y) by the vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

Then, the eigenvectors parametrized by x_1 as $\begin{bmatrix} x_1 \\ \sqrt{2}x_1 \end{bmatrix}$ are the set of points that lie on the line $y = \sqrt{2}x$. Points that lie on this line are transformed by the matrix to other points *also on the same line*, because the effect of the matrix on its eigenvector is to act like a scalar, which does not change the direction of a vector. Moreover, the *degree* of scaling is the associated eigenvalue of $4 + \sqrt{72}$. That is, a point a unit distance from the origin and on this line (a unit eigenvector) would be scaled to point on the line that is a distance of $4 + \sqrt{72}$ from the origin. This is shown in Figure 1.

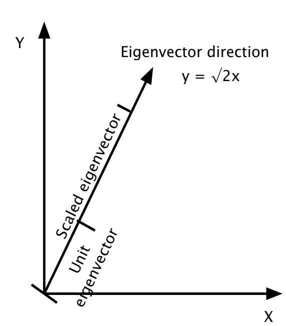


FIGURE 1. The effect of a matrix on an eigenvector is to scale it, preserving its direction.

To obtain the other eigenvectors, we substitute the value $\lambda = 4 - \sqrt{72}$ in Equation to get:

$$(4 - 4 + \sqrt{72})x_1 + 6x_2 = 0$$
$$12x_1 + (4 - 4 + \sqrt{72})x_2 = 0$$

$$\sqrt{72}x_1 + 6x_2 = 0$$
$$12x_1 + \sqrt{72}x_2 = 0$$

This gives us the parametrized eigenvector solution, $\begin{bmatrix} x_1 \\ -\sqrt{2}x_1 \end{bmatrix}$, which can be represented as $a\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$, where a is an arbitrary scalar.

■

3.5.3 Computing the eigenvalues of a matrix

We can compute the eigenvalues and eigenvectors of a matrix by generalizing the method of the previous example. Consider a matrix $\boldsymbol{A}$ with an eigenvector $\boldsymbol{x}$ and a corresponding eigenvalue λ. We know, by definition, that

$$\boldsymbol{Ax} = \lambda\boldsymbol{x}$$

We rewrite this as:

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \qquad \text{(EQ 15)}$$

This is a homogeneous system of equations, so from Section 3.4.5 on page 86 it has non-trivial solutions only if the determinant of the coefficient matrix is zero:

$$|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0 \qquad \text{(EQ 16)}$$

This determinant is called the **characteristic determinant** of the matrix and Equation 16 is called its **characteristic equation**. Expanding the determinant will result, in general, in obtaining a polynomial of the *n*th degree in λ, which is the **characteristic polynomial** of the matrix. As we have seen, the eigenvalues of the matrix are the roots of the characteristic polynomial. This important result allows us to compute the eigenvalues of any matrix. Note also that the value of the characteristic determinant of a matrix does not change if the matrix is transposed. Hence, the eigenvalues of a matrix and its transpose are identical.

In general, the roots of a polynomial of degree n can be real or complex. Moreover, the fundamental theorem of algebra tells us that there is at least one root, and at most n distinct roots. Therefore, a square matrix of degree n has between one and n distinct eigenvalues, some of which may be complex (and will form complex conjugate pairs if the matrix is real). Moreover, some eigenvalues may be repeated. Each eigenvalue corresponds to a family of eigenvectors that are parametrized by at least one free variable.

The set of eigenvalues of a matrix are called its **spectrum**. The largest eigenvalue by magnitude is called the **principal eigenvalue** or the **spectral radius** of the matrix. Each eigenvalue corresponds to a family of eigenvectors. It can be shown that the set of eigenvectors corresponding to a set of *distinct* eigenvalues are always linearly independent of each other. The set of all vectors that can be expressed as a linear combination of the eigenvectors is called the **eigenspace** of the matrix.

EXAMPLE 15: COMPLEX EIGENVALUES

Let us compute eigenvalues for the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We do so by setting the characteristic determinant to 0:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

This gives us the characteristic equation:

$$\lambda^2 + 1 = 0$$

so that $\lambda = \pm i$. This is the spectrum of the matrix and the spectral radius is 1.

We find the eigenvector corresponding to $\lambda = i$ by setting

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = i \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = ix_1$$
$$-x_1 = ix_2$$

which is a set of equations with rank 1 (that is, the second equation just restates the first one). One possible vector that satisfies this is $\begin{bmatrix} 1 \\ i \end{bmatrix}$. To check this, note that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix}$. The eigenvector corresponding to the eigenvalue -i can be similarly found to be $\begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix}$.

Because both eigenvector families are complex, the matrix never leaves the direction of a real-valued vector unchanged. Indeed, the matrix corresponds to a rotation by 90 degrees, so this is expected. What is unexpected that the rotation matrix does leave the 'direction' of a complex-valued vector unchanged, which has no obvious intuitive explanation.

■

EXAMPLE 16: EIGENVALUES OF A DIAGONAL MATRIX

Consider a diagonal matrix, such as $\boldsymbol{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. What are its eigenvalues?

The characteristic equation is

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix} = 0$$

A little work shows that this reduces to the equation:

$$(a-\lambda)(b-\lambda)(c-\lambda) = 0$$

which shows that the eigenvalues are simply the diagonal elements a, b, and c. This generalizes: we can read off the eigenvalues as the diagonal elements of any diagonal matrix.

■

3.5.4 Why are eigenvalues important?

The eigenvalues of a matrix become important when we consider the *repeated* application of a transformation to an input vector. Suppose we represent the state of a system by a vector, where each element of the vector corresponds to some aspect of the system, such as the buffer occupancy level (this is discussed at greater length in Section 8.3.2 on page 226). Suppose further that we can represent the transformation of the state in one time step as being equivalent to the application of a state transformation operator (and the equivalent matrix) on the state vector. Then, the 'steady-state' or eventual state of the system can be obtained by the repeated application of the transformation matrix on the initial vector. It turns out that the eigenvalues of the transformation matrix can be used to characterize the steady state.

To see this, first consider the repeated application of a matrix $\boldsymbol{A}$ to its eigenvector $\boldsymbol{x}$ corresponding to an eigenvalue λ. By definition, a single application of $\boldsymbol{A}$ to $\boldsymbol{x} = \boldsymbol{Ax} = \lambda\boldsymbol{x}$. Applying $\boldsymbol{A}$ n times therefore reduces to computing $\lambda^n\boldsymbol{x}$, which is far simpler!

Now, consider an initial state vector v that can be represented as a linear combination of two eigenvectors of $\boldsymbol{A}$, say $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$ like so:

$$\boldsymbol{v} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$$

Suppose these eigenvectors correspond to the eigenvalues λ_1 and λ_2. Then, $\boldsymbol{A}^n v$ can be found as follows:

$$\boldsymbol{A}^n\boldsymbol{v} = \boldsymbol{A}^n(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = c_1\boldsymbol{A}^n\boldsymbol{x}_1 + c_2\boldsymbol{A}^n\boldsymbol{x}_2 = c_1\lambda_1^n\boldsymbol{x}_1 + c_2\lambda_2^n\boldsymbol{x}_2$$

We see that the repeated application of A on v, which is a complex operation, is replaced by the far simpler operation of raising a scalar to the nth power.

This intuition is easily generalized. If we can represent the initial vector as the linear combination of the eigenvectors of a matrix, then the repeated application of the matrix can be found with little effort, as the next example shows.

EXAMPLE 17: COMPUTING $A^N x$ USING EIGENVECTORS

From the previous example, we know that the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has two eigenvalues i, and $-i$, with corresponding unit eigenvectors $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$. Consider a state vector $\begin{bmatrix} 10 \\ 0 \end{bmatrix}$. We can represent this as $5^*\begin{bmatrix} 1 \\ i \end{bmatrix} + 5^*\begin{bmatrix} 1 \\ -i \end{bmatrix}$. Therefore, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{100}\begin{bmatrix} 10 \\ 0 \end{bmatrix}$ (which is computationally complex) reduces to computing $5^*i^{100}*\begin{bmatrix} 1 \\ i \end{bmatrix} + 5^*i^{100}*\begin{bmatrix} 1 \\ -i \end{bmatrix} = 5^*\begin{bmatrix} 1 \\ i \end{bmatrix} + 5^*\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$. Geometrically, this makes sense, because applying $\boldsymbol{A}$ once corresponds to a 90 degree rotation, so applying it for any multiple of four times will leave the initial vector unchanged.

■

This method of computing the effect of a repeated linear transformation is useful when the initial vector can be written as a linear combination of the eigenvectors. In this context the following fact is useful: the eigenvectors corresponding to a set of *distinct* eigenvalues form a **linearly independent set**. We have already see that any set of n linearly independent complex vectors form a basis for the complex hyperspace C^n. Hence, if a square matrix of order n has n distinct eigenvalues, we can express any initial vector as a linear combination of its eigenvectors. This is one reason why knowing the eigenvalues of a matrix is very useful.

What if the initial state vector cannot be written as a linear combination of the eigenvectors? That is, what if the initial state vector is not in the eigenspace of the matrix. It turns out that a somewhat more complex computation, still involving the eigenvalues, allows us to compute $\boldsymbol{A}^n\boldsymbol{x}$. The details are, however, beyond the scope of this text.

3.5.5 The role of the principal eigenvalue

Consider a matrix $\boldsymbol{A}$ that has a set of m eigenvalues λ_i, i = 1,2, ..., m with corresponding unit eigenvectors $\boldsymbol{x}_i$. Suppose that we choose a vector $\boldsymbol{v}$ in the eigenspace of the matrix such that it is expressed as a linear combination of *all* the eigenvectors:

$$\boldsymbol{v} = \sum_{i=1}^{m} c_i\boldsymbol{x}_i$$

Then, n applications of the matrix to this vector results in the vector:

$$\sum_{i=1}^{m} c_i\lambda_i^n\boldsymbol{x}_i$$

As $n \to \infty$, this sum is dominated by the eigenvalue that has the largest magnitude, which is called the **principal** or **dominant** eigenvalue (if more than one eigenvalue has the same magnitude, they are both considered to be the principal eigenvalues). To first approximation, we can ignore all the other eigenvalues in computing the limit.

EXAMPLE 18: PRINCIPAL EIGENVALUE

Consider the matrix $A = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$. Recall from Example 14 that it has two eigenvalues, $\lambda = 4 \pm \sqrt{72}$, which evaluate to 12.48 and -4.48. The unit eigenvectors of this matrix are $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$. Suppose we start with an initial vector $\begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix}$ which we can express as $2^*\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} - 2^*\begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$. Then, $A^{10}\begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix} = 2^*12.48^{10}{}^*\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} - 2^*(-4.48)^{10}{}^*\begin{bmatrix} 1 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$. This evaluates to $\begin{bmatrix} 1.83^*10^{11} \\ 2.59^*10^{11} \end{bmatrix}$. If we ignore the second term (i.e, the contribution due to the eigenvalue -4.48), then the resulting value changes only in the third decimal place of precision (check this!). It is clear that the dominant eigenvalue is the one that matters.

■

3.5.6 Finding eigenvalues and eigenvectors

Computing the eigenvalues and the eigenvectors of a matrix is important in practice. Here are some facts that help identify the nature of the eigenvalues of a matrix.

- If a matrix is square and diagonal, then its eigenvalues are its diagonal elements.
- If a matrix is square and symmetric (that it, $\boldsymbol{A}^T = \boldsymbol{A}$), then its eigenvalues are real.
- **Gerschgorin's 'circle' theorem** states that all the eigenvalues of a complex matrix lie in the set of disks (on the complex plane) centered on the elements of the diagonal, with a radius equal to the sum of the magnitudes of the off-diagonal elements. Intuitively, if the off-diagonal elements are 'not too large,' then the eigenvalues of the matrix are its diagonal elements.

EXAMPLE 19: FINDING EIGENVALUES

The matrix $\boldsymbol{A} = \begin{bmatrix} 9.1 & 0.8 & 0.3 \\ 0.8 & 5.8 & 0.2 \\ 0.3 & 0.2 & 6.5 \end{bmatrix}$ is symmetric. Hence, its eigenvalues are real.

It has three Gerschgorin disks: (1) Center 9.1 + 0*i*, radius 1.1 (2) Center 5.8 + 0*i* and radius 1.0 (3) Center 6.5+0*i* and radius 0.5. Because the eigenvalues are real, they must lie in one of three intervals: [8, 10.2], [4.8, 6.8], and [6, 7]. The second interval overlaps the third, so we know that the eigenvalues lie either in the interval [4.8 7] or the interval [8 10.2].

■

It is possible to approximately compute the dominant[2] eigenvalue of a matrix using the **power** method. In this technique, we start with an *arbitrary* initial vector $\boldsymbol{x}_0$ and repeatedly apply $\boldsymbol{A}$ to it. At each step, we compute the **Rayleigh ratio**

$\dfrac{\boldsymbol{x}_k^T \boldsymbol{A} \boldsymbol{x}_k}{\boldsymbol{x}_k^T \boldsymbol{x}_k} = \dfrac{\boldsymbol{x}^T \boldsymbol{x}_{k+1}}{\boldsymbol{x}_k^T \boldsymbol{x}_k}$, which converges towards the dominant eigenvalue of the matrix. The intuitive idea is that applying $\boldsymbol{A}$ to any vector scales it in multiple dimensions, but the dominant eigenvalue dominates the scaling effect. Repeatedly applying $\boldsymbol{A}$ magnifies the contribution of the dominant eigenvalue, exposing it.

EXAMPLE 20: POWER METHOD TO COMPUTE THE DOMINANT EIGENVALUE

Use the power method to compute the dominant eigenvalue of the matrix $\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$.

Solution:

Suppose we arbitrarily start with the initial vector $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Applying the matrix once, we get $\boldsymbol{x}_1 = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$. The Rayleigh ratio evaluates to ([1 1] * $\begin{bmatrix} 10 \\ 16 \end{bmatrix}$)/([1 1] * $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$) = 26/2 = 13. Repeating, we get $x_2 = \begin{bmatrix} 136 \\ 184 \end{bmatrix}$ and the corresponding Rayleigh ratio evaluates to 4304/356 = 12.08. After one more iteration, we get $x_3 = \begin{bmatrix} 1648 \\ 2368 \end{bmatrix}$, and the Rayleigh ratio evaluates to 659840/52352 = 12.60. Recall from Example 18 that the dominant eigenvalue is 12.48. We obtain an approximation to within 1% of this value in only three iterations.

2. Usually, but not necessarily the dominant eigenvalue. It turns out that in nearly all practical cases, this is the one that will be found.

■

It can be shown the speed of convergence of this method depends on the ‘gap’ between the dominant and second-largest eigenvalue. The bigger the gap, the faster the convergence. Intuitively, if the second-largest eigenvalue is close in magnitude to the dominant eigenvalue, then its scaling effects do not die down easily, requiring more iterations.

The power method can also be used to find the dominant *eigenvector.* To do so, after each iteration, the vector x_i must be *scaled*. That is, we set its largest element to 1 by dividing each element by the largest element, as demonstrated next.

EXAMPLE 21: POWER METHOD FOR COMPUTING THE DOMINANT EIGENVECTOR

Compute the dominant eigenvector of the matrix $\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$ using the power method.

Solution:

From Example 20, we already know that $\boldsymbol{x}_1 = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$. We rescale it by dividing each element by 16 to get the vector $\begin{bmatrix} 0.625 \\ 1 \end{bmatrix}$. Using this as the new value of $\boldsymbol{x}_1$, we get $\boldsymbol{x}_2 = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix} * \begin{bmatrix} 0.625 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.5 \\ 11.5 \end{bmatrix}$. We rescale this again to get $\boldsymbol{x}_2 = \begin{bmatrix} 0.739 \\ 1 \end{bmatrix}$. This allows us to compute $\boldsymbol{x}_3 = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix} * \begin{bmatrix} 0.739 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.956 \\ 12.868 \end{bmatrix}$ which is rescaled to $\begin{bmatrix} 0.696 \\ 1 \end{bmatrix}$. Recall that the eigenvector for this eigenvalue is exactly $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, which we scale to $\begin{bmatrix} 0.707 \\ 1 \end{bmatrix}$. As before, with just three iterations, we are within 1.5% of this final value.

■

3.5.7 Similarity and diagonalization

Two matrices are said to be **similar** if they have the same set of eigenvalues. In some cases, given a matrix $\boldsymbol{A}$, it is useful to be able to compute a similar **diagonal** matrix $\boldsymbol{D}$ (we show a use case in Example 22).

A sufficient, but not necessary, condition for a matrix of size n to be diagonalizable is that it has n distinct eigenvalues (for instance, consider a diagonal matrix with repeated diagonal elements). In this case, let $\boldsymbol{X}$ denote a matrix whose columns are the eigenvectors of $\boldsymbol{A}$. Then, it can be easily shown, by expansion of the underlying terms, that the matrix

$$\boldsymbol{D} = \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} \tag{EQ 17}$$

is a diagonal matrix whose diagonal elements *are the eigenvalues of* $\boldsymbol{A}$.

Knowing the diagonalized version of a matrix makes it trivial to compute its m^{th} power. From Equation 17, note that

$$\boldsymbol{D}^2 = (\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X})(\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X}) = \boldsymbol{X}^{-1}\boldsymbol{A}^2\boldsymbol{X}$$

A simple induction shows that

$$\boldsymbol{D}^m = \boldsymbol{X}^{-1}\boldsymbol{A}^m\boldsymbol{X}$$

so that

$$A^m = XD^mX^{-1}$$

But the right hand side is easily computed because $\boldsymbol{D}$ is diagonal. Hence, we can easily compute $\boldsymbol{A}^m$.

EXAMPLE 22: DIAGONALIZATION

Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}$. Recall from Example 14 that it has two eigenvalues, $\lambda = 4 \pm 6\sqrt{2}$ corresponding to the eigenvectors $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$. This allows us to write out the matrix $\boldsymbol{X}$ as $\begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$. From Equation 13 we find that $\boldsymbol{X}^{-1}$ = $\frac{1}{(-\sqrt{2}-\sqrt{2})}\begin{bmatrix} -\sqrt{2} & -1 \\ -\sqrt{2} & 1 \end{bmatrix}$. Therefore, we diagonalize $\boldsymbol{A}$ as

$$\frac{1}{-2\sqrt{2}}\begin{bmatrix} -\sqrt{2} & -1 \\ -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 1 \\ \sqrt{2} & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4+6\sqrt{2} & 0 \\ 0 & 4-6\sqrt{2} \end{bmatrix}$$

Note that the diagonal elements of $\boldsymbol{A}$ are its eigenvalues.

Given this transformation, we compute $\boldsymbol{A}^5$ as:

$$XD^mX^{-1} = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 4+6\sqrt{2} & 0 \\ 0 & 4-6\sqrt{2} \end{bmatrix}^5 \frac{1}{-2\sqrt{2}}\begin{bmatrix} -\sqrt{2} & -1 \\ -\sqrt{2} & 1 \end{bmatrix}$$

Due to diagonalization, the matrix power computation reduces to computing the exponential of a scalar value. After simplification (and maintaining 10 digits of precision in the calculations), this reduces to:

$$\begin{bmatrix} 150783.99 & 107903.99 \\ 215807.99 & 150783.99 \end{bmatrix}$$

which is within rounding error of the true value of:

$$\begin{bmatrix} 150784 & 107904 \\ 215808 & 150784 \end{bmatrix}$$

■

If the matrix has repeated eigenvalues, then it cannot be diagonalized. Instead, the best we can do is to put it in the **Jordan Canonical Form**. In this form, the diagonal elements are the eigenvalues of the matrix, as with a diagonal matrix. However, some elements immediately above the main diagonal may also be 1s. More details on the computation of this form can be found in advanced texts on linear algebra.

3.6 Stochastic matrices

We now turn our attention to a special type of matrix called a **stochastic matrix**. A **right stochastic matrix** is a square matrix whose elements are non-negative reals and each of whose *rows* sums to 1. A **left stochastic matrix** is a square matrix whose elements are non-negative reals and each of whose *columns* sums to 1. Stochastic matrices are also called **Markov matrices**. Unless otherwise specified, when we refer to a ‘stochastic matrix,’ we will refer to a *right* stochastic matrix.

Stochastic matrices are important in the context of computer networking because each row of such a matrix $\boldsymbol{A}$ corresponds to the state of a finite state machine (or Markov chain) representing a networking protocol or a buffer in a router. Each element a_{ij} can be viewed as the probability of entering state j from state i. The summation criterion expresses the fact that the result of a transition from a state is to either remain in the same state or to go to some other state. Stochastic matrices arise frequently in the study of Markov chains, stochastic processes, and in queueing theory.

EXAMPLE 23: STOCHASTIC MATRIX

The matrix $\boldsymbol{A} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.1 & 0.9 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$ is a stochastic matrix because it is square, the elements are non-negative reals, and each row sums to 1. We interpret row 1 to mean that if the system is in state 1, then after one transition, it remains in state 1 with probability 0.25, goes to state 2 with probability 0.5, and goes to state 3 with probability 0.25. Note that if the system enters state 3, then it can never leave that state (Why?). We call such a state an **absorbing** state.

■

3.6.1 Computing state transitions using a stochastic matrix

Consider a $n \times n$ stochastic matrix $\boldsymbol{A}$ and a column vector $\boldsymbol{p}$ having dimension n with non-negative real elements such that its elements sum to 1. We think of the ith element of $\boldsymbol{p}$ as representing the probability of being in state i at some point in time. Then $\boldsymbol{p}' = \boldsymbol{A}^T\boldsymbol{p}$ is a vector whose ith element is the probability of being in state i after one transition (note the pre-multiplication not by $\boldsymbol{A}$, but by its transpose). This is because the ith element of $\boldsymbol{p}'$ is given by $p'_i = \sum_{k=1}^{n} p_k a_{ki}$, which is total probability of being in state i after the transition, conditioning on the probability of being in each prior state k.

EXAMPLE 24: STATE TRANSITIONS

Continuing with the stochastic matrix from Example 23, suppose that we start with the system in state 1. What is the probability of being in state 1 after one and two transitions?

Solution:

The initial state vector is $\boldsymbol{p} = [1.0\ 0\ 0]^T$. After one transition, the state vector is given by $\boldsymbol{p} = \boldsymbol{A}^T\boldsymbol{p} =$

$$\begin{bmatrix} 0.25 & 0.1 & 0 \\ 0.5 & 0.9 & 0 \\ 0.25 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

and after two transitions $\boldsymbol{p} = \boldsymbol{A}^T(\boldsymbol{A}^T\boldsymbol{p}) = (\boldsymbol{A}^T)^2\boldsymbol{p} =$

$$\begin{bmatrix} 0.1125 & 0.115 & 0 \\ 0.575 & 0.86 & 0 \\ 0.3125 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1125 \\ 0.575 \\ 0.3125 \end{bmatrix}$$

Thus, after two transitions, the system is in state 1 with probability 0.1125. Note that this probability is larger than the simple probability of staying in state 1 (which is just 0.25 * 0.25 = 0.0625), because it takes into account the probability of transitioning from state 1 to state 2 and then back from state 2 to state 1, which has an additional probability of 0.5*0.1 = 0.05.

■

As this example shows, if $\boldsymbol{A}$ is a stochastic matrix, then the $[i,j]^{\text{th}}$ element of $(\boldsymbol{A}^T)^2$ represents the probability of going from state i to state j in two steps. Generalizing, the probability of going from state i to state j in k steps is given by $(\boldsymbol{A}^T)^k$.

3.6.2 Eigenvalues of a stochastic matrix

We now present three important results concerning stochastic matrices.

First, *every* stochastic matrix has an eigenvalue of 1. To prove this, consider the $n \times n$ stochastic matrix A and column vector $\boldsymbol{x} = [1\ 1\ ...\ 1]^T$. Then, $\boldsymbol{Ax} = \boldsymbol{x}$, because each element of $\boldsymbol{Ax}$ multiplies 1 with the sum of a row of $\boldsymbol{A}$, which, by definition, sums to 1. Because a matrix and its transpose have the same eigenvalues, the transpose of a stochastic matrix also has an eigenvalue of 1. However, the eigenvector of the transposed matrix corresponding to this eigenvalue need not be (and rarely is) the **1** vector.

Second, every (possibly complex) eigenvalue of a stochastic matrix must have a magnitude no greater than 1. To prove this, consider some diagonal element a_{jj}. Suppose this element takes the value x. Then, by definition of a stochastic matrix, it must be the case that the sum of the off-diagonal elements is 1-x. From Gerschgorin's circle theorem, we know that all the eigenvalues lie within a circle in the complex plane centered at x with radius 1-x. The largest magnitude eigenvalue will be a point on this circle (see Figure 2). Although the truth of the proposition is now evident by inspection, we now formally prove the result.

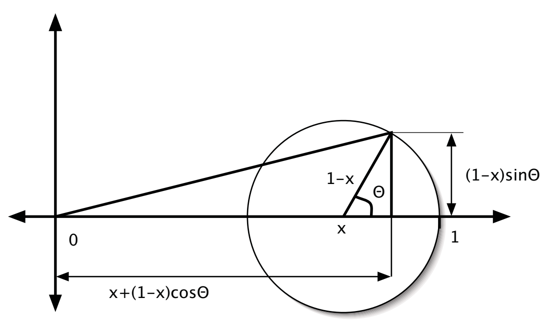


FIGURE 2. Largest possible eigenvalue of a stochastic matrix

Suppose that this point subtends an angle of θ. Then, its coordinates are $(x+(1-x)\cos\theta, (1-x)\sin\theta)$. Therefore, its magnitude is $((x+(1-x)\cos\theta)^2 + ((1-x)\sin\theta)^2)^{1/2}$, which simplifies to $(x^2 + (1-x)^2 + 2x(1-x)\cos\theta)^{1/2}$. This quantity is maximized when $\theta = 0$ so that we merely have to maximize the quantity $x^2 + (1-x)^2 + 2x(1-x)$. Taking the first derivative with respect to x and setting it to zero shows that this expression reaches its maximum of 1 independent of the value of x. So, we can pick x to be a convenient value, such as 1. Substituting $\theta = 0$ and $x = 1$ into $((x+(1-x)\cos\theta)^2 + ((1-x)\sin\theta)^2)^{1/2}$, we find that the magnitude of the maximum eigenvalue is 1.

Third, it can also be shown, although the proof is beyond the scope of this text, that under some mild assumptions[3], only *one* of the eigenvalues of a stochastic matrix is 1, which also must therefore be its dominant eigenvalue.

These three facts lead to a remarkable insight. Recall that we compute one step in the state evolution of a stochastic system described by the transition matrix $\boldsymbol{A}$ by multiplying the state vector by $\boldsymbol{A}^T$. We already know that (under some mild assumptions) every stochastic transition matrix $\boldsymbol{A}$ has a unique eigenvalue of 1. We also know that a matrix and its transpose have the same eigenvalues. Therefore, $\boldsymbol{A}^T$ also has a unique eigenvalue of 1. What if the state vector was the eigenvector of $\boldsymbol{A}^T$corresponding to this eigenvalue? Then, one step in the state evolution would leave the corresponding eigenvector *unchanged*. That is, a system in a state corresponding to that eigenvector would never leave that state. We denote this special state by the vector π. It is also called the **stationary probability distribution** of the system. π can be found by solving the system of linear equations given by

$$A^T\pi = \pi \qquad \text{(EQ 18)}$$

Because every stochastic matrix has an eigenvalue of 1, every stochastic system must have a unique stationary probability distribution, which is the eigenvector corresponding to the unique eigenvalue of 1. Instead of solving a system of linear equations, we can also compute the stationary probability distribution of $\boldsymbol{A}$ using the power method because 1 is also its *dominant* eigenvalue. We put these results together to state that *the power method can be used to compute the stationary probability distribution of a stochastic matrix.* In our study of queueing theory, we will study the conditions on the matrix $\boldsymbol{A}$ which guarantee that its stationary probability distribution is reached independent of the initial state of the system. Roughly speaking, these are the conditions under which the matrix $\boldsymbol{A}^T$ is said to be **ergodic**.

EXAMPLE 25: GOOGLE PAGE RANK ALGORITHM

The power technique of finding the dominant eigenvector of a stochastic matrix can be used to rank a set of web pages. More precisely, given a set of web pages, we would like to identify certain pages as being more important than others. A page can be considered to be important using the recursive definition that (a) many other pages point to it and (b) the other pages are also important.

The importance of a page can be quantified according the actions of a ‘random web surfer’ who goes from web page i to a linked web page j with probability a_{ij}. If a page is ‘important,’ then a random web surfer will be led to that page more often than to other less-important pages. That is, we consider a population of a large number of surfers, then a larger fraction of web surfers will be at a more important page, compared to a less important page. Treating the ratio of the number of web surfers at a page to the total number of surfers as approximating a probability, we see that the importance of a page is just the stationary probability of being at that page.

To make matters more precise, let the matrix $\boldsymbol{A}$ represent the set of all possible transition probabilities. If the probability of the surfer being at page i at some point is p_i then the probability that the surfer is at page i after one time step is $\boldsymbol{A}^T p_i$. The dominant eigenvector of $\boldsymbol{A}^T$ is then the ‘steady state’ probability of a surfer being at page i. Given that $\boldsymbol{A}$ is a stochastic matrix, we know that this dominant eigenvector exists, and that it can be found by the power method.

What remains is to estimate the quantities a_{ij}. Suppose page i has links to k pages. Then, we set $a_{ij} = 1/k$ for each page j to which it has a link, and set $a_{ij} = 0$ for all other j. This models a surfer going from a page uniformly randomly to one of its linked pages. What if a page has no links? Or if two pages link only to each other? These issues can be approximately modelled by assuming that, with constant probability, the surfer ‘teleports’ to a randomly chosen page. That is, if there is a link from page i to page j, $a_{ij} = \alpha/n + (1-\alpha)/k$, where α is a control parameter; otherwise $a_{ij} = \alpha/n$. It can be easily shown that these modified a_{ij}s form a stochastic matrix, so that we can extract the dominant eigenvalue, and thus the page rank, using the power method. A slightly modified version of this algorithm is the publicly-described algorithm used by Google to rank web pages[4].

■

3. These assumptions eliminate matrices such as the identity matrix, which is a degenerate Markov matrix in that all its states are absorbing states.
4. A rather curious fact is that Google’s algorithm to rank pages, called the **Page rank** algorithm, was developed, in part, by its co-founder, Larry Page.

3.7 Exercises

1 Transpose

Compute the transpose of $\begin{bmatrix} 4 & 0 & -3 \\ 7 & 82 & 12 \\ 3 & -2 & 2 \end{bmatrix}$.

2 Matrix multiplications

Find the product of the matrices $\begin{bmatrix} 10 & -4 & 12 \\ -5 & 3 & 9 \\ 8 & 0 & -4 \end{bmatrix}$ and $\begin{bmatrix} -4 & 5 & 3 \\ 7 & -2 & 9 \\ 2 & 5 & -4 \end{bmatrix}$.

3 Exponentiation

Prove that if $\boldsymbol{A}$ is a diagonal matrix the $(i,i)^{\text{th}}$ element of the kth power of $\boldsymbol{A}$ is a_{ii}^k.

4 Linear combination of scalars

Compute the linear combination of the scalars 10, 5, 2, -4 with weights 0.5, 0.4, 0.25, 0.25.

5 Linear combination of vectors

Compute the linear combination of the vectors [1 2 8 5], [3 7 3 1], [7 2 1 9], [2 6 3 4] with weights 0.5, 0.4, 0.25, 0.25.

6 Linear independence and rank

Are the three vectors:

$\mathbf{x}_1$ = [12 2 -4]

$\mathbf{x}_2$ = [2 2 -24]

$\mathbf{x}_3$ = [2.5 0 5]

independent? Determine this from the rank of the corresponding coefficient matrix.

7 Basis and dimension

Give two possible bases for the three vectors in Exercise 6. What is the dimension of the vector space generated by these bases?

8 Gaussian elimination

Use row operations and Gaussian elimination to solve the system given by $\begin{bmatrix} 6 & 4 & -8 & 5 \\ -8 & 2 & 4 & -2 \\ 10 & 0 & 4 & 1 \end{bmatrix}$.

9 Rank

Prove that the rank of an $n \mathrm{x} n$ non-zero diagonal matrix is n.

10 Determinant

Compute the determinant of the matrix $\begin{bmatrix} 4 & 0 & -3 \\ 7 & 8 & 12 \\ 3 & -2 & 2 \end{bmatrix}$.

11 Inverse

Compute the inverse of the matrix $\begin{bmatrix} 4 & 0 & -3 \\ 7 & 8 & 12 \\ 3 & -2 & 2 \end{bmatrix}$.

12 Matrix as a transformation

Using the fact that sin(A+B) = sin(A)cos(B) + cos(A)sin(B) and cos(A+B) = cos(A)cos(B) - sin(A)sin(B), compute the matrix that corresponds to the rotation of a vector joining the origin to the point (x, y) by an angle p.

13 Composing transformations

Compute the composition of a rotation of t degrees followed by a rotation of p degrees.

14 Eigenvalues and eigenvectors

Compute the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}$.

15 Computing $A^n x$

Find the value of $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}^5 \begin{bmatrix} 8 \\ 0 \end{bmatrix}$.

16 Finding eigenvalues

Bound the interval(s) in which the eigenvalues of the matrix $\begin{bmatrix} 4 & 1 & 0.5 \\ 1 & 6 & 0.3 \\ 0.5 & 0.3 & 5 \end{bmatrix}$ lie.

17 Power method

Use the power method to compute the dominant eigenvalue and corresponding eigenvector of the matrix $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}$.

Iterate four times.

18 Diagonlization

What is the diagonal matrix similar to $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}$?

19 Stochastic matrix

Is the matrix $\begin{bmatrix} 0.1 & 0.8 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.3 \end{bmatrix}$ stochastic?

20 State transitions

Consider a system described by the stochastic matrix $\begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.1 & 0.9 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$. Let the ith row of this matrix correspond to state i. If the initial state is known to be state 1 with probability 0.5 and state 2 with probability 0.5, compute the probability of being in these two states after two time steps.

CHAPTER 4 *Optimization*

This chapter presents an overview of optimization: mathematical techniques that can be used to improve the performance of a computer system. We begin with an overview of techniques for mathematically modelling a system. We then survey the elements of linear optimization, including linear programming and dynamic programming, concluding with an introduction to some techniques for non-linear optimization.

4.1 System modelling and optimization

A necessary prerequisite to the use of optimization techniques is to **mathematically model** a system. In doing so, it is necessary to identify the following five elements:

1. The **fixed parameters**. These are aspects of the system that cannot be changed, and therefore, from the perspective of the model, are constants.
2. The **control parameters**. These are the "tuning knobs" or settings that can be chosen to optimize the behaviour of the system. Control parameters are typically constrained to lie within some range. A set of control parameters where each parameter is within its valid range is called a **feasible set** of control parameters.
3. **Input variables**. These are external and uncontrollable inputs to the system. Note that when studying a *particular* instance of a general model, an input variable can be considered to be a fixed parameter. This is the assumption made in a typical optimization problem. When the inputs may vary over time, then the system is better described using control theory, which is the topic of Chapter 8. Mathematical modelling from the perspective of control theory can be found in Section 8.3 on page 225.
4. **Output variables**. These are the observable outputs of the system. A subset of the output variables are chosen as **performance metrics** to quantify the performance of the system.
5. The **transfer function**. Intuitively, this is a function that maps from fixed parameters, control parameters, and input variables to output variables. Control theory lets us precisely define a transfer function as the Laplace transform of the impulse response function. This is discussed in Section 8.3.2 on page 226.

Optimization is the process of choosing a feasible set of control parameters so that an **objective function** O defined over the output variables is either maximized or minimized. We use the transfer function to rewrite the objective function as a mathematical function of the fixed parameters, the control parameters and the input variables. When studying a specific optimization instance, as is the case in this chapter, the fixed parameters and input variables can be considered to be system-defined constants. Therefore, the objective function is typically represented as a function whose variables are the control parameters and whose constants are the fixed parameters and input variables.

EXAMPLE 1: MATHEMATICAL MODELLING:

Consider a communication path from a source to a destination that has a capacity of 100 packets/s so that if packets are sent at a rate $r > 100$ packets/s, packets are dropped at the *drop rate* of (r-100) packets/s. How fast should a source send to maximize its performance, if this performance is quantified as the difference between the carried load and the drop rate?

Solution

We model the system as follows:

- *Fixed parameter*: capacity of the path, i.e., 100 packets/s
- *Control parameter:* the source sending rate
- *Input variables*: none
- *Output variables*: carried load, drop rate
- *Objective function:* carried load - drop rate
- *Transfer function*:

$$\textit{carried load} = \min(\textit{sending rate}, 100) \text{ packets/s}$$

$$\textit{drop rate} = \max(0, \textit{sending rate} - 100) \text{ packets/s}$$

The objective function can be written in terms of the control parameters and fixed parameters as:

$$\begin{aligned} \textit{objective function} &= \textit{carried load} - \textit{drop rate} \\ &= \min(\textit{sending rate}, 100) - \max(0, \textit{sending rate}-100) \text{ packets/s} \end{aligned}$$

which is shown in Figure 1. The objective function is maximized when the sending rate is 100 packets/s.

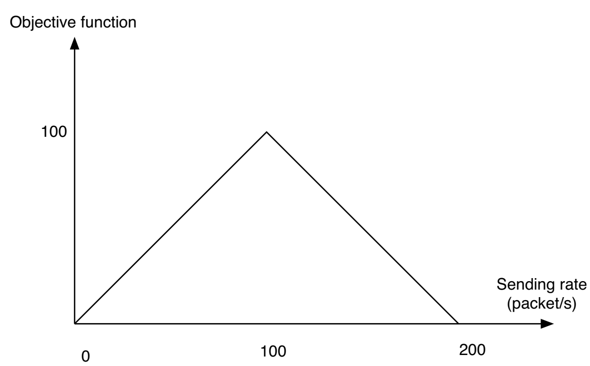


FIGURE 1. The objective function for Example 1. The maximum occurs when the control variable is set to 100 packets/s.

■

Note that the mathematical model in Example 1 was easily derived because the underlying system was trivial. In practice, determining an appropriate mathematical model for a complex system is an art that is only learned with experience and a considerable degree of trial and error. Note also that we could easily graph the objective function because it depended only on one control parameter., the sending rate. We cannot easily graph more complex systems that may have many hundreds of control parameters. For such systems, we must resort to a more sophisticated mathematical analysis, which is the topic of the remainder of this chapter.

4.2 An introduction to optimization

We present an intuitive approach to optimization with the following example.

EXAMPLE 2: OPTIMIZING A SYSTEM WITH TWO CONTROL PARAMETERS

Consider a system whose objective function O can be expressed in terms of two scalar control parameters x_1 and x_2 as:

$$O = 2x_1 - x_2$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

We can obtain the maximal value of O analytically as follows. The partial derivative of O with respect to x_1 is positive and with respect to x_2 is negative. Hence, O increases when x_1 increases and decreases when x_2 increases. Therefore, to maximize O, we should set x_2 to 0 which is its smallest possible value. This implies that $x_1 = 1$, and $O = 2$.

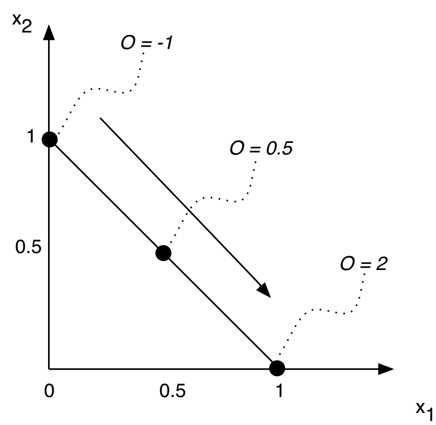


FIGURE 2. Maximizing a function of two variables.

Geometrically, as shown in Figure 2, we interpret the constraint on the sum of the x_i s to mean that their permissible values lie on a line defined by the points (0,1) and (1,0). At (0,1), O is -1. As we move down the line, x_1 increases and x_2 simultaneously decreases, so that O monotonically increases, reaching its maximum value of 2 at (1,0).

■

EXAMPLE 3: OPTIMIZING A SYSTEM WITH THREE VARIABLES

Example 2 can be generalized to three variables as follows. Consider the system whose objective function O can be expressed as:

$$O = 3x_1 - x_2 - x_3$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Analytically, we see that the partial derivative of O with respect to x_1 is positive and with respect to x_2 and x_3 is negative. Therefore, O attains its maximum when x_1 is as large as possible and x_2 and x_3 are as small as possible. This is at the point (1,0,0), where O attains the value 3.

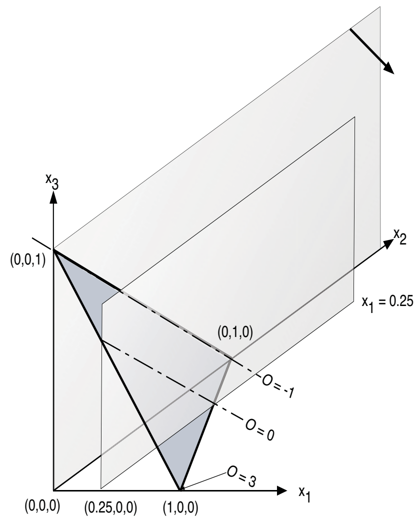


FIGURE 3. **Maximizing a function of three variables The constraint plane is shown with a bold outline.**

The geometric interpretation is shown in Figure 3. Note that the constraints are of two types. Equality constraints force the solution to lie on a plane. For example, the first equality constraint requires the x_is to lie on a **constraint plane** defined by the three points (0,0,1), (0,1,0), and (1,0,0). In contrast, inequality constraints force the solution to *one side* of a constraint plane. For example, the constraint $x_1 \geq 0$ defines a constraint plane along the x_2 and x_3 axes and the solution must lie to the right of this plane. A solution that meets all the constraints must, therefore, lie somewhere in the triangular sub-plane shaded in Figure 3.

Consider a point in this region that lies on the line defined by the two points (0,0,1) and (0,1,0). This point is at the intersection of the two constraint equations: $x_1 + x_2 + x_3 = 1$ and $x_1 = 0$. Therefore, at every point on this line, $x_2 + x_3 = 1$, which implies that $O = 3x_1 - x_2 - x_3 = 0 - 1 = -1$ at every point on this line. We call such a line along which the objective function is constant an **isoquant**.

Now, consider a point that lies at the intersection of the shaded triangular plane and the plane defined by the equation $x_1 = 0.25$. At every point on this line, $x_2 + x_3 = 1 - 0.25 = 0.75$. Therefore, at every point on this line, $O = 3x_1 - x_2 - x_3 = 0.75 - 0.75 = 0$. This is another isoquant of the system.

Continuing in this manner, we can compute an isoquant for each plane defined by the equation $x_1 = c$, $0 \leq c \leq 1$. As c increases, we can imagine a plane sweeping to the right along the x_1 axis, generating a series of isoquants, with the value of O rising monotonically and attaining its maximum value $O = 3$ when the plane departs from the constraint plane at the vertex (1,0,0).

■

It is easy to see that for any two points A and B that are both on the constraint plane, one of three conditions must hold. Either the value of O is the same at both A and B in which case they are on the same isoquant, or O is greater at A, or O is greater at B. This suggests the following optimization procedure: start at a random point A on the constraint plane and generate a set of neighbours of this point that are also on the constraint plane. Then evaluate O at those neighbours. If the value of O at A is the same as or higher than at all of its neighbours, then A is local maximum. Otherwise, the process is repeated from the neighbouring point that has the greatest value of O. This process, as long as it does not get stuck in a local maximum, finds the optimal value of *O*. This is the basis of the optimization technique called **hill climbing** that we will return to in Section 4.7 on page 117.

4.3 Optimizing linear systems

We often wish to maximize[1] an objective function *O* that is a linear combination of the control parameters x_1, x_2,... x_n and can therefore be expressed as:

$$O = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \text{(EQ 1)}$$

where the c_i s are real scalars. Moreover, the x_is are typically constrained by linear constraints in the following **standard form**:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \qquad \text{(EQ 2)}$$

$$\begin{gathered} a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{gathered} \qquad \text{(EQ 3)}$$

or, using matrix notation:

$$\boldsymbol{Ax} = \boldsymbol{b} \qquad \text{(EQ 4)}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \qquad \text{(EQ 5)}$$

where $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ are column vectors with *n* and *m* elements respectively, with $n \geq m$. Let ***A***'s rank be *r*. To allow optimization, the system must be underconstrained with $r < n$, so that some of the x_is can be written as a linear combination of the others, which form a basis for ***A***[2].

Generalizing from Example 3, each equality corresponds to a hyperplane, which is a plane in more than two dimensions (this is not as intuitive as it sounds: for instance, a 3-plane is a solid that fills the entire Euclidean space). The constraints ensure that valid x_is lie at the intersection of these hyperplanes.

Note that we can always transform an inequality of the form

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$$

to an equality by introducing a new variable s_i called the **surplus variable** such that

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - s_i = b_i \qquad \text{(EQ 6)}$$

By treating the s_i as a virtual control parameters, we can convert a constraint that has a greater-than inequality into the standard form (we ignore the value assigned to a surplus variable). Similarly, introducing a **slack** variable converts lesser-than inequalities to equalities. Therefore, any linear system of equal and unequal constraints can be transformed into the standard form that has only equality constraints. Once this is done, we can use **linear programming** (discussed below) to find the value of **x** that maximizes the objective function.

1. In this chapter, we always seek to maximize the objective function. Identical techniques can be used for minimization.
2. To understand this section more fully, the reader may wish to review Section 3.4 on page 82.

EXAMPLE 4: REPRESENTING A LINEAR PROGRAM IN STANDARD FORM

Consider a company that has two network connections to the Internet through two providers (this is also called *multi-homing*). Suppose that the providers charge per-byte and provide different delays. For example, the lower-priced provider may guarantee that transit delays are under 50ms, and the higher-priced provider may guarantee a bound of 20ms. Suppose the company has two commonly used applications, A and B, that have different sensitivities to delay. Application A is more tolerant of delay than application B. Moreover, the applications, on average, generate a certain amount of traffic every day, which has to be carried by one of the two links. The company wants to allocate *all* the traffic from the two applications to one of the two links, maximizing their benefit while minimizing its payments to the link providers. Represent the problem in standard form.

Solution:

The first step is to decide how to model the problem. We must have variables that reflect the traffic sent by each application on each link. Call the lower-priced provider l and the higher priced provider h. Then we denote the traffic sent by A on l as x_{Al} and the traffic sent by A on h as x_{Ah}. Define x_{Bl} and x_{Bh} similarly. The traffic sent is non-negative, so we have:

$$x_{Al} \geq 0\ ;\ x_{Ah} \geq 0\ ;\ x_{Bl} \geq 0\ ;\ x_{Bh} \geq 0\ ;$$

If the traffic sent each day by application A is denoted TA and the traffic sent by B by TB, we have:

$$x_{Al} + x_{Ah} = TA\ ;\ x_{Bl} + x_{Bh} = TB$$

Suppose that the providers charge c_l and c_h monetary units per byte. Then, the cost to the company is:

$$x_{Al}c_l + x_{Bl}c_l + x_{Ah}c_h + x_{Bh}c_h = C$$

What is the benefit to the company? Suppose that application A gains a benefit of b_{Al} per byte from sending traffic on link l and b_{Ah} on link h. Using similar notation for the benefits to application B, the overall benefit (i.e., benefit - cost) that the company should maximize, which is its objective function, is:

$$O = (b_{Al} - c_l)x_{Al} + (b_{Ah} - c_h)x_{Ah} + (b_{Bl} - c_l)x_{Bl} + (b_{Bh} - c_h)x_{Bh}$$

Thus, in standard form, the linear program is the objective function above, and the constraints on the variables expressed as:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{Al} \\ x_{Ah} \\ x_{Bl} \\ x_{Bh} \end{bmatrix} = \begin{bmatrix} TA \\ TB \end{bmatrix} ;\ \begin{bmatrix} x_{Al} \\ x_{Ah} \\ x_{Bl} \\ x_{Bh} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that, in this system, $n = 4$ and $m = 2$. To allow optimization, the rank of the matrix $\boldsymbol{A}$ must be smaller than $n = 4$. In this case, the rank of $\boldsymbol{A}$ is 2, so optimization is feasible.

■

How can we find values of the x_{ij} such that O is maximized? Trying every possible value of $\mathbf{x}$ is an exponentially difficult task, so we have to be cleverer than that. What we need is an algorithm that systematically chooses x_is that maximize or minimize O.

To solve a linear system in standard form, we draw on the intuition developed in Examples 2 and 3. Recall that in Example 3, the optimal value of O was reached at one of the vertices of the constraint plane. This is because any other point has a neighbour that lies on a better isoquant. It is only at a vertex that we ‘run out’ of better neighbours[3]. Of course, in some cases, the isoquant can be parallel to one of the hyperedges of a constraint hyperplane. In this case, the O attains a minimum or maximum along an entire edge.

In a general system, the constraint plane corresponds to a mathematical object called a **polytope** that is defined as a convex hyperspace bounded by a set of hyperplanes. In such a system, it can be shown that the extremal value of the objective function is attained at one of the vertices of the constraint polytope. It is worth noting that a polytope in more than three dimensions is rather difficult to imagine: for instance, the intersection of two four-dimensional hyperplanes is a three-dimensional solid. The principal fact about a polytope that is needed when carrying out an optimization is that each of its vertices is defined by n coordinates, which are the values assumed by the x_is at that vertex. The optimal value of O is achieved for the values of the x_is corresponding to the optimal vertex.

The overall approach to finding the optimal vertex is first, to locate any one vertex of the polytope, second, to move from this vertex to the neighbouring vertex where the value of the objective function is the greatest, and finally, to repeat this process until it reaches a vertex such that the value of the objective function at this vertex is greater than the objective function's value at all of its neighbours: this must be the optimal vertex. This algorithm, developed by G. Dantzig, is the famous **simplex** algorithm.

The simplex algorithm builds on two underlying procedures: finding any one vertex of the polytope and generating all the neighbours of a vertex. The first procedure is carried out by setting n - r of the x_is to 0 so that the resulting system has rank n, and solving the resultant linear system using, for example, Gaussian elimination. The second procedure is carried out using the observation that because $\boldsymbol{A}$'s rank is $r < n$, it is always possible to compute a new basis for $\boldsymbol{A}$ that differs from the current basis in only one column. It can be shown that this basis defines a neighbouring vertex of the polytope.

To carry out simplex in practice, we have to identify if the program has incompatible constraints. This is easy because, if this is the case, then the Gaussian elimination in the first procedure fails. A more subtle problem is that it is possible for a set of vertices to have the same exact value of O, which can lead to infinite loops. We can eliminate this problem by slightly jittering the value of O at these vertices or using other similar **anti-looping** algorithms.

From the perspective of a practitioner, to use linear programming, all that needs to be done is to specify the objective function and the constraints to a program called a **Linear Program Solver** or LP Solver. CPLEX and CS2 are two examples of well-known LP Solvers. A solver returns either the optimal value of the objective function and the vertex at which it is achieved or declares the system to be unsolvable due to incompatible constraints. Today's LP Solvers can routinely solve systems with more than 100,000 variables and tens of thousands of constraints.

The simplex algorithm has been found to work surprisingly well in dealing with most real-life problems. However, in the worst case, it can take time exponential in the size of the input (i.e, the number of variables) to find an optimal solution. Another LP solution algorithm, called the **ellipsoidal method**, is guaranteed to terminate in $O(n^3)$ time, where n is the size of the input, although its performance for realistic problems is not much faster than simplex. Yet another competitor to the simplex algorithm is the **interior point method** that finds the optimal vertex not by moving from vertex to vertex, but by using points interior to the polytope.

Linear programming is a powerful tool. With an appropriate choice of variables, it can be used to solve problems, that, at first glance, may not appear to be linear programs. As an example, we now consider how to set up the network flow problem as a linear program.

4.3.1 Network flow

The network flow problem models the flow of goods in a transportation network. Goods may be temporarily stored in warehouses. We represent the transportation network by a graph. Each graph node corresponds to a warehouse and each directed edge, associated with a capacity, corresponds to a transportation link. The **source** node has no edges entering it and the **sink** node has no edges leaving it. The problem is to determine the maximum possible throughput between the source and the sink. We can solve this problem using LP, as the next example demonstrates.

EXAMPLE 5: NETWORK FLOW

3. For non-linear objective functions, we could 'run out' of better points even within the constraint plane, so the optimal point may not lie at a vertex.

Consider the network flow graph in Figure 4. Here, the node s represents the source and has a total capacity of 11.6 leaving it. The sink, denoted t, has a capacity of 25.4 entering it. The maximum capacity from s to t can be no larger than 11.6, but may be smaller, depending on the intermediate paths.

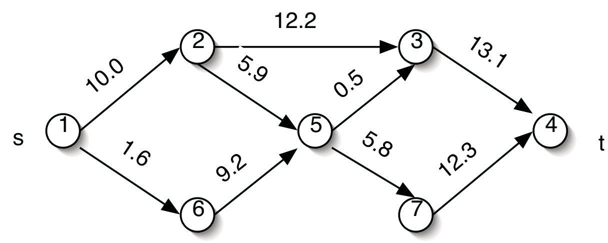


FIGURE 4. Example of a network flow problem

We can compute the maximal flow that can be sustained on a network flow graph using linear programming. Denote the capacity of the link ij from i to j by c_{ij} and the amount of traffic assigned to that link (as part of a flow from s to t) by f_{ij}. For example, in Figure 4, $c_{12} = 10.0$ and we may assign $f_{12} = 2.3$ on it as part of the overall flow from s to t. There are three types of constraints on the f_{ij}s:

1. Capacity constraints: the flow on a link cannot exceed its capacity, that is, $f_{ij} \le c_{ij}$.
2. Conservation conditions: all the flow entering a node (other than the sink) must exit it; that is, for all nodes j other than s and t, $\sum_{\forall i|\exists ij} f_{ij} = \sum_{\forall k|\exists jk} f_{j}$.
3. Non-negativity: $f_{ij} \ge 0$

Given these constraints, the objective function maximizes the flow leaving s. That is $O = \sum_{\forall i|\exists si} f_{si}$. The LP is now easy to frame. It consists of the capacity inequalities (written as equalities after introducing slack variables), the conservation conditions (with the right hand side carried over to the left and adding slack variables), and the conditions on the flows being non-negative. Some examples of these constraints are: on edge 5-7, $f_{57} \le 5.8$ and at vertex 3, $f_{23} + f_{53} = f_{34}$.

■

4.4 Integer linear programming

Linear programming allows variables to assume real values. In Integer Linear Programming, or ILP, variables are only allowed to assume integer values. Although this may appear to be a small difference, this difference makes the solution of ILP *much* harder. More precisely, though LP can be solved in time polynomial in the size of the input, no polynomial-time solution to ILP is known: on some inputs, an ILP solver can take time exponential in the size of the input. In practice, this means that LP can be used to solve problems with hundreds of thousands of variables, but solving an ILP may take a long time even with a few tens of variables.

Nevertheless, ILP arises naturally in a number of cases. In networking, the most common use of ILP is for the **scheduling** problem, where discrete time slots must be assigned to job requests. Since requests cannot be allocated fractional time slots, the problem is naturally posed as one with integer constraints, as the next example shows.

EXAMPLE 6: A SCHEDULING PROBLEM

Two users, Alice and Bob, can schedule jobs on a machine in one of two time periods, Period 1 or Period 2. If Alice schedules a job during Period 1, she gains a benefit of 20 units, and incurs a cost of 10 units, and during Period 2, she gains a benefit of 10 units and incurs a cost of 20 units. If Bob schedules a job during Period 1, he gains a benefit of 100 units and incurs a cost of 10 units, and during Period 2, he gains a benefit of 10 units and incurs a cost of 200 units. Each user may schedule at most one job in one time unit and in each time period, at most one job can be scheduled. There are only four jobs to schedule. Express this system in standard form to maximize the benefit derived from the assignment of user jobs to time periods (also called a **schedule**).

Solution:

The control parameters here are the choice of assignments of user jobs to time periods.We have four jobs and only two time periods. Let x_{ij} be a control parameter that is set to 1 if user i is assigned to schedule a job in time period j. A user can schedule at most two jobs (one in Period one, and one in Period two), so:

$$x_{11} + x_{12} \le 2 \,;\; x_{21} + x_{22} \le 2$$

In each time period, we can have at most one job scheduled, so:

$$x_{11} + x_{21} \le 1 \,;\; x_{12} + x_{22} \le 1$$

If Alice's job is scheduled in Period 1, the net benefit is (20-10) and if it isn't, the benefit is 0. The benefit to Alice in time Period 1, therefore, as: $x_{11}(20\text{-}10)$. Similarly, taking into account the other costs and benefits, the overall objective function is:

$$O = x_{11}(20-10) + x_{12}(10-20) + x_{21}(100-10) + x_{22}(10-200)$$

$$= 10x_{11} - 10x_{12} + 90x_{21} - 190x_{22}$$

Note that the constraints are not in standard form because of the inequalities. We can rewrite the constraints using the slack variables s_1 - s_4 as:

$$x_{11} + x_{12} + s_1 = 2 \,;\; x_{12} + x_{22} + s_2 = 2 \,;\; x_{11} + x_{21} + s_3 = 1 \,;\; x_{12} + x_{22} + s_4 = 1$$

or

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \;;\; \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This expresses the system in standard matrix form. Note that all the x_{ij}s must be 0 or 1, and therefore the system is an ILP.

■

There are three alternatives to solving an ILP. One approach is to carefully model the system so that the input size is small which would minimize the time taken by an ILP solver. A second approach is to not require the solution to be optimal and accept sub-optimal solutions found using heuristic approaches such as those discussed in Section 4.7 on page 117. A third approach, discussed next, allows certain ILPs to be solved as if they were LPs.

4.4.1 Total unimodularity

An ILP $Ax = b$ can be solved as a LP, ignoring integer constraints, if A is totally unimodular. A square, integer matrix A is called **unimodular** if its determinant is either 0, +1 or -1. An integer matrix (which may itself not be square) is called **totally unimodular** if every square, nonsingular submatrix is unimodular.

In practice, there is a simple test for unimodularity:

1. Every entry is either 0, 1, or -1.
2. There are zero, one, or two non-zero entries in any *column*.
3. The *rows* can be partitioned into two sets *A* and *B* such that:
   (a) If a column has two entries of the same sign, one of these is in A, and the other is in B.
   (b) If a column has two entries of different signs, both entries are in either A or B.

EXAMPLE 7: A TOTALLY UNIMODULAR MATRIX

Here is an example of a totally unimodular matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

The matrix can be divided into two sets of two rows (the first and second and the third and fourth) that meet the test for unimodularity.

■

Like LP, ILP can also be used to model a variety of problems. As an example, we study the use of ILP to solve the weighted bipartite matching problem.

4.4.2 Weighted bipartite matching

A **bipartite** graph is a graph where the vertices can be divided into two sets such that all the edges in the graph have one vertex in one set and another vertex in another set (see Figure 5). Such graphs arise naturally in many problems. For instance, one set of vertices could represent a set of demands and the other set could represent a set of resources. Edges then show the resources that could meet each demand. A weight on such an edge could represent the goodness of fit or perhaps the cost of the resource.

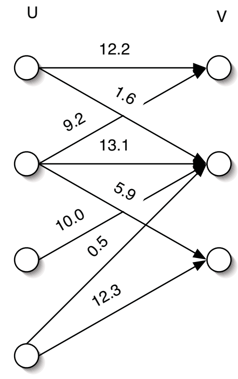


FIGURE 5. **Example of a weighted bipartite graph.**

A **matching** M on a graph is the subset of edges such that no two edges in this set share a common vertex. 'Matching' indicates that each edge in M matches one vertex in one set (say, a demand) to a vertex in another set (say, a resource). A **maximal weighted matching** is a matching such that the sum of the edge weights, summed over the edges in M, is the greatest. If only one such matching exists, it is also the *maximum* weighted matching.

ILP can be used to find the maximal weighted matching in a bipartite graph. Let U and V be the two sets of vertices and let uv refer to an edge that goes from some element u in U to some element v in V. Let w_{uv} be the weight of such an edge.

Define integer variables x_{uv} that are set to 1 if the corresponding edge is in the matching M and 0 otherwise. The total weight of a matching is $\sum_{u,v} w_{uv}x_{uv}$ and this is the value we wish to maximize in the LP (i.e., this is the objective function).

In a matching, there should be at most one edge from an element in U to an element in V and *vice versa* (note that there may be no edge incident at some node in U or V in M if $|U|$ and $|V|$ are not equal). It is convenient to convert the original graph, where not every element in U has an edge to an element in V, to a *complete* graph, where we add extra zero-weight edges so that every element in U has $|V|$ edges. Then, the constraints are:

$$\forall u \sum_{w} x_{uw} \le 1 \; ; \; \forall v \sum_{w} x_{wv} \le 1$$

The first constraint ensures that at most one edge in M leaves every node in U. If this is an extra edge, it adds zero weight to M's weight and can be ignored in the final solution. Similarly, the second constraint ensures that at most one edge is incident at every node in V. With these constraints, an ILP solver can be used to solve the maximal weighted matching problem.

4.5 Dynamic programming

We now turn our attention to **dynamic programming**, a powerful optimization technique applicable when a problem can be decomposed into sub-problems such that the optimal solution to the original problem is a composition of the optimal solution to each sub-problem (this is also called the **optimal substructure** of the problem). The technique, then, is to decompose the problem into two or more sub-problems, solve each sub-problem recursively, and then put the solutions together again to obtain the final answer. Of course, the recursion needs to end in a 'small' problem that can be easily solved. Moreover, it is critical that the recursion does not result in a proliferation of sub-problems.

EXAMPLE 8: FIBONACCI COMPUTATION

Although not an optimization problem, the Fibonacci sequence clearly demonstrates the meaning of substructure, composition, and the need to limit the number of sub-problems. This sequence is defined by $F(1) = F(2) = 1$, and for $n > 2$, $F(n) = F(n\text{-}1) + F(n\text{-}2)$. The first few terms of the sequence are 1, 1, 2, 3, 5, 8, 13.

$F(k)$ can be computed by solving two sub-problems (computing $F(k\text{-}1)$ and computing $F(k\text{-}2)$) and composing the solutions by addition. Note that the computation of $F(k\text{-}1)$ *reuses* the computation of $F(k\text{-}2)$, because $F(k\text{-}1) = F(k\text{-}2) + F(k\text{-}3)$. The solution to $F(k\text{-}2)$ is used twice: once to compute $F(k)$ and once to compute $F(k\text{-}1)$. In fact, a little thought shows that we need to compute each sub-problem (that is, $F(k\text{-}i)$ for $i \in [1, k-1]$) only *once*. This makes dynamic programming efficient.

■

Dynamic programming is useful only when we can strictly limit the number of underlying sub-problems. Moreover, it is necessary to remember the solution to the sub-problems so that they are not repeatedly re-solved. This is called **memoization**.

There are two standard approaches to dynamic programming. The first approach is **bottom-up**, where sub-problems are solved starting from the simplest one. They are then composed to find the required solution. In the case of the Fibonacci sequence, this would correspond to computing $F(1)$, $F(2)$, and so on, until $F(k)$ is found. In contrast, the **top-down** approach decomposes problems into sub-problems as shown in Example 8.

EXAMPLE 9: FLOYD-WARSHALL ALGORITHM FOR ALL-PAIRS SHORTEST PATHS

A well-known example of the use of dynamic programming in a networking context is the Floyd-Warshall algorithm for simultaneously computing the shortest paths between all pairs of nodes in a graph in only $O(N^3)$ time. This uses the bottom-up approach to dynamic programming.

The algorithm operates on an undirected graph G with N nodes whose nodes are numbered 1...N. Define a *path* as a sequence of nodes such that no node index is repeated. The length of the path is the number of edges in it. We want to find the shortest path from any node i to any node j.

Consider all paths from node i to node j that only contains nodes numbered from 1...k. Let $s(i,j,k)$ denote the shortest of these paths. For the moment, we will abuse notation and use s to denote both the path and its length. Set $s(i,j,k)$ to ∞ if no such path exists.

Taking the bottom up approach, assume that $s(i,j,k-1)$, that is, the shortest path from i to j that only uses nodes numbered 1...k-1, is known. We will use this to compute $s(i,j,k)$.

The solution follows from the following observation: either the shortest path from i to j includes the node numbered k or it doesn't. If it doesn't, then $s(i,j,k) = s(i,j,k-1)$. If it does, then there must be a path from i to j passing through k, which means that k must be reachable from both i and j using only nodes numbered 1...k-1. Moreover, the shortest path is composed from the shortest path from i to k using only nodes 1...k-1 and the shortest path from k to j using only nodes 1...k-1. Therefore, $s(i,j,k) = s(i,k, k-1) + s(k, j, k-1)$.

We now have the decomposition we need:

$$s(i, j, k) = min(s((i, j, k-1), s(i, k, k-1) + s(k, j, k-1)))$$

To solve the problem, we first compute the values (and paths) $s(i,j,1)$ for all i,j. We can use these values to compute $s(i,j,2)$ for all values of i,j and repeat for increasing values of k, until, when $k = N$, we have the desired solution. Dynamic programming is effective here because of the optimal substructure, the ease of composition, and the limited number of sub-problems.

■

4.6 Nonlinear constrained optimization

So far, we have been examining the use of optimization techniques where the objective function and the set of constraints are both linear functions. We now consider situations where these functions are not linear.

How does non-linearity change the optimization problem? In a (non-integer constrained) linear system, the objective function attains its maximum or minimum value at one of the vertices of a polytope defined by the constraint planes. Intuitively, because the objective function is linear, we can always ‘walk along’ one of the hyper-edges of the polytope to increase the value of the objective function, so that the extremal value of the objective function is guaranteed to be at a polytope vertex.

In contrast, with non-linear optimization, the objective function may both increase and decrease as we walk along what would correspond to a hyper-edge (a contour line, as we will see shortly). Therefore, we cannot exploit polytope vertices to carry out optimization. Instead, we must resort to one of a large number of non-linear optimization techniques, some of which we study next.

Non-linear optimization techniques fall into roughly into two categories.

- When the objective function and the constraints are mathematically 'nice', that is, continuous and at least twice differentiable, there are two well-known techniques, Lagrangian optimization and Lagrangian optimization with the Karush-Kuhn-Tucker conditions.
- When the objective functions are not continuous or differentiable, we are forced to use heuristic techniques such as hill-climbing, simulated annealing, and ant algorithms.

We will first look at Lagrangian techniques (Section 4.6.1 on page 115), a variant called the KKT conditions that allows inequality constraints (Section 4.6.2 on page 116) then briefly consider several heuristic optimization techniques (Section 4.7 on page 117).

4.6.1 Lagrangian techniques

Lagrangian optimization computes the maximum (or minimum) of a function *f* of several variables subject to one or more constraint functions denoted g_i. We will assume that *f* and all the g_i are continuous, at least twice-differentiable, and are defined over the entire domain, that is, do not have 'boundaries.'

Formally, *f* is defined over a vector **x** drawn from R^n and we wish to find the value(s) of **x** for which *f* attains its maximum or minimum, subject to the constraint function(s): $g_i(\mathbf{x}) = c_i$, where the c_i are real constants.

To begin with, consider a function *f* of two variables *x* and *y* with a single constraint function. We want to find the set of tuples of the form (*x*,*y*) that maximize *f*(*x*,*y*) subject to the constraint $g(x,y) = c$. The constraint $g_i(\mathbf{x}) = c_i$ corresponds to a **contour** or **level set**, that is, a set of points where *g*'s value does not change. Imagine tracing a path along such a contour. Along this path, *f* will increase and decrease in some manner. Imagine the contours of *f* corresponding to $f(\mathbf{x}) = d$ for some value of *d*. The path on *g*'s contour touches successive contours of *f*. An extremal value of *f* on *g*'s contour is reached exactly when *g*'s contour grazes an extremal contour of *f*. At this point, the two contours are tangential, so that the gradient of *f*'s contour (a vector that points in a direction perpendicular to the contour) has the same direction as the gradient of *g*'s contour (though it may have a different absolute value). More precisely, if the gradient is denoted by $\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, then, at the constrained extremal point,

$$\nabla_{x,y} f = -\lambda \nabla_{x,y} g$$

Define an auxiliary function:

$$F(x, y, \lambda) = f(x, y) + \lambda(g(x, y) - c) \qquad \textbf{(EQ 7)}$$

The stationary points of *F*, that is the points where $\nabla_{x,y,\lambda} F(x, y, \lambda) = 0$, are points that

(a) satisfy the constraint *g*, because the partial derivative with respect to λ, i.e., $g(x, y) - c$ must be zero, and

(b) are also constrained extremal points of *f*, because $\nabla_{x,y} f = -\lambda \nabla_{x,y} g$

Thus, the extremal points of *F* are also the points of constrained extrema of *f* (i.e minima or maxima). From Fermat's theorem, the maximum or minimum value of any function is attained at one of three types of points: (a) a boundary point (b) a point where *f* is not differentiable and (c) at a stationary point where its first derivative is zero. Because we assume away the first two situations, the maximum or minimum is attained at one of the stationary points of *F*. Thus, we can simply solve $\nabla_{x,y,\lambda} F(x, y, \lambda) = 0$ and use the second derivative to determine the type of extremum.

This analysis continues to hold for more than two dimensions and more than one constraint function. That is, to obtain a constrained extremal point of *f*, take the objective function and add to it a constant multiple of each constraint function to get the auxiliary. This constant is called a **Lagrange multiplier**. The resulting system of equations is solved by setting the gradient of the auxiliary function to 0 to find its stationary points.

EXAMPLE 10: LAGRANGIAN OPTIMIZATION

Consider a company that purchases capacity on a link to the Internet and has to pay for this capacity. Suppose that the cost of a link of capacity *b* is *Kb*. Also suppose that the mean delay experienced by data sent on the link, denoted by *d*, is inversely proportional to *b*, so that *bd = 1*. Finally, let the benefit *U* from using a network connection with capacity *b*, and delay *d*, $0 \le d < \infty$ be described by *U = -Kb -d*, that is, it decreases both with cost and with the delay. We want to maximize *U* subject to the constraint *bd = 1*. Both *U* and the constraint function are continuous and twice-differentiable. Therefore, we can define the auxiliary function:

$$F = -Kb - d + \lambda(bd - 1)$$

Set the partial derivatives with respect to *b, d* and λ to zero, to obtain, respectively:

$$-K + \lambda d = 0$$
$$-1 + \lambda b = 0$$
$$bd = 1$$

From the second equation, $b = 1/\lambda$ and from the first equation, $d = K/\lambda$ and from the third equation, substituting the values for *b* and *d*, $\lambda = \sqrt{K}$. Substituting these values into the equations for *b* and *d*, $b = 1/(\sqrt{K})$ and $d = \sqrt{K}$.This gives a value of *U* at (*b*, *d*) to be $-2\sqrt{K}$. Since *U* is clearly unbounded in terms of a smallest value (when *b* approaches 0), this is also its maximum.

■

4.6.2 Karush-Kuhn-Tucker conditions for nonlinear optimization

The Lagrangian method is applicable when the constraint function is of the form $g(x) = 0$. What if the constraints are of the form $g(\mathbf{x}) \le 0$? In this case, we can use the Karush-Kuhn-Tucker conditions, often called the KKT or Kuhn-Tucker conditions, to determine whether the stationary point of the auxiliary function is also a global minimum.

As a preliminary, we define what is meant by a **convex** function. A function *f* is convex if, for any two points *x* and *y* in its domain, and for *t* in the closed interval [0,1], $f(tx + (1-t)y) \le tf(x) + (1-t)y$. That is, the function always lies *below* a line drawn from *x* to *y*.

Consider a convex objective function *f*: $R^n \to R$ with both *m* inequality and *l* equality constraints. Denote the inequality constraints by $g_i(\mathbf{x}) \le 0, 1 \le i \le m$ and the equality constraints by $h_j(\mathbf{x}) = 0$, $1 \le j \le l$. The KKT conditions require all the g_i to be convex and all the h_j to be linear. Then, if *a* is a point in R^n, and there exist *m* and *l* constants respectively, denoted μ_i and ν_j such that the following conditions hold, then we can guarantee that *a* is a globally constrained minimum of *f*:

$$\begin{aligned} \nabla f(a) + \sum_{i=1}^{m} \mu_i \nabla g_i(a) + \sum_{j=1}^{l} \mu_j \nabla h_j(a) &= 0 \\ g_i(a) &\le 0 \forall i \\ h_j(a) &= 0 \forall j \\ \mu_i &\ge 0 \forall i \\ \mu_i g_i(a) &= 0 \forall i \end{aligned} \tag{EQ 8}$$

If these conditions are met, then the stationary points of the auxiliary function (which is the first equation above) yield the minima of *f*.

4.7 Heuristic non-linear optimization

We now turn to the situation where the objective function and the constraints may not be mathematically ‘nice,’ that is, linear or convex. In such cases, we need to rely on heuristic approaches to optimization. Many heuristic approaches have been proposed in the literature: we will outline only two common ones- Hill climbing and Genetic algorithms.

4.7.1 Hill climbing

Hill climbing is perhaps the simplest technique for heuristic optimization. Its simplest variant does not even support constraints, seeking only to find the value of **x** that maximizes $f(\boldsymbol{x})$. Hill climbing requires only two primitives: a way to evaluate $f(\boldsymbol{x})$ given **x** and a way to generate, for each point **x,** another point **y,** that is ‘near’ **x** (assume that **x** and **y** are embedded in a suitable metric space).

Start by randomly choosing a point **x** in the domain of *f* and labelling it the candidate maximum (we might just get lucky!). Evaluate *f* on **x**, then generate a point **y** that is ‘close’ to **x**. If the value of *f* is higher at **y,** then **y** is the new candidate maximum, else **x** remains the candidate. We continue to generate and evaluate *f* on neighbouring points of the candidate maximum until we find a point **x** all of whose neighbours have a lower value of *f* than at **x**. We declare this to be the maximum.

The analogy to climbing a hill is clear. We start somewhere on the hill and take a step in a random direction. If it is higher, we step up. If not, we stay where we are. This way, assuming the hill has a single peak, we will eventually get to the top, where every neighbouring step must lead downhill.

Although simple, this approach to hill climbing leaves much to be desired. These concerns are addressed by the following variants of the basic approach:

- Generate more than one neighbour of ***x*** and choose the best of these. This variant is also called the **steepest-gradient method**.
- Memorize some or all of the values of ***y*** that we discarded in a **tabu list**. Subsequently, if any value in the tabu list is generated, it can be immediately discarded. This variant is called **tabu search**.
- To find the maximum value of *f* subject to constraint *g*, choose the initial candidate maximum ***x*** to be a value that also satisfies *g*. Then, when generating neighbours of ***x***, ensure that the neighbouring values also satisfy *g*. This allows us to use hill climbing for constrained optimization.

The single biggest problem with hill climbing is that it fails when *f* has more than one maximum. In this case, an unfortunate initial choice of ***x*** will cause the algorithm to be stuck in a local maximum, instead of finding the global maximum. This is illustrated in Figure 6 which shows a function with multiple peaks. Starting at the base of any of the lesser peaks will result in hill climbing stopping at a local maximum.

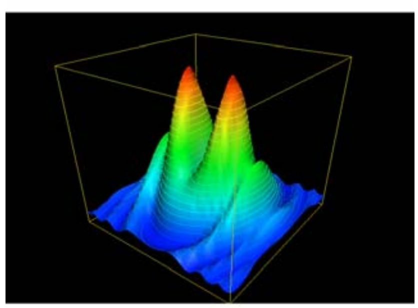


FIGURE 6. Example of a function where hill climbing can get stuck in a local maximum.

There are several ways to get around this problem. One approach is called **shotgun** hill climbing. Here, the hill climbing algorithm is started from several randomly chosen candidate maxima. The best result from among these should be the global maximum as well. This approach is widely used.

A second approach, called **simulated annealing**, varies the 'closeness' of a selected neighbour dynamically. Moreover, it allows for some steps of the climb to be downhill. The idea is that if the algorithm is stuck at a local maximum, the only way out is to go down before going up, therefore downhill steps should be allowed. The degree to which downhill steps are permitted varies over the climb. At the start even large downhill steps are permitted. As the climb progresses, however, only small downhill steps are permitted.

More precisely, the algorithm evaluates the function value at the current candidate point **x** and at some neighbour **y**. There is also a control variable, called the temperature, T, that describes how large a downhill step is permitted. The **acceptance function** $A(f(\mathbf{x}), f(\mathbf{y}), T)$ determines the probability with which the algorithm moves from **x** to **y** as a function of their values and the current temperature, with a non-zero probability even when $f(\mathbf{y}) < f(\mathbf{x})$. Moreover, the acceptance function tends to zero when T tends to zero and $f(\mathbf{y}) < f(\mathbf{x})$. The choice of the acceptance function is problem-specific and therefore usually hand-crafted.

4.7.2 Genetic algorithms

The term 'genetic algorithm' applies to a broad class of approaches that share some common attributes. The key idea is to encode a candidate maximum value **x** as a bit string. At the start, hundreds or thousands of such candidate values are randomly generated. The function f is then evaluated at each such value and the best ones are **selected** for propagation in one of two ways. With **mutation**, some bits of a selected candidate value are randomly perturbed to form the next generation of candidates. With **crossover**, bits from two selected candidate values are randomly exchanged. In this way, the best **features** of the population are **inherited** by the next generation. The algorithm proceeds by forming generation after generation of candidates, until adequate solutions are found.

There is an extensive literature on algorithms for encoding candidates, introducing mutations, and making effective crossovers. Genetic algorithms have been found to produce surprisingly good results in areas ranging from antenna design to job scheduling. However, the approach has also been criticized for many shortcomings, such as its sensitivity to numerous tuning parameters and a tendency to converge to local optima.

4.8 Exercises

1 Modelling

You have been hired as the head of a hot air balloon company's flight operations. Too much money is being spent for each flight! Your job is to make flight profitable again (the number of flights is not negotiable).

For each flight, you can control where you take off from (there is a finite set of take-off locations) and the duration of the flight, as long as the flight lasts at least 15 minutes. The cost of a flight depends on its duration (to pay for natural gas, the pilot's wages, and for the chase vehicle), where the balloon takes off from, and how far the landing site is from a road (the further away it is from a road, the more it has to be dragged over a farmer's field). Moreover, you can have up to 9 passengers (in addition to at least one pilot), and charge them what you wish. Of course, the number of passengers decreases (say linearly) with the cost of the ticket.

What are the fixed parameters? What are the input and output parameters? What are the control variables? Come up with plausible transfer and objective functions. How would you empirically estimate the transfer function?

2 Optimizing a function of two variables

Consider the following system:

$$O = 10x_1 - 3x_{2,} \text{ where}$$

$$2x_1 - x_2 = 1 \text{ and}$$

$$x_1 \geq 0\ ;\ x_2 \geq 0$$

Geometrically find the optimal value of O.

3 Optimizing a function of three variables

Geometrically find the optimal value of O where

$$O = 5x_1 + 2x_2 - x_3 \text{ and}$$

$$x_1 + x_2 + x_3 = 1 \text{ and}$$

$$x_1 \geq 0\ ;\ x_2 \geq 0\ ;\ x_3 \geq 0$$

4 Network flow

Model the network flow problem of Example 5, where the warehouses have finite bounded capacity, as a linear program.

5 Integer linear programming

Generalize Example 6 to the case where n users can schedule jobs on one of k machines, such that each user incurs a specific cost and gains a specific benefit on each machine at each of m time periods. Write out the ILP for this problem.

6 Weighted bipartite matching

Suppose you have K balls that need to placed in M urns such that the payoff from placing the kth ball in the mth urn is p_{km}, and no more than 2 balls can be placed in each urn. Model this as a weighted bipartite matching problem to maximize the payoff.

7 Dynamic programming

You are given a long string L of symbols from a finite alphabet. Your goal is to find the matching substrings of L with a shorter string S from the same alphabet. However, matches need not be exact: you can delete one element of L or S for a penalty of 1, and you can also substitute an element of L for an element of S for a penalty of 1. So, for example, the match between the string L = “text” and S = “tx” is “te” with a penalty of 1 (one substitution), “tex” with a penalty of 1 (one deletion), “ex” with a penalty of 2 (two substitutions), “ext” with a penalty of 2 (one substitution and one deletion) etc. Use dynamic programming to output all matching substrings of L along with the corresponding penalty.

8 Lagrangian optimization

Use Lagrangian optimization to find the extremal values of $z=x^3 + 2y$ subject to the condition that $x^2+y^2 = 1$ (i.e. the points (x,y) lie on the unit circle.

9 Hill climbing

Suppose you know that the objective function you are trying to maximize has no more than K local optima. Outline an algorithm that is guaranteed to find the global optimum using hill climbing.

CHAPTER 5 *Signals, Systems, and Transforms*

5.1 Introduction

This chapter focuses on **transforms**: a family of techniques that are used to study a series of values or **signals**. Transforms give insight into the nature of signals and to help design effective **systems** to manipulate signals. They form the basis for the study of control theory, which is discussed in more detail in Chapter 8.

We begin by covering background material on complex numbers and Euler's formula. We then discuss different types of signals and systems. Subsequently, the bulk of the chapter is devoted to the study of Fourier series, Fourier transforms, Laplace transforms, and the Z transform.

5.2 Background

5.2.1 Sinusoids

A communication network is composed from communication links. A widely-used approach to transferring information over a wired or wireless link is to modify an underlying continuous **carrier** signal to represent the symbol values '0' and '1.' In nearly all cases of interest, the carrier signal is **sinusoidal**. That is, it is associated with a time period T such that at time t, the amplitude of the signal is described by the function $\cos\left(\frac{2\pi t}{T}\right)$, as shown in Figure 1.

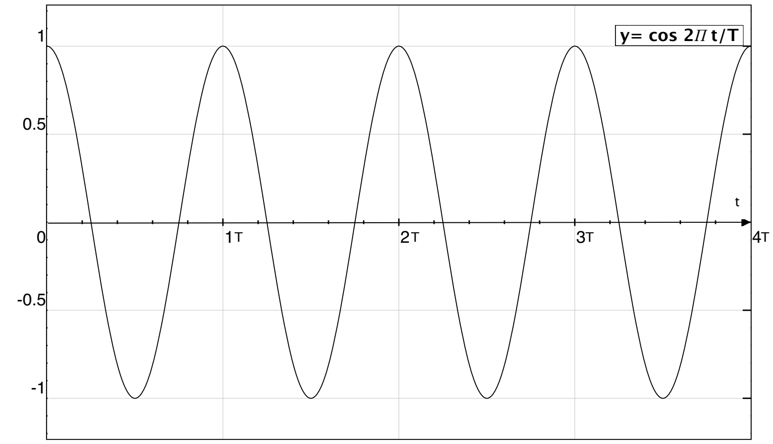


FIGURE 1. A typical carrier wave described by a sinusoidal curve

How can we generate such a time-varying signal? Consider a disk with a radius of one unit that has an axle at its center. Suppose that it rotates in the counter-clockwise direction with a uniform angular velocity and with a period of *T* seconds. Let us paint a small dot on its rim and draw a vector from the center to this dot. The angle that this rotating vector makes with the X axis at time *t* is denoted by θ and given by $\theta = \frac{2\pi t}{T}$ radians (Figure 2). At time *t*, the distance of the dot from the Y axis is given by $\cos(\theta) = \cos\left(\frac{2\pi t}{T}\right)$, which is precisely the equation describing a sinusoidal carrier. In other words, the unidimensional *time-varying* signal strength of a sinusoidal carrier can be thought of as a projection of a *fixed-length* two-dimensional rotating vector on the X axis. This motivates a more careful study of the mathematics of rotating two-dimensional vectors.

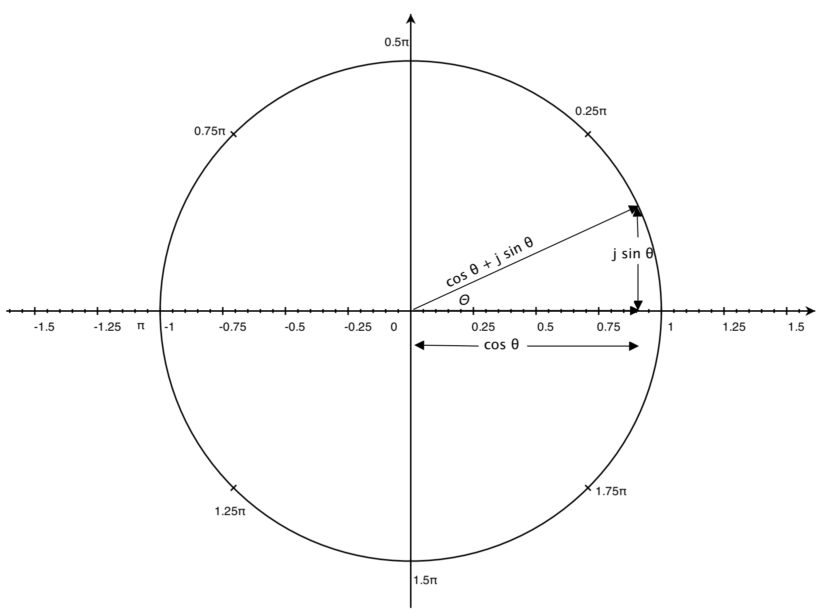


FIGURE 2. Generating a sinusoid from a rotating vector

5.2.2 Complex numbers

Recall that any vector on a two-dimensional plane can be written as the vector sum of two *basis* vectors, one lying on the X axis and one parallel to the Y axis. Let us consider how to represent these basis vectors 'naturally.' A real value *x* on the number line naturally corresponds to a vector lying on the X axis drawn from (0,0) to (0, *x*). Thus, the first of these basis vectors corresponds to, and can be represented by, a real number. How can we represent the second vector? Intuitively, we need a way to 'raise' a vector off the real axis. Suppose we denote the operation 'rotate a vector counter-clockwise by 90 degrees about the origin' by the letter *j*[1]. Then, the vector *j*5, for example, corresponds to a vector of magnitude 5 lying on the Y axis. Given this operation, we can denote a vector on the unit disk making an angle θ with the X axis as the vector sum $\cos(\theta) + j\sin(\theta)$, as shown in Figure 2.

We denote the application of the rotation operator *j* twice in a row like this: *j.j* or j^2. Note that the application of this combined operator to the vector 1 (which is the vector from (0,0) to (0,1)) results in the vector -1. We represent this by the equation:

$$j.j.1 = j^2.1 = -1 \qquad \textbf{(EQ 1)}$$

With a slight abuse of notation, we can take the square root of either side to get:

$$j = \sqrt{-1} \qquad \textbf{(EQ 2)}$$

revealing *j* to be the unit imaginary vector. It is unfortunate that most students encounter *j* as a ghostly, ethereal entity, rather than as the far more intuitive 90-degree rotation operator.

Given the basis vectors 1 and *j*, it is easy to see that we can represent a vector from the origin to any point (*a*,*b*) on the plane as the vector sum $a + jb$. The corresponding tuple (*a*,*b*) is called a **complex** number, with *a* as its **real** part and *b* as its **imaginary** part. If $z = a+jb$ is a complex number, then we denote $a = \mathrm{Re}(z)$; $b = \mathrm{Im}(z)$. We denote by z^* the **complex conjugate** of *z*, defined as $z^* = a-jb$.

5.2.3 Euler's formula

Given the importance of rotating vectors (also called **phasors**), it is desirable to compactly represent the current position of the vector on the unit disk. Of course, this can be easily specified by the single value θ. By tradition, and for sound mathematical reasons that we will see later, we instead represent a rotating vector of magnitude *c* making an angle θ with the X axis using the notation $ce^{j\theta}$. This is rather counter-intuitive notation! What does it mean to raise *e* to an imaginary power? The answer is: it does not matter. All that matters is that the following formula, known as **Euler's formula**[2], has been found to be consistent with the foundations of real analysis:

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \textbf{(EQ 3)}$$

EXAMPLE 1: EULER'S FORMULA

1. Mathematicians and engineers differ in the symbol used for this operation. Engineers use '*j*' rather than the mathematician's '*i*' because by universal engineering convention the symbol '*i*' represents an electric current.

2.The Euler formula has been called the most beautiful formula in mathematics, especially when written for the special value of $\theta = \pi$, where, using standard mathematical convention, it reduces to

$$e^{i\pi} + 1 = 0$$

relating five elementary quantities in mathematics: *e*, *i*, 0, 1, and π.

Use Euler's formula to represent $e^{j\frac{\pi}{2}}$ in the form *a* + *jb*.

Solution:

By definition, $e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$.

■

EXAMPLE 2: EULER'S FORMULA

Use Euler's formula to represent $e^{-j\frac{\pi}{10}} + 2e^{-j\frac{2\pi}{10}}$ in the form *a* + *jb*.

Solution:

$$\begin{aligned}
& e^{-j\frac{\pi}{10}} + 2e^{-j\frac{2\pi}{10}} \\
&= \left(\cos\left(-\frac{\pi}{10}\right) + j\sin\left(-\frac{\pi}{10}\right)\right) + 2\left(\cos\left(-\frac{2\pi}{10}\right) + j\sin\left(-\frac{2\pi}{10}\right)\right) \\
&= \left(\cos\left(\frac{\pi}{10}\right) + 2\cos\left(\frac{2\pi}{10}\right)\right) - j\left(\sin\left(\frac{\pi}{10}\right) + 2\sin\left(\frac{2\pi}{10}\right)\right) \\
&= (0.95 + 1.618) - j(0.309 + 1.175) \\
&= 2.568 + j1.484
\end{aligned}$$

■

We have already seen that the projection of a phasor $ce^{j\theta}$ on the X axis is given by $c\cos(\theta)$, which is the first (or real) component in the expansion of $ce^{j\theta}$ using Euler's formula. We denote this as Re($ce^{j\theta}$). Recalling that $\theta = \frac{2\pi t}{T}$, the strength of a carrier signal at time *t* is therefore given by Re($ce^{j\left(\frac{2\pi t}{T}\right)}$). The factor $\frac{2\pi}{T}$ occurs so frequently in the study of rotating vectors that it has its own symbol, ω, where $\omega = \frac{2\pi}{T}$. ω has a physical interpretation as the angular frequency of rotation and has units of radians/second. With the notation and concepts built up so far, we can succinctly denote a carrier signal as Re($ce^{j\omega t}$).

We have seen two ways of representing a complex number: as a sum *a* + *jb*, or as a phasor $ce^{j\theta}$. It is straightforward to go from one representation to the other using the formulae below:

$$\begin{aligned} a &= c\cos\theta \qquad & c &= \sqrt{a^2 + b^2} \\ b &= c\sin\theta \qquad & \theta &= \operatorname{atan}\left(\frac{b}{a}\right) \end{aligned} \qquad \text{(EQ 4)}$$

Note that we have arbitrarily described the phasor as rotation in the counter-clockwise direction. A sinusoidal carrier can equally well be generated by a phasor rotating in the counter-clockwise direction. To distinguish between the two, we denote the angular frequency of a phasor rotating in the clockwise direction as a negative frequency. Thus, a phasor with a frequency of -20π radians/second is rotating 10 times a second in the clockwise direction and the second hand of a clock has a frequency of $-\frac{\pi}{30}$ radians/second.

EXAMPLE 3: KEYING

We now consider how to mathematically represent the transmission of information on a link by modification of the phasor $ce^{j\omega t}$. This is also called **keying**. For simplicity, focus on the problem of sending just one bit: either a '0' or a '1.'

Amplitude shift keying: In amplitude shift keying, we choose two phasor magnitudes c_1 and c_2. To send a '0', the transmitter transmits a signal $c_1\mathrm{Re}(e^{j\omega t})$ and to send a '1', its transmits a signal $c_2\mathrm{Re}(e^{j\omega t})$. In practice, this is accomplished by an electronic circuit that sets the degree of amplification of the carrier signal to either c_1 or c_2. If either of the constants c_1 and c_2 are 0, then this is also called **on-off keying**.

Frequency shift keying: In frequency shift keying, we choose two phasor frequencies ω_1 and ω_2. To send a '0', the transmitter transmits a signal $c\mathrm{Re}(e^{j\omega_1 t})$ and to send a '1', its transmits a signal $c\mathrm{Re}(e^{j\omega_2 t})$. In practice, this is accomplished by either by coupling the output signal to one of two oscillators tuned to the frequencies ω_1 and ω_2 or by sending control signals to a voltage-controlled oscillator.

Phase shift keying: The **phase** of a sinusoid at time *t* is the angle it makes with the X axis at that time. With phase shift keying, the transmitter and receiver are assumed to be **phase-locked**, that is, they each have local oscillators that have identical frequencies and phases. The transmitter sends a '0' or a '1' by modifying the phase of the carrier signal. With **binary phase shift keying**, for example, the transmitter transmits the unmodified carrier signal $ce^{j\omega t}$ to send a '0' and $-ce^{j\omega t}$, which is a signal that is exactly 180 degrees out of phase with the receiver's local oscillator, to send a '1.' This general idea can be easily extended. For example, with **quadrature phase shift keying**, the transmitter shifts the transmitting phasor's phase by 0, 90, 180, or 270 degrees using the phasors $ce^{j\omega t}$, $ce^{j\left(\omega t+\frac{\pi}{2}\right)}$, $ce^{j(\omega t+\pi)}$, and $ce^{j\left(\omega t+\frac{3\pi}{2}\right)}$. These allow the receiver to extract *two* bits for each received signal. In practice, phase shift keying is accomplished by slightly delaying the transmitted carrier signal so that it is received out of phase.

One might ask: what is the limit to phase shift keying? Would it be possible to send larger and larger numbers of bits by using more fine-grained phase shifting? Could we send, for example, 32 bits at a time by using 65536 phases? The answer is that as we increase the number of phase levels, even a slight distortion of a signal along the path from the transmitter to the receiver corrupts the signal. Therefore, the limit to the *capacity* of the path comes from its inherent *noise*. The relationship between the capacity of a channel and its noise characteristics form the basis of *information theory*, which is discussed in Chapter 9.

■

It is sometimes useful to express sinusoids as exponentials. We can do so as follows. First, recall that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Now, because $e^{j\theta} = \cos\theta + j\sin\theta$, it follows that $e^{-(j\theta)} = e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$. Therefore, $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, so that

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \qquad \text{(EQ 5)}$$

Similarly,

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \qquad \text{(EQ 6)}$$

5.2.4 Discrete-time convolution and the impulse function

Consider the discrete-time function $x(t)$, where t is a non-negative integer that represents the number of HTTP requests made to a web server over intervals of 1 second. Suppose this function, over an interval of 10 seconds, is given by $x(t),\ 0 \le t \le 9\ =$ 1, 3, 5, 2, 5, 8, 7, 3, 9, 4. That is, in the zeroth interval, the web server receives $x(0) = 1$ request, in the first interval, it receives $x(1) = 3$ requests, in second interval it receives $x(2) = 5$ requests and so on. The histogram of this function is shown in Figure 3(a).

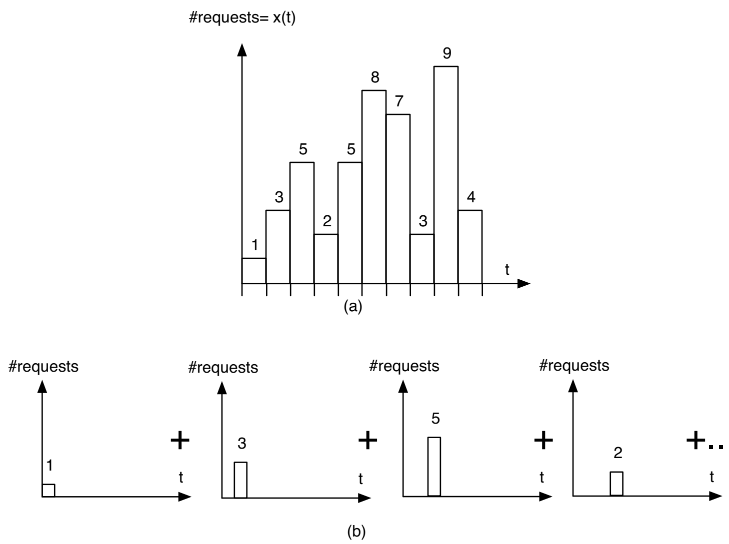


FIGURE 3. **Decomposing a discrete-time function**

The same histogram can also be represented as the sum of a set of smaller histograms, one per time interval, as shown in Figure 3(b). Each smaller histogram is the product of two functions, $x(t)$ and a selector function $sel(t, \tau)$, which is defined as $sel(t, \tau) = 1$ when $t = \tau$ and = 0 otherwise. For instance, the first small histogram corresponds to $x(t)*sel(t,0)$ because it is zero everywhere, except when $t = 0$, when $x(t)$ takes the value $x(0)$ and $sel(t,0)$ is 1. Similarly, the second histogram corresponds to $x(t)*sel(t,1)$ and so on. Because their sum is $x(t)$, we can write

$$x(t) = \sum_{\tau = 0}^{\infty} x(t)sel(t, \tau) \qquad \text{(EQ 7)}$$

We make a small digression to consider the relationship between the function $x(t)$ and the function $x(t\text{-}a)$. For concreteness, consider the function $x(t)$ shown in Figure 3(a) and let $a = 2$. The function $x(t\text{-}a)$ is undefined for times before time 2, and, for convenience, we define it to have the value 0. At time 2, $x(t\text{-}a) = x(2\text{-}2) = x(0) = 1$. At time 3, $x(t\text{-}a) = x(1) = 3$ and so on. We plot this in Figure 4.

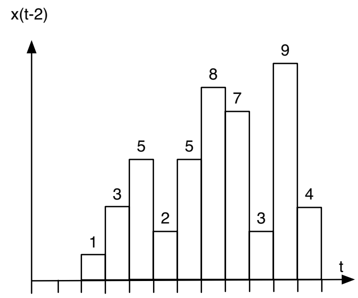


FIGURE 4. **Time-shifting a time series**

Note that $x(t-2)$ is just $x(t)$ shifted two steps to the right. A little thought shows that, in general, the time series $x(t-a)$ is the time series $x(t)$ shifted by a time interval a to the right.

Now, consider the function $\delta(t)$, also called the **impulse function**, defined as $\delta(0) = 1$ and $\delta(t) = 0$ for all other values of t. We see that, trivially, $sel(t, 0) = \delta(t)$. From our observation about time-shifting, we can define $sel(t, 1) = \delta(t-1)$, which is the one-step time-shifted version of $\delta(t)$. Generalizing, we see that

$$sel(t, \tau) = \delta(t-\tau) \qquad \text{(EQ 8)}$$

Therefore, we can rewrite Equation 7 as

$$x(t) = \sum_{\tau=0}^{\infty} x(t)\delta(t-\tau) \qquad \text{(EQ 9)}$$

We have already seen that for each term in the summation, the value of the product of $x(t)$ and $\delta(t-\tau)$ is zero except when $\tau = t$, so we can also write the summation as

$$x(t) = \sum_{\tau=0}^{\infty} x(\tau)\delta(t-\tau)$$

Finally, for functions $x(t)$ that are defined over the both positive and negative values of t, we can generalize the summation over the entire number line, to get

$$x(t) = \sum_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau) \qquad \text{(EQ 10)}$$

This summation is called the **convolution** of $x(t)$ with $\delta(t)$ and is denoted $x(t) \otimes \delta(t)$[3].

In general, the convolution of two discrete functions $x(t)$ and $y(t)$ is defined as

$$x(t) \otimes y(t) = \sum_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau) \qquad \text{(EQ 11)}$$

3. In electrical engineering texts, convolution is represented by ‘*’. Unfortunately, this is universally used in computer science to denote multiplication, thus our choice.

Note that *each* value of the convolution of $x(t)$ and $y(t)$ (i.e., at time t) is the result of adding *all* product pairs $x(\tau)$ and $y(t-\tau)$. This is clearly a computationally expensive operation. Moreover, we need to be careful in dealing with values of $x(t)$ and $y(t)$ that lie outside their range of definition.

EXAMPLE 4: DISCRETE-TIME CONVOLUTION

Compute the convolution of $x(t),\ 0 \le t \le 9 = 1, 3, 5, 2, 5, 8, 7, 3, 9, 4$ and $y(t),\ 0 \le t \le 9 = 3, 1, 7, 4, 5, 9, 7, 1, 3, 8$.

Solution: By convention, we assume that both functions are zero outside their range of definition. Let $z(t) = x(t) \otimes y(t)$.

$z(0) = \sum_{\tau=-\infty}^{\infty} x(\tau)y(-\tau)$. The only value of τ for which both $x(t)$ and $y(t)$ are non-zero is $\tau = 0$, so the sum reduces to $x(0)*y(0) = 3$. $z(1) = \sum_{\tau=-\infty}^{\infty} x(\tau)y(1-\tau)$. The only values of τ for which both $x(t)$ and $y(t)$ are non-zero are $\tau = 0, 1$ so the sum reduces to $x(0)*y(1) + x(1)*y(0) = 1*1 + 3*3 = 10$. It is left as an exercise to the reader to show that $z(2) = 7+3+15 = 25$, $z(3) = 4+21+5+6 = 36$, $z(4) = 5+12+35+2+15 = 69$,....

■

5.2.5 Continuous-time convolution and the Dirac delta function

The definition of convolution above can be extended for continuous-time functions. We start, as before, with the definition of a selector function with a parameter τ that picks out the value of $x(t)$ at time τ. Thus, we would like to define a function that is zero everywhere except at a single point, where it has a value of 1. Obviously, such a function is discontinuous at that point. which makes it mathematically troublesome. To avoid this problem, we define the selector function as the limiting case of a rectangular function of unit area defined over a support of length T and with height $1/T$ as $T \to 0$. At the limit, this represents an infinitely long rectangle that nevertheless has a finite area! Nevertheless, this function is continuous everywhere except at the limit so that its properties at the limit can be extrapolated from the limiting series. Setting aside a mathematically rigorous development, which is beyond the scope of this text, we define the continuous-time impulse function as

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$
$$\delta(t) = 0 \quad \text{for } t \neq 0 \qquad \textbf{(EQ 12)}$$

Intuitively, a continuous-time impulse function has a unit mass concentrated entirely at $t = 0$. It was first defined by the physicist P.A.M Dirac and therefore is also called the **Dirac delta** in his honour. The Dirac delta is represented graphically by a vertical arrow at the origin. The integral of the delta function is called the **unit-step function** and written $u(t)$. $u(t)$ is zero for $t < 0$ and 1 for $t > 0$. Its value at zero is undefined (Figure 5).

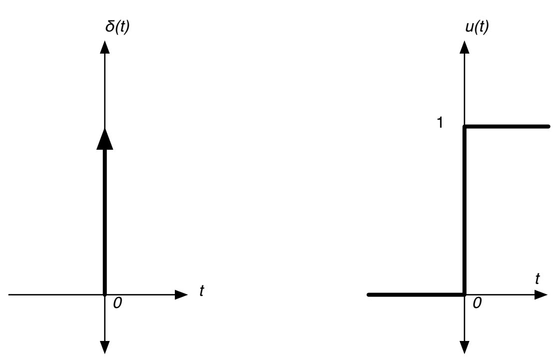


FIGURE 5. The Dirac delta and its integral, the unit step function

For any function *x*(*t*) that is continuous in the neighbourhood of the origin, the delta function acts as a selector, so that

$$x(t)\delta(t) = x(0)\delta(t) \tag{EQ 13}$$

We interpret this to mean that the multiplication of *x*(*t*) with the delta function results in a delta function whose strength is the value of *x*(*t*) at the origin (compare this with the discrete-time selector function). Similarly,

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau) \tag{EQ 14}$$

Moreover, analogous to Equation 10, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \\ x(\tau) &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt \end{aligned} \tag{EQ 15}$$

where the second equation can be obtained by swapping the variables t and τ.

Finally, we can define the convolution of any two continuous-time functions *x*(*t*) and *y*(*t*) as

$$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \tag{EQ 16}$$

EXAMPLE 5: CONTINUOUS-TIME CONVOLUTION

Compute the convolution of the functions *x*(*t*) and *y*(*t*) defined graphically below.

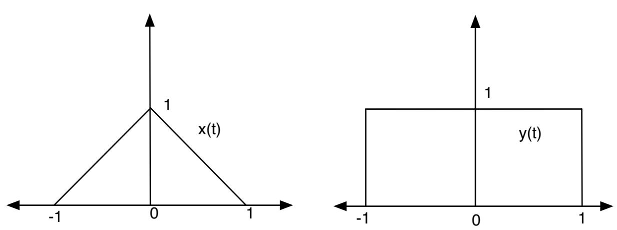


FIGURE 6. Example continuous time functions for convolution

Solution: By convention, we assume that both functions are zero outside their range of definition. Thus

$$x(\tau) \neq 0 \text{ in the range } -1 \leq \tau \leq 1$$
$$y(t-\tau) \neq 0 \text{ in the range } -1 \leq (t-\tau) \leq 1 \Rightarrow t-1 \leq \tau \leq t+1$$

Therefore, both *x*(*t*) and *y*(*t*) are simultaneously non-zero only in the range $max(t-1, -1) \leq \tau \leq min(t+1, 1)$ and

$$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{max(t-1,-1)}^{min(t+1,1)} x(\tau)y(t-\tau)d\tau$$

It is necessary to evaluate this integral separately in the ranges $[-\infty, -2]$, [-2, -1], [-1, 0], [0,1], [1,2], and $[2, \infty]$. In all the ranges, $y(t) = 1$, but the limits of integration are modified as follows:

$$x(t) \otimes y(t) = \begin{cases} 0 & -\infty \le t \le -2 \\ \int_{-1}^{(t+1)} (1+\tau)d\tau & -2 \le t \le -1 \\ \int_{0}^{(t+1)} (1-\tau)d\tau + \frac{1}{2} & -1 \le t \le 0 \\ \int_{(t-1)}^{0} (1+\tau)d\tau + \frac{1}{2} & 0 \le t \le 1 \\ \int_{t-1}^{1} (1-\tau)d\tau & 1 \le t \le 2 \\ 0 & 2 \le t \le \infty \end{cases}$$

$$x(t) \otimes y(t) = \begin{cases} 0 & -\infty \le t \le -2 \\ \frac{(t+2)^2}{2} & -2 \le t \le -1 \\ 1 - \frac{t^2}{2} & -1 \le t \le 0 \\ 1 - \frac{t^2}{2} & 0 \le t \le 1 \\ \frac{(t-2)^2}{2} & 1 \le t \le 2 \\ 0 & 2 \le t \le \infty \end{cases}$$

Note that, in this case, the convolution happens to be the same across the entire range. Nevertheless, it is a tricky task to compute it. As we will see later, the use of transforms greatly reduces the complexity of computing the convolution of two functions.

■

5.3 Signals

A signal is a set of data, usually a function of time. We denote a signal as $x(t)$, where t is nominally time. We have already seen two types of signals: **discrete** signals, where the values of t take integer values, and **continuous** signals, where t takes on real values. In contrast, **digital** signals are those where the signal $x(t)$ takes on one of a quantized set of values, nominally 0 and 1, and **analog** signals are those where the signal $x(t)$ takes on real values.

Some signals are **periodic**, so that their values repeat after every time period T. The smallest such period is called the **fundamental period** of the signal. Signals that do not have such a period are called **aperiodic**.

Most signals only exist for a limited period in time. They are called **time-limited** signals. Others can be modelled as eternal signals, and they are called **time-unlimited** signals.

Given a signal $x(t)$, we define a **time-shifted** signal that is time-shifted a time units to the right (i.e., it is delayed by a time units as compared to the original signal) to be $x(t-a)$. We similarly define a **time-scaled** signal where the signal is expanded or compressed in time by the scalar value a to be $x(at)$.

EXAMPLE 6: (SIGNALS)

Here are some examples of signals that arise in computer networks:

- The series of packet round-trip-times measured at a network source (discrete, analog, aperiodic, time-limited)
- The buffer occupancy over time at a router (continuous, digital, aperiodic, time-unlimited)
- The samples collected at an analog-to-digital transducer during a voice call (discrete, digital, aperiodic, time-limited)
- The physical carrier signal on a wired or wireless link (continuous, analog, periodic, time-unlimited)
- The number of HTTP requests received at a web server over intervals of 1 second (discrete, digital, aperiodic, time-unlimited)
- The number of routing updates received by a BGP daemon at a router over intervals of 5 minutes (discrete, digital, aperiodic, time-unlimited)

Digital systems such as computer networks approximate a continuous signal by a discrete signal whose time instants are sufficiently closely spaced and an analog signal by a digital signal whose quantization levels are sufficiently closely spaced. We will study what is precisely meant by ‘sufficiently closely spaced’ later in this chapter. For the remainder of this chapter, rather than these specific signals, we will consider signals in the abstract.

■

5.3.1 The complex exponential signal

A signal that frequently crops up in the study of transforms is the complex exponential signal denoted ke^{st}, where k is a real number and s is a complex quantity that can be written as $s = \sigma + j\omega$. We can expand ke^{st} as $ke^{st} = ke^{(\sigma + j\omega)t} = ke^{\sigma t}e^{j\omega t} = ke^{\sigma t}(\cos\omega t + j\sin\omega t) = ke^{\sigma t}\cos\omega t + jke^{\sigma t}\sin\omega t$. This is a signal whose real and imaginary components are both exponentially-modulated sinusoids having frequency ω. With carefully chosen values of σ, ω and k, this expression and its conjugate can represent a variety of seemingly unrelated signals:

- When $s = 0$, this reduces to the constant k (Figure 7 (a))
- When $\omega = 0$, this reduces to the real monotone exponential $ke^{\sigma t}$ (Figure 7 (b))
- When $\sigma = 0$, setting $s = \pm j\omega$ gives us the sum $k((\cos\omega t + j\sin\omega t) + (\cos\omega t - j\sin\omega t)) = 2k\cos\omega t$, which is a real sinusoid scaled by a factor k (Figure 7 (c))
- When $\sigma \neq 0$, setting $s = \pm j\omega$ gives us the sum $2ke^{\sigma}\cos\omega t$, which is a scaled real sinusoid that is modulated by a monotone exponential (Figure 7 (d))
- When $\sigma = 0$, the expression reduces to $k(\cos\omega t + j\sin\omega t)$, which is a helix oriented along the t axis, whose projections on the real-t and imaginary-t planes are sinusoids (Figure 8 (a))
- In the general case, the function represents an exponentially modulated helix oriented along the t axis, whose projections on the real-t and imaginary-t planes are exponentially modulated sinusoids (Figure 8 (b))

It is worth studying these figures carefully because they provide deep insight into the nature of a complex exponential that will greatly help in understanding the nature of transforms.

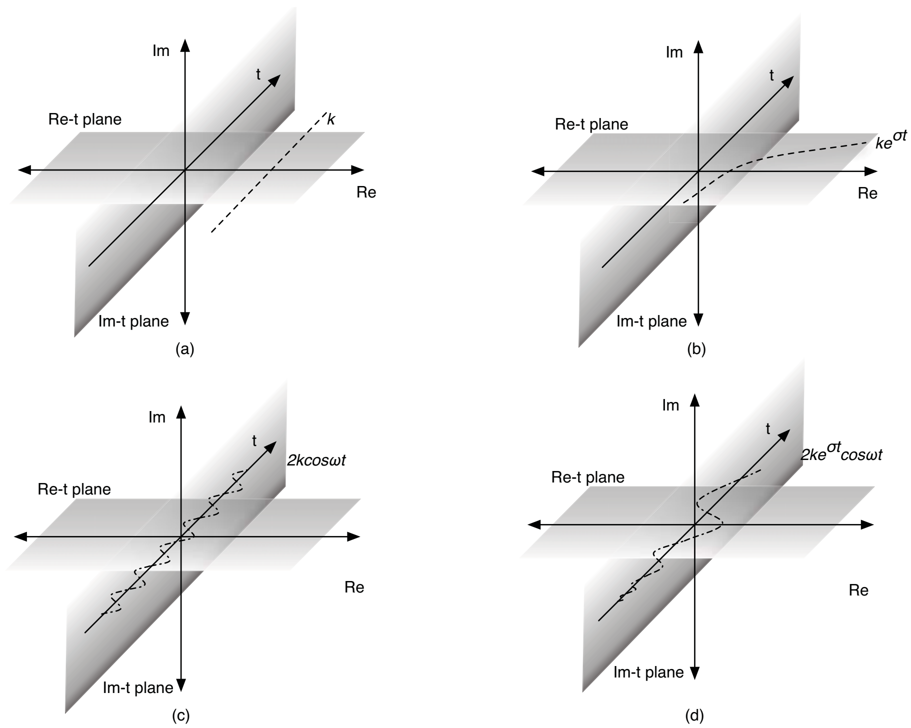


FIGURE 7. Elementary cases of the complex exponential signal

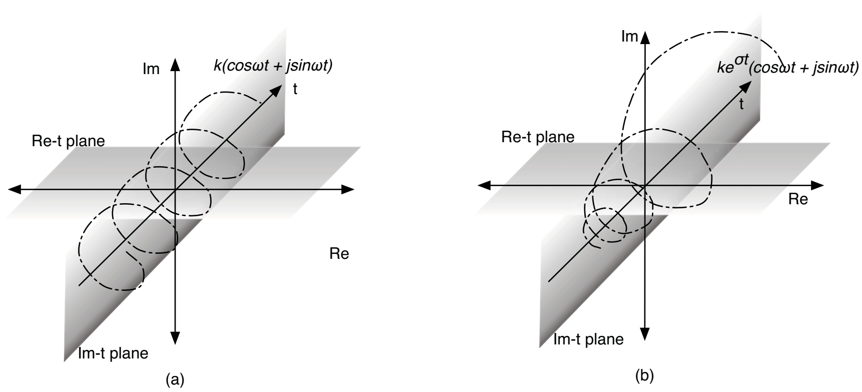


FIGURE 8. More general cases of the complex exponential signals

5.4 Systems

We define a system to be an entity that acts on a set of input signals to produce a set of output signals. Mathematically speaking, it is a function whose domain and ranges are both sets of functions. We typically denote the input to the system by the vector $\boldsymbol{x}(t)$ and its output by the vector $\boldsymbol{y}(t)$. The **transfer function** $\boldsymbol{H}(.)$ converts the input to the output, so that $\boldsymbol{y}(t) = \boldsymbol{H}(\boldsymbol{x}(t))$[4].

When studying a system, we have one of two goals in mind: to determine the output from an existing system when it receives a specific input (the **analysis** problem) or to design a system that exhibits a desired transformation from a set of input signals to a set of output signals (the **design** problem). In both cases, we use transforms as an essential analytical and design tool. This chapter focuses on the tools of analysis, and Chapter 8 on control theory focuses on the use of these tools.

5.4.1 Types of systems

As with signals, systems can be categorized in several ways. These include **continuous time** and **discrete time** systems and **analog** and **digital** systems, whose definitions mirror those for signals. We discuss four categories in more detail.

Causal systems are those that act only on inputs from the past. That is, they cannot know the future. In contrast, **acausal** systems are aware of the future. Although acausal systems sound improbable, they are actually rather easy to build if they operate on logs or traces of inputs, rather than inputs presented to the system in real time.

Memoryless systems act only on the inputs presented at any point in time, whereas the output of a **dynamic** system may depend both on current and past inputs.

A **time-invariant** system is one whose parameters do not change with time. So, if the system has an output $\boldsymbol{y}(t)$ at time t for an input $\boldsymbol{x}(t)$, then it has the output $\boldsymbol{y}(t\text{-}T)$ for an input $\boldsymbol{x}(t\text{-}T)$. That is, if $\boldsymbol{y}(t) = \boldsymbol{H}(\boldsymbol{x}(t))$, then $\boldsymbol{y}(t\text{-}T) = \boldsymbol{H}(\boldsymbol{x}(t\text{-}T))$.

Finally, and most importantly, a **linear** system is one that exhibits the properties of **additivity** and **homogeneity**.

Additivity: A system is additive if, given that input $\boldsymbol{x_1}$ leads to output $\boldsymbol{y_1}$ and input $\boldsymbol{x_2}$ leads to output $\boldsymbol{y_2}$, then the input $(\boldsymbol{x_1} + \boldsymbol{x_2})$ leads to the output $(\boldsymbol{y}_1+\boldsymbol{y}_2)$. That is, if $\boldsymbol{y}_1 = \boldsymbol{H}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \boldsymbol{H}(\boldsymbol{x}_2)$ then $\boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{H}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$.

Homogeneity (scaling): A system is homogeneous if, given that the input $\boldsymbol{x}$ leads to the output $\boldsymbol{y}$**,** then the input $k\boldsymbol{x}$, where k is a scalar, leads to the output $k\boldsymbol{y}$. That is, if $\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{x})$ then $k\boldsymbol{y} = \boldsymbol{H}(k\boldsymbol{x})$.

These two conditions can be combined into the single condition of **superposition**.

Superposition: A system exhibits superposition if the input $\boldsymbol{x}_1$ leads to output $\boldsymbol{y}_1$ and input $\boldsymbol{x}_2$ leads to output $\boldsymbol{y}_2$, then for all constants k_1 and k_2, the input $(k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2)$ leads to the output $(k_1\boldsymbol{y}_1+k_2\boldsymbol{y}_2)$. That is, if $\boldsymbol{y}_1 = \boldsymbol{H}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \boldsymbol{H}(\boldsymbol{x}_2)$ then $k_1\boldsymbol{y}_1 + k_2\boldsymbol{y}_2 = \boldsymbol{H}(k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2)$.

EXAMPLE 7: LINEAR SYSTEM

Is the system defined by the transfer function $H(x) = x^2$ linear?

Solution: $H(k_1x_1 + k_2x_2) = (k_1x_1 + k_2x_2)^2$, which is not the same as $k_1y_1 + k_2y_2 = k_1^2x_1^2 + k_2^2x_2^2$. Hence, the system is not linear.

■

Note that any system that limits the output to some maximum or minimum is not linear, because such a system is not homogeneous. But, in practice, outputs are almost always bounded (that is, they *saturate* at some limiting value) due to physical limitations. Therefore, most real systems are not linear. However, it is often possible to model the system behaviour, around its operating point, as being approximately linear. Specifically, we can use the first-order Taylor expansion of the transfer function in the region of the operating point to approximate the behaviour of the system. This technique, called **linearization**, is discussed in Section 8.3.2 on page 226.

4. We are using the term transfer function loosely. As discussed in Section 8.3.2 on page 226, in the context of control theory, the transfer function is more precisely described as the Laplace transform of the impulse response of the system. At this stage, however this loose but intuitively appealing description of the transfer function suffices.

5.5 Analysis of a linear time-invariant system

The class of linear, time-invariant (**LTI**) systems is important both because it is relatively easy to analyze and because many common systems can be approximated by LTI systems. Here, we study three aspects of an LTI system: the effect of an LTI system on a complex exponential input, the output of an LTI system with a zero input, and the output of an LTI system for an arbitrary input. We will revisit the third topic later to demonstrate the power of using transform domain techniques in analysis.

Note that the discussion here is for continuous time systems. The analysis of discrete time systems is nearly identical and we defer this discussion to our study of control theory in Chapter 8.

5.5.1 The effect of an LTI system on a complex exponential input

Consider an LTI system that is described by

$$\boldsymbol{y}(t) = \boldsymbol{H}(\boldsymbol{x}(t))$$

where $\boldsymbol{y}$ and $\boldsymbol{x}$ are, in general, vector functions of time and $\boldsymbol{H}$ is the transfer function. For simplicity of exposition, in the remainder of this discussion, we will assume that both x and y are scalar functions of time (so we will no longer use bold font to represent them). Suppose the input is a complex exponential function of time, so that $x(t) = ke^{st}$, where s is complex and k is real. Then,

$$y(t) = H(ke^{st}) \tag{EQ 17}$$

Now, consider the input $(e^{s\tau})(ke^{st})$, where τ is independent of t. Because the system is LTI and $e^{s\tau}$ is a scalar with respect to t, the corresponding output will be $(e^{s\tau})y(t)$. But note that the input $(e^{s\tau})(ke^{st})$ can be rewritten as $ke^{s(t+\tau)}$, which we recognize as a time-shifted input (i.e., t is replaced by $t+\tau$). Therefore, the corresponding output, because of time-invariance, must be $y(t+\tau)$. This gives us the relationship:

$$y(t+\tau) = e^{s\tau}y(t) \tag{EQ 18}$$

This must hold for all t, so it is true for $t = 0$, where

$$y(\tau) = e^{s\tau}y(0) \tag{EQ 19}$$

Differentiating both sides with respect to τ, we get

$$\frac{dy(\tau)}{d\tau} = sy(0)e^{s\tau} = sy(\tau) \tag{EQ 20}$$

where, for the last step, we invoke Equation 19. Using standard techniques for the solution of ordinary differential equations, it is easy to show that the unique solution to this differential equation is given by

$$y(t) = f(s)e^{st} \tag{EQ 21}$$

where $f(s)$ is an arbitrary function independent of t. This is an important result! It shows that for *any* LTI system, if the input is a complex exponential, so is the output. It is impressive that we can characterize all LTI systems so simply: after all, we have absolutely no idea what the system looks like 'on the inside,' yet we are able to state how it will respond to a complex exponential input.

We can gain additional insight into the nature of this phenomenon by comparing Equation 21 with Equation 17. We see that for the special input $x(t) = ke^{st}$, the effect of the LTI system is to act as a scalar multiplier, with the multiplication factor being $f(s)$. Therefore, $x(t) = ke^{st}$ is an **eigenfunction** of an LTI system, that is, an input function such that the effect of the system is simply to scale it. The corresponding **eigenvalue** is $f(s)$.

We can summarize this discussion by stating that the input $x(t) = ke^{st}$ is an eigenfunction of any LTI system for any value of s. Therefore, if we can represent an arbitrary input as the sum of such eigenfunctions, then the corresponding output will be the sum of the scaled inputs. From this perspective, the transforms we are about to study can be viewed as a way to represent an arbitrary input as the sum of complex exponentials.

5.5.2 The output of an LTI system with a zero input

A wide range of LTI systems can be described in the form of a **linear differential equation**, whose general form is:

$$\frac{d^N y(t)}{dt^N} + a_1\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_{N-1}\frac{dy(t)}{dt} + a_N y(t) = b_{N-M}\frac{d^M x(t)}{dt^M} + b_{N-M+1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \ldots + b_{N-1}\frac{dx(t)}{dt} + b_N x(t) \quad \text{(EQ 22)}$$

In nearly all practical systems, $M > N$. Here, we study the output of this class of LTI system when its input $x(t)$ is zero, so that the right hand side is zero. Using the operator D to denote differentiation, with D^N representing $\frac{d^N}{dt^N}$, we can rewrite the equation, for zero input, as

$$(D^N + a_1 D^{N-1} + \ldots + a_{N-1}D + a_N)y(t) = 0$$

where the polynomial in D on the left hand side is called the **characteristic polynomial** of the system. Note that this characteristic polynomial is of degree N. Therefore, from the fundamental theorem of algebra, it can be factorized as

$$((D-\lambda_1)(D-\lambda_2)\ldots(D-\lambda_N))y(t) = 0$$

where the λ_is (in general complex quantities) are the roots of the polynomial, and some of the roots may be repeated. Assume for the moment that the roots are distinct. In this case, each solution of this equation (that is, a value of $y(t)$ for which the equation is true) is given by

$$(D-\lambda_i)y(t) = 0$$

which we can expand as

$$\frac{dy(t)}{dt} = \lambda_i y(t)$$

Comparing this with Equation 20, we see that the solution of this equation is given by

$$y(t) = c_i e^{\lambda_i t}$$

where c is a constant independent of t (but may be a function of λ_i). Because each root of the equation generates a solution of this form, and the system is linear, by superposition the general solution (assuming distinct roots) is given by their sum:

$$y(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} \quad \text{(EQ 23)}$$

We interpret this as follows: if a linear differential LTI system (irrespective of its internal organization) receives no input, then, if the roots of its characteristic polynomial are distinct, its output can be expressed as the sum of complex exponentials. Recall that any complex exponential has an intrinsic frequency. Each root of Equation 23 corresponds to a **natural frequency** of the LTI system: a frequency at which the system vibrates given zero input. The solution in Equation 23 is therefore also called the **natural response** of the system. If a system is subjected to an input that then ceases (such as a burst of packets entering a router's buffer), the subsequent response of the system, because there is no additional input, will be its natural response. The natural response is a combination of its natural frequencies, which are intrinsic to the system.

A system that is given an input at a frequency that coincides with one of its natural frequencies will *resonate*, amplifying the input. This important phenomenon is widely employed in practical systems. For example, the basis of radio reception is to tune the resonant frequency[5] of an electronic circuit so that it selectively amplifies signals corresponding to a desired transmitter, ignoring the rest of the radio spectrum.

EXAMPLE 8: REAL NATURAL RESPONSE OF AN LTI SYSTEM

Compute the natural response of the LTI system given by $\frac{d^2y(t)}{dt^2} + \frac{5dy(t)}{dt} + 6y(t) = 8x(t)$.

Solution: The natural response is given by the differential equation $(D^2 + 5D + 6)y(t) = 0$. This can be factored as $((D+3)(D+2))y(t) = 0$. Thus, the natural response is given by $c_1e^{-3t} + c_2e^{-2t}$, where the two constants can be determined from the initial conditions $y(0)$ and $\dot{y}(0)$. The two phasors corresponding to this solution have no complex component, and therefore both correspond to a natural frequency of 0 (i.e., do not have an oscillatory behaviour).

■

EXAMPLE 9: COMPLEX NATURAL RESPONSE OF AN LTI SYSTEM

Compute the natural response of the LTI system given by $\frac{d^2y(t)}{dt^2} + y(t) = 8x(t)$.

Solution: The natural response is given by the differential equation $(D^2 + 1)y(t) = 0$. This can be factored as $((D+j)(D-j))y(t) = 0$. Thus, the natural response is given by $c_1e^{-jt} + c_2e^{jt}$, where the two constants can be determined from the initial conditions $y(0)$ and $\dot{y}(0)$. To get more insight into this solution, we use Euler's equation to expand this solution as

$$c_1(\cos(-t) + j\sin(-t)) + c_2(\cos(t) + j\sin(t))$$

$$= c_1(\cos(t) - j\sin(t)) + c_2(\cos(t) + j\sin(t))$$

$$= (c_1 + c_2)\cos(t) + j(c_2 - c_1)\sin(t)$$

For this solution to be real, it is necessary that $(c_1 + c_2)$ have no imaginary component, and $(c_2 - c_1)$ have no real component. This is only possible if $c_1 = a + jb$ and $c_2 = a - jb$ where *a* and *b* are real numbers. Recall that pairs of numbers of this form are called **complex conjugate pairs**. In general, for practical systems, the constants associated with the natural response come in complex conjugate pairs.

Continuing with our solution, we therefore substitute $a \pm jb$ for the constants, to get the natural response as $2a\ \cos(t) + 2b\ \sin(t)$. We recognize this as an *oscillatory* response, with a angular frequency of 1 radian/second. Interestingly, this natural response does not decay: the system oscillates indefinitely at this frequency of its own accord!

■

5. Strictly speaking, we tune the receiver to a range centered on the resonant frequency.

We have thus far assumed that the roots of the characteristic equation are distinct. This need not necessarily be the case. For example, if the characteristic equation is $D^2 + 2D + 1$, then the roots are -1 and -1, which are repeated. In the case of repeated roots, the form of the solution is slightly different. For a root λ that is repeated r times, it can be shown that the corresponding solution is given by

$$y(t) = (c_1 e^{\lambda t} + c_2 t e^{\lambda t} + \ldots + c_r t^{r-1} e^{\lambda t}) \qquad \textbf{(EQ 24)}$$

5.5.3 The output of an LTI system for an arbitrary input

We now study how an LTI system responds to a non-zero input. We use the notation $x(t) \to y(t)$ to denote that if the input to a system is $x(t)$ then its output is $y(t)$. Suppose that we are given $h(t)$, the **impulse response** of the system. This is the response to the system when the input is the Dirac delta $\delta(t)$. Using our notation, we write:

$$\delta(t) \to h(t) \qquad \textbf{(EQ 25)}$$

What does the impulse response look like? By definition, because the input is a delta function, it ceases immediately after time 0. Therefore, the response of the system is its natural response (other than at time 0 itself). We have already seen how to compute this response.

Recall from Equation 13 that multiplying $\delta(t)$ by $x(t)$ results in scaling the impulse by the value $x(0)$, that is, $x(t)\delta(t) = x(0)\delta(t)$. Because the system is linear, the response of the system to the scaled impulse $x(0)\delta(t)$ will be the scaled output $x(0)h(t)$, so that

$$x(t)\delta(t) \to x(0)h(t) \qquad \textbf{(EQ 26)}$$

Recall from Equation 14 that if we time-shift the impulse, we select a different 'slice' of $x(t)$. Specifically, $x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$. Because the system is both linear and time-invariant, the response to this time-shifted scaled impulse $x(\tau)\delta(t-\tau)$ will be the scaled and time-shifted impulse response $x(\tau)h(t-\tau)$, so that

$$x(\tau)\delta(t-\tau) \to x(\tau)h(t-\tau) \qquad \textbf{(EQ 27)}$$

Finally, Equation 15 tells us that we can assemble $x(t)$ by integrating these small 'slices' together, so that $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$. Clearly, this will result in the integration of the corresponding responses together as follows:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \to \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) \otimes h(t) \qquad \textbf{(EQ 28)}$$

This is important result tells us that if we know the impulse response $h(t)$ of an LTI system, then we can compute its response to *any* input $x(t)$ by convolving the input signal with the impulse response. Therefore, it is important to be able to compute the convolution of any two functions. This, however, is a difficult and complicated operation, as we saw in Section 5.2.5 on page 128. An important outcome of transform domain techniques is to convert the difficult convolution operation to a simple multiplication.

5.5.4 Stability of an LTI system

We briefly consider the stability of an LTI system. This topic is covered in more detail in Section 8.9 on page 245. Intuitively, a system is stable if, when all inputs are removed, the output either eventually dies down, or, at most, oscillates with a bounded amplitude. Here, we study how to characterize the stability of an LTI system.

We have seen that the behaviour of an LTI system is described by its natural frequencies, which are expressed as complex exponentials. In Section 5.3.1 on page 131 we saw that a complex exponential can show only one of three behaviours over time: it can grow without bound, it can decay to zero, or it can oscillate with a constant amplitude and frequency. Therefore, an LTI system can be easily characterized as stable, oscillatory (or marginally stable), or unstable as follows.

Consider the complex exponential $ke^{(\sigma + j\omega)t}$ which represents a solution to an LTI system's characteristic equation (Equation 23). If $\sigma > 0$, then this exponential grows without bound. On the other hand, $\sigma < 0$, then the exponential decays to zero. Finally, if $\sigma = 0$, then the exponential reduces to a pure sinusoidal oscillation.

Given this observation, we can now easily characterize the stability of an LTI system. If *all* its roots have a value of $\sigma < 0$, then the system is **asymptotically stable**. If even one root has a value of $\sigma > 0$, then it is **unstable**. Finally, if all the values of σ are 0, then the behaviour of the system depends on whether there are repeated roots. If there are no repeated roots, then the system is **purely oscillatory**. On the other hand, if there are repeated roots, then the system is unstable.

This observation allows us to great insight into the design of stable LTI systems: we must ensure that the system is such that all the roots of the characteristic polynomial (also called the **poles** of the system) lie in the left-half of the complex plane. This lies at the heart of control theory and we will consider it in more detail in Chapter 8.

5.6 Transforms

We are finally ready to embark on a study of transforms. Before plunging in, it is worth recapitulating why transforms are important. Transforms allow us to achieve three goals:

- The convolution of two functions, which arises in the computation of the output of an LTI system, can be computed more easily by transforming the two functions, multiplying the transformed functions, and then computing the inverse transform
- Transforms convert a linear differential equation into a simpler algebraic equation
- They give insight into the natural response of a system: the natural frequencies at which it oscillates. This allows us to quickly determine if there are special frequencies at which the system becomes unstable, if there are special frequencies that cause the system output to become zero, and the frequencies at which most of the output magnitude can be found.

We will consider five different transforms: the Fourier series, the Fourier transform, the Discrete Fourier Transform, the Laplace transform, and the Z transform. All the transforms share some common properties. For now, we observe that a transform converts a function of time into either a infinite sum or an integral of exponentials of a complex quantity. The transforms differ in whether the complex quantity is a pure sinusoid or an exponentially modulated sinusoid, and whether the input signal is continuous or discrete, as shown in Table 1. Although this may appear confusing at first glance, these concepts should become clear by the end of this chapter.

Transform	**Signal type**	**Transform description**
Fourier Series	Continuous, periodic	Sum of sinusoids
Fourier Transform	Continuous, periodic or aperiodic	Integral of sinusoids
Discrete-time Fourier Transform	Discrete, periodic or aperiodic	Sum of sinusoids
Laplace	Continuous, periodic or aperiodic	Integral of complex exponentials
Z	Discrete, periodic or aperiodic	Sum of complex exponentials

TABLE 1. Summary of transforms described in this chapter

5.7 The Fourier series

We first study **eternal periodic** signals, that is, signals that last for an infinitely long duration and whose values repeat identically after every period of duration T_0. All signals are, of course, time-limited, so this signal cannot be achieved in practice. However, they are still worth studying because the insight gained from the Fourier series allows us to define the Fourier transform for aperiodic signals.

Fourier showed that nearly every periodic signal $x(t)$, other than some pathological signals, can be represented at nearly every value of t as an infinite sum of sinusoids as follows:

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \quad \text{(EQ 29)}$$

where the **fundamental angular frequency** ω_0 is given by

$$\omega_0 = \frac{2\pi}{T_0} \quad \text{(EQ 30)}$$

and the constants a_0, a_k, and b_k are real numbers uniquely determined by $x(t)$. This is a remarkable equation! It shows that <u>any</u> periodic function, not just a function that looks sinusoidal, can be represented as the sum of sinusoids. It is as fundamental as the observation that any natural number can be denoted using only the ten symbols 0-9. More formally, using the appropriate vector space, we can show that the set of pairs of sinusoids in a Fourier series form an (infinite) orthogonal basis set, so that every vector (corresponding to an eternal periodic function), can be represented as a linear combination of this basis.

Each pair of terms in the series has a frequency that is a multiple of the fundamental frequency and is therefore called a **harmonic** of that frequency. Once the (potentially infinite) set of values associated with the constants a_0, a_k, and b_k is known, the function itself is completely specified, and can be synthesised from them alone. The accuracy of representation quickly improves with the number of terms[6].

Note that the constant a_0 can be viewed as a degenerate sinusoid with zero frequency. It is called the **DC** component of the signal by analogy to a direct-current electrical system that, unlike an alternating-current or AC system, does not have a sinusoidally oscillating voltage or current.

There is a graphical interpretation of a Fourier series that may add some additional insight. Recall from 5.2.1 on page 121 that a sinusoid can be thought of as a being generated by a rotating phasor. We see that each harmonic in the Fourier series corresponds to two rotating phasors that are offset by 90 degrees and with a common rotational frequency of $k\omega_0$ with magnitudes of a_k and b_k respectively. The sinusoid generated by these phasors add up in just the right way to form $x(t)$.

So far, we have only studied the form of the Fourier series. We do, of course, need a way to determine the constants corresponding to each term in a Fourier series. It can be shown that these are given by the following equations

6. However, note that due to the **Gibb's phenomenon**, the Fourier series can have an error of about 10%, even with many tens of terms, when representing a sharply changing signal such as a square wave.

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$
$$a_k = \frac{2}{T_0}\int_0^{T_0} x(t)\cos k\omega_0 t dt \quad \text{(EQ 31)}$$
$$b_k = \frac{2}{T_0}\int_0^{T_0} x(t)\sin k\omega_0 t dt$$

where the integral is taken over any period of length T_0 (because they are all the same).

The form of the Fourier series presented so far is called the **sinusoidal form**. These sinusoids can also be expressed as complex exponentials, as we show next. Note that the *k*th term of the Fourier series is

$$x_k(t) = a_k \cos k\omega_0 t + b_k \sin k\omega_0 t$$

Using Equation 5 and Equation 6, we can rewrite this as

$$x_k(t) = \frac{a_k}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t}) + \frac{b_k}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

Collecting like terms, and defining $c_0 = a_0$, $c_k = \frac{1}{2}(a_k - jb_k)$, $c_{-k} = \frac{1}{2}(a_k + jb_k)$, k>0 , we can rewrite this as

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=1}^{\infty} c_{-k} e^{j(-k)\omega_0 t}$$
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{(EQ 32)}$$

This compact notation shows that we can express *x*(*t*) as an infinite sum of complex exponentials. Specifically, it can be viewed as a sum of an infinite number of phasors, each with a real magnitude c_k and a rotational frequency of $k\omega_0$. It can be shown that the constants c_k can be found, if we are given *x*(*t*), by the relation:

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt \quad \text{(EQ 33)}$$

EXAMPLE 10: FOURIER SERIES

Find the Fourier Series corresponding to the series of rectangular pulses shown in Figure 9.

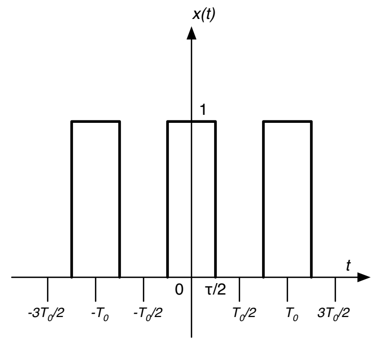


FIGURE 9. **An infinite series of rectangular pulses**

Solution: The *k*th coefficient of the Fourier series corresponding to this function is given by $c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$.

Instead of choosing the limits from 0 to T_0, we will choose the limits from $-T_0/2$ to $T_0/2$ because it is symmetric about 0. Note that in this range, the function is 1 in the range [-τ/2, τ/2] and 0 elsewhere. Therefore, the integral reduces to

$$c_k = \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T_0}e^{-jk\omega_0 t}\Big|_{-\frac{\tau}{2}}^{\tau/2} = \frac{1}{jk\omega_0 T_0}\left(e^{jk\omega_0\frac{\tau}{2}} - e^{-jk\omega_0\frac{\tau}{2}}\right) \quad \textbf{(EQ 34)}$$

Using Equation 6, and multiplying the numerator and denominator by τ/2, we can rewrite this as

$$c_k = \frac{\tau}{T_0}\left(\frac{\sin\left(\frac{k\omega_0\tau}{2}\right)}{\frac{k\omega_0\tau}{2}}\right) \quad \textbf{(EQ 35)}$$

Note that the coefficients c_k are real functions of τ (not *t*), which is a parameter of the input signal. For a given input signal, we can treat τ as a constant. Observing that $T_0 = \frac{2\pi}{\omega_0}$, the coefficients can be obtained as the values taken by a continuous function of ω, defined as

$$X(\omega) = \frac{\tau\omega_0}{2\pi}\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \quad \textbf{(EQ 36)}$$

for the values $\omega = k\omega_0$. That is, $c_k = X(k\omega_0)$. Thus, c_k is a *discrete* function of ω defined at the discrete frequency values $\omega = k\omega_0$ as shown in Figure 10.

The function *sin(x)/x* arises frequently in the study of transforms and is called the **sinc** function. It is a sine wave whose value at 0 is 1 and elsewhere is the sine function linearly modulated by distance from the *Y* axis. This function forms the envelope of the Fourier coefficients, as shown in Figure 10. Note that the function is zero (i.e., has zero-crossings) when a sine function is zero, that is, when $\omega\tau/2 = m\pi,\ m \neq 0$, or $\omega = 2m\pi/\tau,\ m \neq 0$.

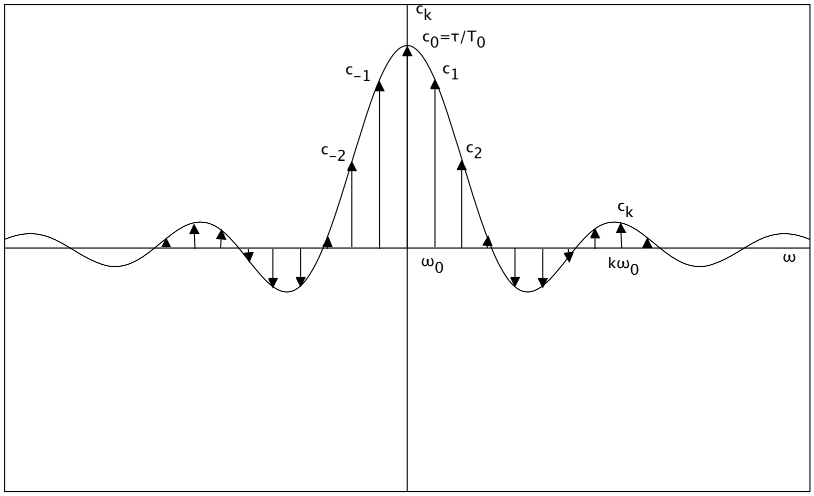


FIGURE 10. The sinc function as a function of ω

■

5.8 The Fourier Transform

A Fourier series can represent any *eternal* periodic function as an infinite sum of real sinusoids. However, it cannot be used to represent aperiodic functions. These are represented, instead, using a Fourier transform.

To motivate the Fourier transform, consider the behaviour of the pulse series of Example 10 when increasing the inter-pulse spacing T_0 while keeping the pulse width τ constant. Recall that for this signal the *k*th Fourier series coefficient c_k is given by $X(\omega) = \frac{\tau}{T_0}\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$ for $\omega = k\omega_0$ where $\omega_0 = \frac{2\pi}{T_0}$. As T_0 increases, ω_0 decreases, which brings the coefficients under the sinc curve (which are spaced ω_0 apart) closer together (see Figure 10). In the limit, as $T_0 \rightarrow \infty$, the periodic pulse train becomes a single pulse and the coefficients are spaced infinitely closely together, converging with the sinc curve. Thus, in the limit, the sinc curve corresponds to the Fourier series representing a single pulse and is called its **Fourier transform**.

Arguing along the same lines, we can analyse the limiting behaviour of the generic *k*th Fourier series coefficient $c_k = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} x(t)e^{-jk\omega_0 t}dt$ as $\omega_0 \rightarrow 0$ to define the Fourier transform of a signal *x*(*t*) as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{EQ 37}$$

This transforms a signal (a function of time) to a function of the complex quantity $j\omega$. The value of the transform for a specific value of $\omega = w$ can be viewed as the amount of signal energy at the frequency *w*. The magnitude at $\omega = 0$, for example, is the DC component of the signal. We represent the fact that $X(j\omega)$ is the transform of the signal *x*(*t*) as:

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) \quad \textbf{(EQ 38)}$$

The double-headed arrow indicates that $x(t)$ can be recovered from its transform, $X(jw)$ using the *inverse Fourier transform* discussed later.

EXAMPLE 11: FOURIER TRANSFORM

Compute the Fourier transform of a single rectangular pulse of height 1 and width τ centered on the origin (Figure 11). This signal is also denote *rect*(t/τ).

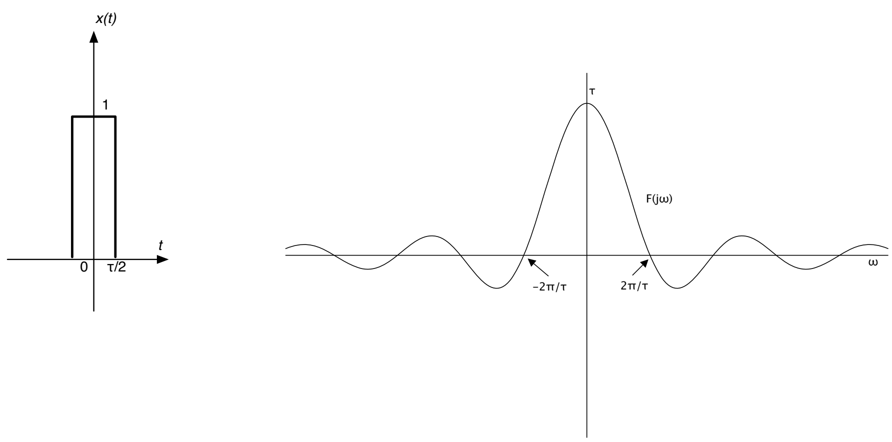


FIGURE 11. A single pulse and its Fourier transform

Solution: The Fourier transform of this signal $x(t)$ is given by $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. The function is zero outside of the interval [-τ/2, τ/2] and 1 inside this interval. Therefore, it reduces to $X(j\omega) = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}dt = \left.\frac{-e^{-j\omega t}}{jw}\right|_{\frac{-\tau}{2}}^{\tau/2} = \frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{j\omega}$.

Multiplying the number and denominator by $\frac{\tau}{2}$ as before, and using Euler's formula, we can rewrite this as

$$X(j\omega) = \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \tau\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \quad \textbf{(EQ 39)}$$

Therefore, the Fourier transform of $rect\left(\frac{t}{\tau}\right)$ is a sinc function that has a peak of height τ at the origin and zero crossings when $\omega\frac{\tau}{2} = k\pi$, $k \neq 0$ that is, when $\omega = \frac{2k\pi}{\tau}$, as shown in Figure 11. Note that the signal has no energy at frequencies corresponding to its zero crossings. This allows us to introduce a narrow-frequency signal at these frequencies and not interfere with the signal. This is the basis of **Orthogonal Frequency Division Multiplexing (OFDM)**, which is widely used in technologies such as 4G and WiFi.

■

The Fourier transform of a signal may not always exist. The existence of a transform is guaranteed by the **Dirichlet conditions**, which are that:

- the signal $x(t)$ is **absolutely integrable**, that is, $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
- $x(t)$ has a finite number of maxima and minima and a finite number of discontinuities in any finite interval

These conditions are satisfied by most signals that arise in practical situations. Note that signals with 'jumps,' such as rectangular pulses, do have Fourier transforms, as we have already seen. Indeed, we can even compute the Fourier transform of a highly discontinuous function, such as an impulse, as in the next example.

EXAMPLE 12: FOURIER TRANSFORM OF AN IMPULSE

Compute the Fourier transform of an impulse.

Solution: The transform is given by $X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$. From Equation 15, setting $\tau = 0$, we see that this integral is simply $X(\tau) = e^{-j\omega\tau}|_0 = 1$. Thus, the Fourier transform of an impulse is the constant 1 or $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$.

■

Given a transformed signal, we can recover it using the inverse Fourier transform, given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \textbf{(EQ 40)}$$

Compare this with Equation 32 to see how the summation in the Fourier series is transformed into its limiting integral.

EXAMPLE 13: INVERSE FOURIER TRANSFORM

What is the signal corresponding to the transformed signal $\delta(\omega - \omega_0)$?

Solution: The signal is given by $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega$. But, from Equation 15, the integral reduces to $e^{j\omega_0 t}$.

Therefore, we have $\frac{e^{j\omega_0 t}}{2\pi} \overset{\mathcal{F}}{\leftrightarrow} \delta(\omega - \omega_0)$ or $e^{j\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$.

■

5.8.1 Properties of the Fourier transform

We will now state some useful properties of the Fourier transform. Proofs of these properties can be derived from first principles using the definition of the transform and can be found in standard textbooks on the subject. These properties, along with the table of Fourier transforms in Table 2 allow us to compute the transforms of most functions that arise in practice without having to compute the integral in Equation 37 explicitly.

1. Linearity: If $x_1(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)$ and $x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_2(j\omega)$, then for any constants *a* and *b*,

$ax_1(t) + bx_2(t) \overset{\mathcal{F}}{\leftrightarrow} aX_1(j\omega) + bX_2(j\omega)$. Note that *a* and *b* can be complex numbers.

EXAMPLE 14: LINEARITY OF THE FOURIER TRANSFORM

Compute the Fourier transform of the eternal sinusoid $\cos(\omega_0 t)$.

Solution: We use Equation 5 to rewrite $\cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$. From the previous example and the linearity property, we get $\cos(\omega_0 t) \overset{\mathcal{F}}{\leftrightarrow} \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$. Thus, a sinusoid in the time domain corresponds to two impulses in the frequency domain. These are at the two frequencies ω_0 and $-\omega_0$ and represent counter-rotating phasors with the same frequency. The imaginary components of these phasors are always equal and opposite and therefore their sum is a real sinusoid.

■

2. Time-shifting: If $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$ then $x_1(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0} X_1(j\omega)$. That is, shifting the signal forward in time by t_0 units results in multiplying the transform by $e^{-j\omega t_0}$.

EXAMPLE 15: FOURIER TRANSFORM OF A TIME-SHIFTED IMPULSE

Compute the Fourier transform of an impulse that occurs at time t_0.

Solution: Since $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, we immediately get $\delta(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0}$.

■

3. Time-scaling: If $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$ then $x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$, where *a* is an arbitrary constant. Intuitively, *x*(*at*) is *a* times ‘faster’ than *x*(*t*). This results in the transform having a magnitude that is *a* times smaller and a frequency scale that is *a* times ‘slower.’For instance, if *a* = 5, what happens at time 10 in the unscaled function happens at time 2 in the scaled function. In contrast, a feature present at 100 radians/second in the unscaled transform is present at 500 radians/second in the scaled transform.

EXAMPLE 16: FOURIER TRANSFORM OF A SCALED FUNCTION

Compute the Fourier transform of a rectangular pulse as in Figure 11 but with a pulse width of 2τ.

Solution: The transform of a pulse of width τ is $\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$. When the pulse is twice as wide, the pulse starts at time $-\tau$ instead of at time $-\frac{\tau}{2}$, expanding the time-scale by a factor of 2, so that *a* = 1/2. The transform of the longer pulse is $2\tau\operatorname{sinc}(\omega\tau)$. This has zero crossings when $\omega\tau = k\pi$, that is, when $\omega = \frac{k\pi}{\tau}$, which is twice as fast as before.

■

4. Duality: If $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$ then $X(t) \overset{\mathcal{F}}{\leftrightarrow} 2\pi x(-j\omega)$ and $X(jt) \overset{\mathcal{F}}{\leftrightarrow} 2\pi x(-\omega)$. This property allows us to use knowledge of a transform to compute new transforms.

EXAMPLE 17: DUALITY

What is the Fourier transform of the constant function 1(*t*)?

Solution: Since $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, $1(t) \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(-j\omega)$. But $\delta(.)$ is symmetric about the origin, so, we have $1(t) \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(j\omega)$ which is an impulse centered on the origin, i.e., at zero frequency. This means that all the energy of the constant function is at a frequency of 0, which is what one would expect.

■

5. Differentiation: If $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega)$ then $\frac{dx^n(t)}{dt^n} \overset{\mathcal{F}}{\leftrightarrow} (j\omega)^n X(j\omega)$.

EXAMPLE 18: FOURIER TRANSFORM OF A DERIVATIVE

Compute the Fourier transform of the signal $j\omega_0 e^{j\omega_0 t}$.

Solution: Since $j\omega_0 e^{j\omega_0 t} = \frac{de^{j\omega_0 t}}{dt}$, $j\omega_0 e^{j\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} 2\pi(j\omega)\delta(\omega - \omega_0)$.

■

Table 2 presents some standard transform pairs. These are derived using both first principles as well as the properties of the Fourier transform discussed earlier.

No.	*x*(*t*)	*X*(*j*ω)	Notes
1	$\delta(t)$	1	
2	1	$2\pi\delta(\omega)$	
3	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
4	$\cos\omega_0 t$	$\pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$	Note that the Fourier transform of a cosine results in impulses in both the positive and negative frequency axis
5	$\sin\omega_0 t$	$j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$	See note above
6	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	
7	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$	$a > 0$, and this is an exponential defined only for positive time values
8	$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) + \frac{j\omega}{\omega_0^2 - \omega^2}$	This is a sinusoid defined only for positive time values

No.	x(t)	X(jω)	Notes
9	$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)) + \frac{\omega_0}{\omega_0^2-\omega^2}$	This is a sinusoid defined only for positive time values
10	$rect\left(\frac{t}{\tau}\right)$	$\tau\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$	This is a pulse symmetric about the origin with width τ

TABLE 2. Some standard Fourier transforms

6. Convolution property: If $x_1(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)$, $x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_2(j\omega)$, $y(t) \overset{\mathcal{F}}{\leftrightarrow} Y(j\omega)$ and $y(t) = x_1(t) \otimes x_2(t)$ then $Y(jw) = X_1(jw)X_2(jw)$. That is, convolution in the time domain corresponds to multiplication in the transform domain. For a LTI system with an impulse response of *h*(*t*), we know from Equation 28 that an input of *x*(*t*) leads to an output of $y(t) = x(t) \otimes h(t)$. Therefore, $Y(jw) = X(jw)H(jw)$. We can recover *y*(*t*) from this by taking the inverse transform.

The symmetric convolution property also holds: multiplication in the time domain corresponds to convolution in the transform domain. That is, if $x_1(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(j\omega)$, $x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_2(j\omega)$, $y(t) \overset{\mathcal{F}}{\leftrightarrow} Y(j\omega)$ and $y(t) = x_1(t)x_2(t)$ then $Y(jw) = \frac{1}{2\pi}X_1(jw) \otimes X_2(jw)$.

EXAMPLE 19: SOLVING A SYSTEM USING CONVOLUTION

Consider a system such that the Fourier transform *H*(*j*ω) of its transfer function *h*(*t*) is $\frac{1}{j\omega+2}$. What is its response to the input $e^{-t}u(t)$?

Solution: Because $x(t) = e^{-t}u(t)$, from Row 7 of Table 2, $X(j\omega) = \frac{1}{1+j\omega}$. Therefore, $Y(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)}$. Using the method of partial fractions (i.e., writing the expression as $\frac{a}{1+j\omega} + \frac{b}{2+j\omega}$ and solving for *a* and *b*)[7], we find that $Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$. To find the inverse transform, recall that the Fourier transform is linear, so we only need to find the inverse transform of each term in isolation. From Row 7 of Table 2 again, we get $y(t) = (e^{-t} - e^{-2t})u(t)$.

■

It is clear that the Fourier transform greatly simplifies the analysis of LTI systems: instead of having to deal with the complex convolution operator, we merely need to deal with multiplying two transforms then taking the inverse. In general, we can always find the inverse of the product from first principles by using Equation 40. However, because the Fourier transform may not always exist, system analysis is typically performed using the Laplace transform.

7. More generally, for $F(x) = \frac{P(x)}{(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_r)}$, we can write $F(x) = \frac{k_1}{(x-\lambda_1)} + \ldots + \frac{k_r}{(x-\lambda_r)}$ where $k_r = [(x-\lambda_r)F(x)]\big|_{x=\lambda_r}$. Also see Section 8.12 on page 255.

This concludes our discussion of the Fourier transform. We will now turn our attention to the more general Laplace transform, which shares all the properties of the Fourier transform, and can be applied to wider range of functions.

5.9 The Laplace Transform

The Fourier transform gives us insight into the frequencies where a signal's energy is concentrated and also simplifies the computation of convolutions. Its main problem is that it is not guaranteed (though it may exist) for functions that do not satisfy the Dirichlet conditions. We now study the Laplace transform that shares all the properties of the Fourier transform, and, additionally, is defined for a wider class of functions.

Recall that the Fourier transform is defined by $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. The Laplace transform is defined by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad \text{(EQ 41)}$$

where $s = \sigma + j\omega$. Intuitively, instead of taking the limiting sum of a series of pure sinusoids, as with the Fourier transform, the Laplace transform takes the limiting sum of a series of complex exponentials. Alternatively, writing e^{-st} as $e^{-\sigma t}e^{-j\omega t}$, we can write

$$X(s) = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}dt \quad \text{(EQ 42)}$$

so that the Laplace transform is identical to the Fourier transform of the function $(x(t)e^{-\sigma t})$. Intuitively, given a signal $x(t)$ that does not satisfy the Dirichlet conditions, we multiply it by an exponentially decreasing function $e^{-\sigma t}$, so that the product becomes absolutely integral. Note that, in this process, we must be careful to select a large enough value of σ to assure convergence. Therefore, unlike the Fourier transform, in specifying the Laplace transform, it is essential to give the **region of convergence**, that is, the values of σ for which the Laplace transform is defined.

As with the Fourier transform, we denote the relationship between a signal $x(t)$ and its transform $X(s)$ using the notation

$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \quad \text{(EQ 43)}$$

Note that the Laplace transform of a signal is, in general, a *complex* function, with both real and imaginary parts, hence its representation as $X(s)$.

The inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j}\int_{(\sigma - j\infty)}^{(\sigma + j\infty)} X(s)e^{st}ds \quad \text{(EQ 44)}$$

The limits of the integral denote integrating from $-\infty$ to ∞ along the imaginary axis for a value of σ chosen in the transform's region of convergence. In practice, the inverse operation is rarely performed from first principles, because the inverse can be found from a table of standard transforms and the (nice!) properties of the Laplace transform.

5.9.1 Poles, Zeroes, and the Region of convergence

We now examine, in greater depth, the somewhat mysterious notion of a region of convergence by referring to a concrete example. Observe that the unit step signal $u(t)$ is not absolutely integrable (the area under this signal is infinite). Therefore it does not satisfy the Dirichlet conditions and a Fourier transform is not guaranteed. Nevertheless, it can be shown that its Fourier transform is $\pi\delta(\omega) + \frac{1}{j\omega}$. In the next example, we will compute its Laplace transform.

EXAMPLE 20: LAPLACE TRANSFORM OF A UNIT STEP

Compute the Laplace transform of the unit step signal.

Solution: By definition, $X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^{\infty}$. Now, $e^{-s\infty}$ =0 iff $Re(s) > 0$. Assuming this is the case, we can evaluate the integral as $\frac{1}{s}$ $\quad Re(s) > 0$. Thus, $u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}$ $\quad Re(s) > 0$. The region of convergence of this transform is the set of all values of s where the condition $Re(s) > 0$ holds. Recall that s is complex, so this is the right half plane of the complex plane.

■

The next example reinforces this notion.

EXAMPLE 21: LAPLACE TRANSFORM OF A REAL EXPONENTIAL

Compute the Laplace transform of the signal $x(t) = u(t)e^{at}$ where a is a real constant.

Solution: By definition, $X(s) = \int_{-\infty}^{\infty} u(t)e^{at}e^{-st}dt = \int_{0}^{\infty} e^{-(s-a)t}dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^{\infty}$. As before, $e^{-(s-a)\infty}$ is 0 iff $Re(s-a) > 0$, that is, $Re(s) > a$. If this condition holds, then the integral evaluates to $\frac{1}{(s-a)}$. Therefore, $u(t)e^{-at} \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{(s-a)}$ $\quad Re(s) > a$. In this case, the region of convergence is the complex half plane defined by $Re(s) > a$.

What if $s = a$? In this case, the denominator becomes zero and the transform's value is infinite. This is called a **pole** of the system (the pole in the previous example was at 0).

It is easy to show that the transform pair is valid even for a complex a, where $a = \sigma + j\omega$ as long as $Re(s) > \sigma$.

■

The values of s for which the transform vanishes are called the **zeroes** of the transform. This is illustrated by the next example.

EXAMPLE 22: LAPLACE TRANSFORM OF A SINUSOID

Compute the Laplace transform of the sinusoid $u(t)\cos\omega_1 t$.

Solution: We use Euler's formula to rewrite the signal as $u(t)\left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right)$. By definition,

$$X(s) = \int_{-\infty}^{\infty} u(t)\left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right)e^{-st}dt$$

$$= \frac{\left(\int_0^{\infty} e^{j\omega_1 t}e^{-st}dt + \int_0^{\infty} e^{-j\omega_1 t}e^{-st}dt\right)}{2}$$

$$= \frac{1}{2}\left(\frac{1}{s - j\omega_1} + \frac{1}{s + j\omega_1}\right)$$

where, in the last step, we used the result from the previous example, and we are assuming that $Re(s) > 0$. This reduces to $\frac{s}{s^2 + \omega_1^2}$, with the region of convergence is $Re(s) > 0$. Note that the transform becomes infinite for $s = \pm j\omega_1$ (these are the poles of the transform) and is zero for $s = 0$, which is the zero of the transform.

■

It is important to keep track of the region of convergence of the transform. Two time-domain functions that are completely different may have an identical transform and may differ only in their region of convergence. Therefore, the Laplace transform is unique only if the region is also specified.

If the region of convergence of the Laplace transform of a signal includes the imaginary axis, then the Fourier transform of the signal is defined and can be obtained by setting $s = j\omega$. Otherwise, the Laplace transform of the signal exists, but not its Fourier transform.

5.9.2 Properties of the Laplace transform

The Fourier transform is a special case of the Laplace transform. Therefore, all the properties of the Fourier transform, namely linearity, time-shifting, time-scaling, duality, differentiation, and the convolution property, also hold for the Laplace transform, though in a slightly different form. We summarize the corresponding properties in Table 3.

Property	**Pre-condition**	**Post condition**	**Notes**
Linearity	$x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s) \quad \alpha_1 < Re(s) < \beta_1$ $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s) \quad \alpha_2 < Re(s) < \beta_2$	$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$ $max(\alpha_1, \alpha_2) < Re(s) < min(\beta_1, \beta_2)$	*a* and *b* are arbitrary constants and can be complex
Time scaling	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \quad \alpha < Re(s) < \beta$	$x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$ $a\alpha < Re(s) < a\beta$	An compression in the time scale expands the frequency scale

Property	Pre-condition	Post condition	Notes
Frequency scaling	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \qquad \alpha < Re(s) < \beta$	$\frac{1}{\lvert a\rvert}x\left(\frac{t}{a}\right) \overset{\mathcal{L}}{\leftrightarrow} X(as)$ $\frac{\alpha}{a} < Re(s) < \frac{\beta}{a}$	An compression in the frequency scale expands the time scale
Time shifting	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \qquad \alpha < Re(s) < \beta$	$x(t-t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s)$ $\alpha < Re(s) < \beta$	Delaying by a time t_0 multiplies the transform by e^{-st_0}
Frequency shifting	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \qquad \alpha < Re(s) < \beta$	$e^{at}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s-a)$ $\alpha - Re(a) < Re(s) < \beta - Re(a)$	Note the change in the region of convergence due to frequency shifting
Differentiation	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \qquad \alpha < Re(s) < \beta$	$\frac{d^n x(t)}{dt^n} \overset{\mathcal{L}}{\leftrightarrow} s^n X(s)$ $\alpha < Re(s) < \beta$	Differentiation in the time domain corresponds to multiplication by a factor of s in the transform domain
Integration	$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \qquad \alpha < Re(s) < \beta$	$\int_{-\infty}^{t} x(r)dr \overset{\mathcal{L}}{\leftrightarrow} \frac{X(s)}{s}$ $max(\alpha, 0) < Re(s) < \beta$	Integration in the time domain corresponds to division by a factor of s in the transform domain. Note that the region of convergence also changes
Convolution in time domain	$x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s) \qquad \alpha_1 < Re(s) < \beta_1$ $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s) \qquad \alpha_2 < Re(s) < \beta_2$ $y(t) = x_1(t) \otimes x_2(t)$	$y(t) \overset{\mathcal{L}}{\leftrightarrow} Y(s)$ $Y(s) = X_1(s)X(s)_2$ $max(\alpha_1, \alpha_2) < Re(s) < min(\beta_1, \beta_2)$	$y(t)$ is a convolution of two functions $x_1(t)$ and $x_2(t)$. The product of their transforms $X_1(s)$ and $X_s(s)$ determine $Y(s)$, the transform of $y(t)$, and its region of convergence

Property	Pre-condition	Post condition	Notes
Multiplication in time domain	$x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s) \quad \alpha_1 < Re(s) < \beta_1$ $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s) \quad \alpha_2 < Re(s) < \beta_2$ $y(t) = x_1(t)x_2(t)$	$y(t) \overset{\mathcal{L}}{\leftrightarrow} Y(s)$ $Y(s) = \frac{1}{2\pi j} X_1(s) \otimes X_2(s)$ $\alpha_1 + \alpha_2 < Re(s) < \beta_1 + \beta_2$	*y*(*t*) is a product of two functions $x_1(t)$ and $x_2(t)$. The convolution of their transforms $X_1(s)$ and $X_s(s)$ determine *Y*(*s*), the transform of *y*(*t*), and its region of convergence
Final value theorem	$X(s)$	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$	The limiting value of *x(t)* in the time domain as $t \to \infty$ is given by finding the limit of *sX(s)* as $s \to 0$ in the transform domain

TABLE 3. Properties of the Laplace transform

These properties, along with the table of common transforms (Table 4) allow us to derive the transform of most common signals without having to derive them from first principles. Note that these transforms are defined for functions that exist only for $t > 0$, so that the Laplace integral has limits from 0 to ∞. This is also called the **unilateral** Laplace transform. In a practical system, a *causal* signal is 0 for $t < 0$, so the unilateral transform suffices for all practical systems.

No.	Signal	*x*(*t*)	*X*(*s*)	Region of convergence
1	Delta or unit impulse	$\delta(t)$	1	All *s*
2	Unit step	$u(t)$	$1/s$	$Re(s) > 0$
3	Delayed delta	$\delta(t - t_0)$	$e^{-t_0 s}$	All *s*
4	Ramp	$tu(t)$	$\frac{1}{s^2}$	$Re(s) > 0$
5	Exponential decay	$e^{\alpha t}u(t)$	$\frac{1}{s - \alpha}$	$Re(s) > \alpha$
6	*N*th power decay	$\frac{t^n}{n!}e^{\alpha t}u(t)$	$\frac{1}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
7	Sine	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$	$Re(s) > 0$
8	Cosine	$\cos(\omega t)u(t)$	$\frac{s}{s^2 + \omega^2}$	$Re(s) > 0$
9	Exponentially modulated sine	$e^{\alpha t}\sin(\omega t)u(t)$	$\frac{\omega}{(s-\alpha)^2 + \omega^2}$	$Re(s) > \alpha$
10	Exponentially modulate cosine	$e^{\alpha t}\cos(\omega t)u(t)$	$\frac{s-\alpha}{(s-\alpha)^2 + \omega^2}$	$Re(s) > \alpha$

TABLE 4. Some standard Laplace transforms

EXAMPLE 23: SOLVING A SYSTEM USING THE LAPLACE TRANSFORM

Consider a system such that the Laplace transform $H(s)$ of its transfer function $h(t)$ is $\frac{1}{s+2}$. What is its response to the input $e^{-t}u(t)$?

Solution: Since $x(t) = e^{-t}u(t)$, from Row 5 of Table 4, $X(s) = \frac{1}{s+1}$. Therefore, $Y(s) = \frac{1}{(1+s)(2+s)}$. Using the method of partial fractions (i.e., writing the expression as $\frac{a}{1+s} + \frac{b}{2+s}$ and solving for a and b - also see Section 8.12 on page 255), we find that $Y(s) = \frac{1}{1+s} - \frac{1}{2+s}$. To find the inverse transform, recall that the Laplace transform is linear, so we only need to find the inverse transform of each term in isolation. From Row 5 of Table 4 again, we get $y(t) = (e^{-t} - e^{-2t})u(t)$.

This analysis assumes that the system is at rest at time zero. If this is not the case, then the actual response would be the sum of the natural response, which is the way the system behaves assuming that there is no external input, and the forced response, which is the way the system behaves assuming that its initial state is at rest. The details of this analysis are beyond the scope of this text.

■

This concludes our study of the Laplace transform. We will now focus on two transforms that deal with *discrete* signals rather then continuous signals, as we have done so far.

5.10 The Discrete Fourier Transform and Fast Fourier Transform

Most signals in computing systems take on values only at discrete moments in time. Specifically, the signal is defined only at times nT, where n is an integer, and T, a real, is the inverse of the clock frequency. We denote such a signal as $x[nT]$ in contrast to a continuous signal, which is denoted $x(t)$.

5.10.1 The impulse train

We define an impulse train $s_T(t)$ with parameter T to be a signal that consists of an infinite series of delta signals with time-separation T. It is denoted as

$$s_T = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \text{(EQ 45)}$$

What is the Fourier transform of an impulse train? By the linearity property of the Fourier transform, the transform of the infinite sum in Equation 45 is the sum of the individual transforms. From Example 13, $\delta(t-nT) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega nT}$. Therefore, the Fourier transform of an impulse train is given by:

$$s_T \overset{\mathcal{F}}{\leftrightarrow} \sum_{n=-\infty}^{\infty} e^{-j\omega nT} \quad \text{(EQ 46)}$$

5.10.2 The discrete-time Fourier transform

We now consider the Fourier transform of the discrete signal $x[nT]$. We start by studying the product of an impulse train with a *continuous* signal $x(t)$, given by

$$x(t)s_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x(t)\delta(t-nT) \qquad \text{(EQ 47)}$$

From Equation 14, we can rewrite this as

$$x(t)s_T(t) = \sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT) \qquad \text{(EQ 48)}$$

We thus interpret the product of a signal with an impulse train as an impulse train whose value at time nT is an impulse of height $x(nT)$, the value of $x(t)$ at that time. This is called a **modulated** impulse train.

We now return to the function $x[nT]$ that is defined only at the discrete times nT. We argue that this function is equivalent to a modulated impulse train of a function $x(t)$ if the value of $x(t)$ matches the value if $x[nT]$ at every discrete time. From a strictly mathematical perspective, this is not quite correct: $x[nT]$ is discrete and therefore undefined between sample times whereas the product $x(t)s_T(t)$ is defined for all time. However, in the interests of developing intuition, we will ignore this distinction, writing

$$x[nT] = x(t)s_T(t) = \sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT) \qquad \text{(EQ 49)}$$

$$= \sum_{n=-\infty}^{\infty}x[nT]\delta(t-nT) \qquad \text{(EQ 50)}$$

where, in the second step, we use the fact that at the discrete times nT, $x(nT) = x[nT]$. Let us compute the Fourier transform of $x[nT]$. Because of the linearity of the Fourier transform, we can simply compute the transform of a single value $x[nT]\delta(t-nT)$ for a particular value of n then sum these up. Note that the first term in this expression, i.e. $x[nT]$, is *independent* of t. Therefore, from the perspective of the transform, it is a constant multiple of $\delta(t-nT)$. Moreover, from Example 13, $\delta(t-nT) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega nT}$. Therefore, we have

$$x[nT] \overset{\mathcal{F}}{\leftrightarrow} \sum_{n=-\infty}^{\infty}x[nT]e^{-j\omega nT} \qquad \text{(EQ 51)}$$

This defines the **discrete-time Fourier transform** of the discrete function $x[nT]$ and is denoted $\tilde{X}(j\omega)$.

EXAMPLE 24: DISCRETE-TIME FOURIER TRANSFORM OF A DISCRETE EXPONENTIAL SIGNAL

Compute the discrete-time Fourier transform of the signal $a^n u[nT]$ where $0 < a < 1$.

Solution: The transform is given by $\sum_{n=-\infty}^{\infty} a^n u(nT) e^{-j\omega nT} = \sum_{n=0}^{\infty} (ae^{-j\omega T})^n$. We recognize this as a geometric series with parameter $ae^{-j\omega T}$. Now, $e^{-j\omega T}$ is a phasor with unit magnitude and we already know $|a| < 1$, so this parameter is < 1 and the geometric series sums to $\frac{1}{1 - ae^{-j\omega T}}$, which is the desired transform.

■

The corresponding inverse transform is given by

$$x[nT] = \frac{1}{\omega_s} \int_0^{\omega_s} \tilde{X}(j\omega) e^{j\omega nT} d\omega \qquad \text{(EQ 52)}$$

where $\omega_s = \frac{2\pi}{T}$.

5.10.3 Aliasing

The sampled version of a signal may not be unique. Two non-identical signals whose sampled versions are identical are said to be **aliases** of each other. We now study the conditions under which aliasing is possible.

To understand the problem, we need to re-examine the nature of an impulse train, s_T. Recall that this is a periodic function with a period of T. Therefore, its fundamental angular frequency

$$\omega_s = \frac{2\pi}{T} \qquad \text{(EQ 53)}$$

Now, any periodic signal can be expressed as a Fourier series using Equation 32, restated here:

$$s_T = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} \qquad \text{(EQ 54)}$$

The kth term of this series is given by

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s_T e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-jk\omega_s t} dt \end{aligned} \qquad \text{(EQ 55)}$$

where, in the second step, we expand s_T using Equation 45. Note that the integration limits are in the range [-$T/2$, $T/2$]. In this range, the only delta signal in the summation that is non zero is $\delta(t)$. So, the infinite sum reduces to a single term, $\delta(t)e^{-jk\omega_s t}$. Moreover, because $\delta(t)$ is zero valued other than at the origin, we can expand the integral to cover the entire real line, to get

$$c_k = \frac{1}{T}\int_{-\infty}^{\infty} \delta(t)e^{-jk\omega_s t}dt \quad \text{(EQ 56)}$$

But, from Equation 15, the integral is simply the value of $e^{-jk\omega_s t}$ at t = 0, which is 1, so $c_k = \frac{1}{T}$ and we can rewrite Equation 54 as

$$s_T = \sum_{k=-\infty}^{\infty} \frac{1}{T}e^{jk\omega_s t} \quad \text{(EQ 57)}$$

Let us use this to compute the Fourier transform of the impulse train. From the linearity property, we only need to compute the transform of $e^{jk\omega_s t}$. From Row 3 of Table 2, this is given by $2\pi\delta(\omega - k\omega_s)$. Therefore, we have

$$s_T \overset{\mathcal{F}}{\leftrightarrow} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\omega - k\omega_s) \quad \text{(EQ 58)}$$

$$s_T \overset{\mathcal{F}}{\leftrightarrow} \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad \text{(EQ 59)}$$

This is an alternative form of the transform we already saw in Equation 46. However, this form gives us additional insight. Notice that the transform is an infinite series of impulses in the frequency domain that are separated by the fundamental frequency ω_s. In other words, it is an impulse train in the transform domain. Therefore, we have the beautiful result that the *discrete-time Fourier transform of an impulse train in the time domain is an impulse train in the transform domain*!

We can use this result to explain the need for bandlimiting $x(t)$. From Equation 49, we see that the discrete (sampled) signal $x[nT]$ is the product of $x(t)$ and s_T. Therefore from the convolution property of the Fourier transform, $\tilde{X}(j\omega)$, the transform of $x[nT]$ (which we previously computed in Equation 51) can also be written as

$$\begin{aligned}\tilde{X}(j\omega) &= \frac{1}{2\pi}X(j\omega) \otimes \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(j\omega) \otimes \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \otimes \delta(\omega - k\omega_s)\end{aligned} \quad \text{(EQ 60)}$$

Carefully examine the expression within the summation, which is the convolution of $X(j\omega)$ with a frequency-shifted impulse. From Equation 10, we see that this reduces to $X(j\omega - k\omega_s)$, which is the (continuous) Fourier transform of $x(t)$ shifted in frequency by ω_s. Therefore, the summation represents the addition of scaled and frequency-shifted replicas of $X(j\omega)$ (see Figure 12). That is, sampling a signal $x(t)$ with a sampler of period T to produce $x[nT]$ causes the transform of the sampled signal to infinitely replicate the transform of the original signal with a frequency spacing of ω_s.

Suppose that the support of $X(j\omega)$ (the range of values for which it is non-zero) is smaller than ω_s. Then, the shift-and-add operation will result in creating multiple replicas of $X(j\omega)$ that do not touch each other. We can pick any one copy (by multiplying it with a pulse of width ω_s in the frequency domain, also called a **band-pass filter**) and take the inverse transform to recover $x(t)$. In other words, the original *continuous* signal can be recovered despite *digital* sampling! This is quite a neat trick, for we have gone from the continuous domain to a discrete domain with no loss of information. The corresponding condition is called the **Nyquist criterion**.

To get some insight into this condition, recall that the Fourier transform of a cosine signal of frequency ω_L (Row 4 of Table 2) results in the creation of two symmetric impulses at $\pm\omega_L$. Because any signal can be represented as an integral of sinusoids using the Fourier transform, intuitively, if the signal has a highest inherent frequency component of ω_L, then its Fourier transform will have a bandwidth of $2\omega_L$ (see Figure 12). If we wish to sample such a signal in a way that preserves all the information of the original signal, we require $\omega_s > 2\omega_L$. This is an alternative statement of the Nyquist criterion and can be remembered as: *to prevent aliasing, the sampling function should have a frequency that is at least twice that of the highest frequency component of a signal*.

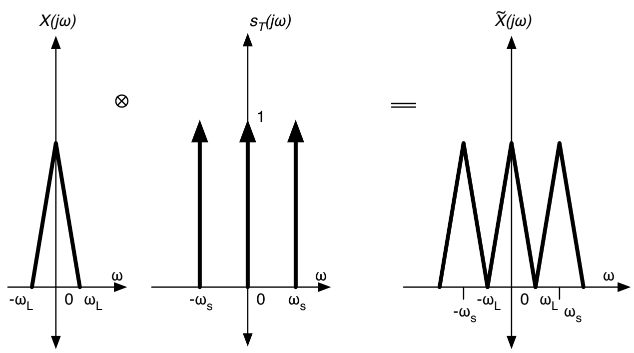


FIGURE 12. **The effect of sampling on the Fourier transform of a signal**

This result is widely applicable to computer networking, where we are nearly always dealing with sampled signals. It is important to ensure that, given as estimate of highest estimated frequency of the underlying signal, the sample rate is at least twice as fast. Recall that any signal with a sharp transition (such as a pulse signal) has a Fourier transform that has an infinite support, so that it is not bandlimited. Sampling any such function is guaranteed to introduce aliasing. Therefore, it is important to ensure, in practice, that most of the signal energy lies within $\pm\omega_s$.

5.10.4 The Discrete-Time-and-Frequency Fourier Transform and the Fast Fourier Transform (FFT)

So far, we have placed no restrictions on the form of the transformed signal. In the discrete-time Fourier transform, for example, the transformed signal extends over all frequencies. Suppose we introduce the restriction that the transformed signal must be represented by a finite set of discrete frequencies. In other words, the transformed signal must modulate a finite impulse train in the frequency domain. In this case, we call the transform the discrete-time-and-frequency Fourier transform, that is usually abbreviated as the ‘discrete Fourier transform’ or DFT. The Fast Fourier Transform (FFT) is a clever technique for rapidly computing the DFT that we will study later in this section.

It can be shown that the DFT of a discrete-time function cannot be uniquely defined unless the function is either time-limited or, if eternal, then periodic with a finite period. Moreover, the duration of the function (or its period) must be an integer multiple of the sampling period *T*. We have already seen that a discrete-time function can be represented as a modulated impulse train, whose discrete-time Fourier transform is a modulated impulse train in the frequency domain. This fact can be used to show that a discrete-time function that is either time-limited or periodic with *N* samples per period is discrete and periodic in the transform domain with *N* component frequencies.

We denote the discrete-time signal, which needs to be specified at *N* instants, as $x[0], x[T], x[2T], ..., x[(N-1)T]$. We denote the period of the signal as T_0, with $T_0 = NT$. Then, the sampling frequency is given by

$$\omega_s = \frac{2\pi}{T} \qquad \text{(EQ 61)}$$

and the signal frequency (corresponding to the period over which the signal repeats) is given by

$$\omega_0 = \frac{2\pi}{T_0} \qquad \text{(EQ 62)}$$

The *k*th frequency of the DFT of this signal is denoted $X[jk\omega_0]$ where the term $jk\omega_0$ indicates that the transform domain is complex and discrete, with a fundamental frequency of ω_0. This term of the DFT is given by

$$X[jk\omega_0] = \frac{1}{NT}\sum_{n=0}^{N-1} x[nT]e^{-jk\omega_0 nT} \qquad \text{(EQ 63)}$$

The corresponding inverse transform is given by

$$x[nT] = T\sum_{k=0}^{N-1} X[jk\omega_0]e^{jk\omega_0 nT} \qquad \text{(EQ 64)}$$

EXAMPLE 25: DFT

Compute the DFT of the function shown in Figure 13. (Assume $T = 1$, $T_0 = 9$).

Solution: We have $x[0] = 0$, $x[1] = 1$, $x[2] = 2$, $x[3] = 3$, $x[4] = 4$, $x[5] = 3$, $x[6] = 2$, $x[7] = 1$, $x[8] = 0$. Also, $N = \frac{T_0}{T} = 9$, and $\omega_0 = \frac{2\pi}{9}$.

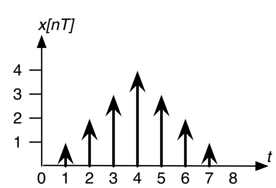


FIGURE 13. **Figure for Example 23**

The first Fourier value, $X[0]$ is given by $\frac{1}{NT}\sum_{n=0}^{N-1} x[nT]e^{-j0.\omega_0 nT} = \frac{1}{N}\sum_{n=0}^{N-1} x[nT] = \frac{1}{9}(1+2+3+4+3+2+1) = \frac{16}{9}$.
This is also the arithmetic mean of the input signal (the first DFT coefficient is the always the average of the signal values because this represents the DC value of the signal).

The second Fourier value, with $k = 1$, is $X\left[j\frac{2\pi}{9}\right]$ is given by

$$\frac{1}{9}\sum_{n=0}^{8} x[nT]e^{-j1.\omega_0 n} = \frac{1}{9}\left(1e^{-j\frac{2\pi}{9}} + 2e^{-j\frac{4\pi}{9}} + 3e^{-j\frac{6\pi}{9}} + \ldots + 2e^{-j\frac{12\pi}{9}} + 1e^{-j\frac{14\pi}{9}}\right).$$

Recall that this can be reduced to a complex quantity of the form $a + jb$ by expanding each term using Euler's formula.

The third Fourier value, with $k = 2$, $X\left[j\frac{4\pi}{9}\right]$ is given by

$$\frac{1}{9}\sum_{n=0}^{8} x[nT]e^{-2.j\omega_0 n} = \frac{1}{9}\left(1e^{-j\frac{4\pi}{9}} + 2e^{-j\frac{8\pi}{9}} + 3e^{-j\frac{12\pi}{9}} + \ldots + 2e^{-j\frac{24\pi}{9}} + 1e^{-j\frac{28\pi}{9}}\right).$$

The remaining values are computed similarly.

■

5.10.5 The Fast Fourier Transform

The Fast Fourier Transform (FFT) is a technique for rapidly computing the terms of a discrete time-and-frequency Fourier transform. It draws upon the observation that the terms in this transform follow a regular recursive structure, as described next.

To see the hidden structure of a DFT we start by simplifying the notation. Suppose time is normalized so that the sampling period T has a unit duration so that $T = 1$ and $T_0 = NT = N$. Then, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N}$. Denote by ω_N the value $e^{j\frac{2\pi}{N}}$.

Finally, we write the signal as $x[n]$ and its transform by $X[k]$. Then, we can rewrite $X[jk\omega_0] = \frac{1}{NT}\sum_{n=0}^{N-1} x[nT]e^{-jk\omega_0 nT}$ as $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\omega_N^{-kn}$. We expand the right hand side as:

$$\frac{1}{N}(x[0] + x[1]\omega_N^{-k} + x[2]\omega_N^{-2k} + x[3]\omega_N^{-3k} + \ldots + x[N-1]\omega_N^{-k(N-1)}) \quad \textbf{(EQ 65)}$$

To obtain this value is computationally difficult because we need to compute N powers of the complex quantity ω_N. The heart of the FFT algorithm is to reduce this computational overhead.

For simplicity, assume that N is even. Then, we can partition this expression into two equal-length series, one with the odd terms and one with the even terms. The even terms form the series:

$$\frac{1}{N}(x[0] + x[2]\omega_N^{-2k} + x[4]\omega_N^{-4k} + \ldots + x[N-2]\omega_N^{-k(N-2)})$$

Denote by ω_E the value ω_N^2. Then, we can rewrite this series as:

$$\frac{1}{N}\left(x[0] + x[2]\omega_E^{-k} + x[4]\omega_E^{-2k} + \ldots + x[N-2]\omega_E^{-\frac{k(N-2)}{2}}\right) \quad \textbf{(EQ 66)}$$

We can interpret this as the DFT of a series with only *N/2* terms, therefore requiring only *N/2* powers of ω_E. Now, let us consider the odd terms of Equation 65:

$$\frac{1}{N}(x[1]\omega_N^{-k} + x[3]\omega_N^{-3k} + x[5]\omega_N^{-5k} + \ldots + x[N-1]\omega_N^{-k(N-1)})$$

We can rewrite this as:

$$\frac{\omega_N^{-k}}{N}(x[1] + x[3]\omega_N^{-2k} + x[5]\omega_N^{-4k} + \ldots + x[N-1]\omega_N^{-k(N-2)})$$

$$= \frac{\omega_N^{-k}}{N}\left(x[1] + x[3]\omega_E^{-k} + x[5]\omega_E^{-2k} + \ldots + x[N-1]\omega_E^{-\frac{k(N-2)}{2}}\right) \quad \text{(EQ 67)}$$

Notice that the powers of ω_E on the right hand side are exactly the same as in Equation 66. Therefore, once we compute the *N/2* powers of ω_E to compute Equation 66, we can obtain the sum of the odd terms with only one additional multiplication.

It is easy to see how this structure can be made to recurse: we rewrite Equation 66 and Equation 67 as odd and even terms, reducing the number of exponentials by another factor of two. If *N* is a power of 2, we can continue in this fashion until we get to the trivial case of compute the DFT of a series with only two terms, which is given by $\frac{1}{2}(x[0] + x[1]\omega_N^{-k})$. We can then unwind the recursion to find the required transform. This is illustrated by the following example.

EXAMPLE 26: (FAST FOURIER TRANSFORM)

Use the FFT technique to compute the Discrete Fourier Transform of the signal $x[i] = i$ for $0 \le i \le 7$.

Solution: We have *N* = 8. Denote by ω_N the value $e^{j\frac{2\pi}{8}}$. The *k*th term of the transform is given by

$$\frac{1}{8}(\omega_N^{-k} + 2\omega_N^{-2k} + 3\omega_N^{-3k} + 4\omega_N^{-4k} + 5\omega_N^{-5k} + 6\omega_N^{-6k} + 7\omega_N^{-7k})$$

$$= \frac{1}{8}(2\omega_N^{-2k} + 4\omega_N^{-4k} + 6\omega_N^{-6k}) + \frac{\omega_N^{-k}}{8}(1 + 3\omega_N^{-2k} + 5\omega_N^{-4k} + 7\omega_N^{-6k})$$

Let $\omega_E = \omega_N^2 = e^{j\frac{4\pi}{8}}$. Then, we can rewrite this sum as

$$= \frac{1}{8}(2\omega_E^{-k} + 4\omega_E^{-2k} + 6\omega_E^{-3k}) + \frac{\omega_N^{-k}}{8}(1 + 3\omega_E^{-k} + 5\omega_E^{-2k} + 7\omega_E^{-3k})$$

$$= \frac{1}{8}((4\omega_E^{-2k}) + (2\omega_E^{-k} + 6\omega_E^{-3k})) + \frac{\omega_N^{-k}}{8}((1 + 5\omega_E^{-2k}) + (3\omega_E^{-k} + 7\omega_E^{-3k}))$$

Let $\omega_F = \omega_E^2 = e^{j\frac{8\pi}{8}}$. Then, we can continue to rewrite the series as:

$$= \frac{1}{8}((4\omega_F^{-k}) + \omega_E^{-k}(2 + 6\omega_F^{-k})) + \frac{\omega_N^{-k}}{8}((1 + 5\omega_F^{-k}) + \omega_E^{-k}(3 + 7\omega_F^{-k})) \quad \text{(EQ 68)}$$

To evaluate this, for each k, we compute ω_F^{-k}, ω_E^{-k}, and ω_N^{-k} and substitute in Equation 68. For example, for k=2, we compute the transform value as follows:

$$\omega_N^{-k} = e^{-j\frac{4\pi}{8}} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = -j$$

$$\omega_E^{-k} = \omega_N^{-2k} = -1$$

$$\omega^{-k}{}_F = \omega_E^{-2k} = 1$$

We substitute this in Equation 68 to find the value to be:

$$\frac{1}{8}((4) - 1(2+6)) + \left(-\frac{j}{8}\right)((1+5) - 1(3+7)) = \frac{1}{8}(-4) + \left(-\frac{j}{8}\right)(-4) = -0.5 + j0.5$$

■

5.11 The Z Transform

The Z transform generalizes the DFT in roughly the same way as the Laplace transform generalizes the Fourier transform. The integral in the Fourier transform can be viewed as the limiting sum of a series of unmodulated complex exponentials as their inter-frequency separation tends to zero, The Laplace transform generalizes this by taking the limiting sum over a series of modulated complex exponentials. In the same way, the DFT is a finite sum of unmodulated complex exponentials. The Z transform generalizes this to an infinite sum of modulated complex exponentials[8]. However, the notation of the Z transform is particularly simple because the complex transform variable, denoted z, is written in the a+jb form rather than the equivalent modulated complex exponential form.

We will consider only causal signals, that is, signals $x[k]$ such that $x[k] = 0$ for k<0. We define the Z transform of such a signal as the infinite sum $\sum_{k=0}^{\infty} x[k]z^{-k}$. We denote this by

$$x[k] \overset{Z}{\leftrightarrow} X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} \quad \text{(EQ 69)}$$

As with the Laplace transform, the infinite series defining the Z transform converges only for certain values of z, which define its **region of convergence**.

The inverse Z transform is given by:

$$x[k] = \frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz \quad \text{(EQ 70)}$$

8. A careful reader may note that we are simultaneously generalizing in two dimensions: from a finite sum to an infinite sum and from a sum of unmodulated complex exponentials to a sum of modulated complex exponentials. Indeed, there is an intermediate step, the discrete Laplace transform, which corresponds to a finite sum of modulated complex exponentials, that we have glossed over in this exposition.

where C is a circle with its center at the origin of the z plane such that all values z such that $X(z)z^{k-1} = \infty$ (its poles) are inside this circle. As with the Laplace transform, the inverse is usually found from a table of standard transforms and the properties of the Z transform. Similarly, the concepts of poles and zeroes introduced for the Laplace transform continue to hold for the Z transform, as the next examples show.

EXAMPLE 27: Z TRANSFORM OF A UNIT STEP

Compute the Z transform of the discrete unit step signal defined by $x[k] = 1$ for $k \geq 0$.

Solution: By definition, $X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$ which converges when $|z| > 1$, which is the region of convergence of this transform. Recall that z is complex, so this is the set of points outside a unit circle in the complex plane centered on the origin. Note that, unlike the Laplace transform, in the case of the Z transform, the region of convergence is expressed in terms of circular regions or their intersections, rather than half-planes.

■

EXAMPLE 28: Z TRANSFORM OF A DISCRETE EXPONENTIAL

Compute the Z transform of the signal $x[k] = e^{-ak}$ where a is a complex constant and $k \geq 0$.

Solution: By definition, $X(z) = \sum_{k=0}^{\infty} e^{-ak}z^{-k} = \sum_{k=0}^{\infty} (e^{-a}z^{-1})^k = \frac{1}{1-e^{-a}z^{-1}} = \frac{z}{z-e^{-a}}$. The series converges and the transform is defined for the region where $e^{-a}z^{-1} < 1 \Rightarrow |z| > e^{-a}$ which is a circle of radius e^{-a} centered at the origin. What if $z = e^{-a}$? In this case, the denominator becomes zero and the transform's value is infinite. This is called a **pole** of the system (the pole in the previous example was at 1). Intuitively, the series e^{-ak} diverges when $a < 1$. We can ensure convergence by modulating this signal with the complex value z^{-1}, but only when the absolute value of z is greater than e^{-a}.

■

The values of z for which the transform vanishes are called the *zeroes* of the transform. This is illustrated by the next example.

EXAMPLE 29: Z TRANSFORM OF A SINUSOID

Compute the Z transform of the discrete sinusoid $\cos[\omega k]$.

Solution: We use Euler's formula to rewrite the signal as $\left(\frac{e^{j\omega k}+e^{-j\omega k}}{2}\right)$. By definition,

$X(z) = \sum_{k=0}^{\infty}\left(\frac{e^{j\omega k}+e^{-j\omega k}}{2}\right)z^{-k} = \frac{1}{2}\sum_{k=0}^{\infty} e^{j\omega k}z^{-k} + \frac{1}{2}\sum_{k=0}^{\infty} e^{-j\omega k}z^{-k} = \frac{1}{2}\left(\frac{1}{1-e^{j\omega}z^{-1}} + \frac{1}{1-e^{-j\omega}z^{-1}}\right)$, where, in the last step, we used the result from the previous example, and we are assuming that $|z| > e^{-j\omega}$. This reduces to $\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$, with the region of convergence $|z| > |e^{-j\omega}| = 1$ (the entire z plane excluding a disk of radius 1). Note that the transform becomes infinite for $z = \pm e^{j\omega}$ (these are the poles of the transform) and is zero for z=0, which is the zero of the transform.

■

It is important to keep track of the region of convergence of the transform. As with the Laplace transform, it can be shown that two time-domain functions that are completely different may have an identical transform, other than the region of convergence. The transform is unique only if the region is also specified. In general, the region of convergence of the *Z* transform is given by the annulus in the complex plane specified by $\alpha < |z| < \beta$.

5.11.1 Relationship between Z and Laplace transform

(This section can be skipped during a first reading) The Z transform is a compact representation of the Laplace transform of a discrete-time signal. We now discuss how the Z transform's *z* auxiliary variable relates to the *s* auxiliary variable used in the Laplace transform.

Recall the following facts:

- The Laplace transform of a continuous signal is the Fourier transform of the signal after it has been modulated by the real exponential $e^{-\sigma t}$
- A discrete-time signal whose values are separated in time by *T* seconds can be viewed as the product of a corresponding continuous-time signal and an impulse train with impulses spaced *T* seconds apart
- The transform of a discrete-time impulse train with impulses spaced *T* seconds apart is an impulse train in the frequency domain with impulses spaced $2\frac{\pi}{T}$ Hz apart
- The Fourier transform of the product of two signals is the convolution of their Fourier transforms
- The convolution of a signal with an impulse train replicates the signal

Given these facts, we see that the Laplace transform of a discrete-time signal with values spaced apart by *T* seconds results in infinitely replicating the transform of the signal modulated by the real exponential $e^{-\sigma t}$ with a period of $2\frac{\pi}{T}$. Therefore, it is possible to fully describe the signal in the frequency domain considering only frequencies in the range $0 \pm \frac{\pi}{T}$. This corresponds to values of the Laplace variable $s = \sigma + j\omega$ that lie in band $\sigma - j\frac{\pi}{T} \le s \le \sigma + j\frac{\pi}{T}$, as shown by the two dashed horizontal lines in Figure 14.

The Z transform denotes these values with the *z* variable, using the relationship $z = e^{sT} = e^{\sigma T}e^{j\omega T}$. Representing *z* in the form $z = Ae^{j\theta}$, we see that

$$\begin{aligned} A &= e^{\sigma T} \\ \theta &= \omega T \end{aligned} \tag{EQ 71}$$

As a result of this mapping, lines parallel to the Y axis in the *s* plane correspond to circles in the *z* plane and lines parallel to the X axis in the *s* plane correspond to radial lines in the *z* plane. This is shown in Figure 14, where the two vertical lines marked by a diamond and a cross in the *s* plane are transformed to circles in the *z* plane and the two horizontal dashed lines marked A and B map to the same radial line in the *z* plane.

The Y axis in the *s* plane corresponds to the unit circle in the *z* plane. The left half-plane in the *s* domain corresponds to points within the unit circle in the *z* plane. Note that the vertical line marked with a diamond lies in the left half-plane of the *s* plane. Therefore, it lies within the unit circle in the *z* plane. Similarly, point X in the *s* plane, that lies in the left half-plane, is mapped to a point within the unit circle in the *z* plane, and point Y in the *s* plane, that lies in the right half-plane is mapped to a point outside the unit circle. Moreover, because Y has a larger ω value than X, it is rotated further in the anti-clockwise direction in the *z* plane than X.

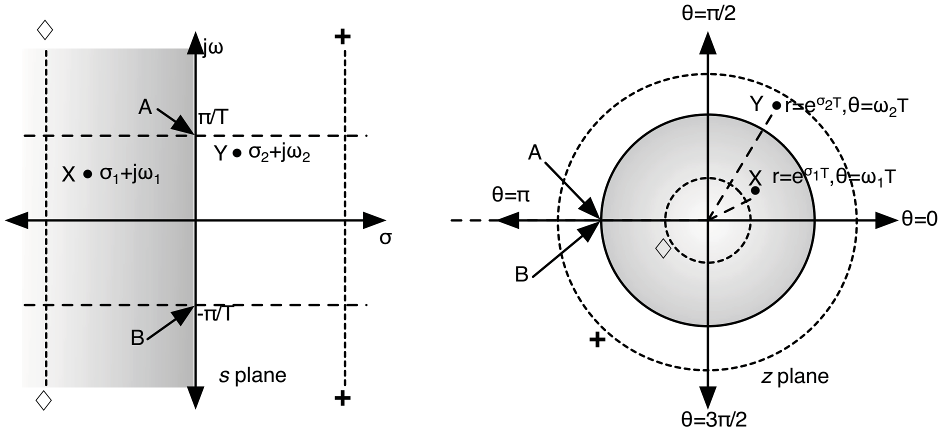


FIGURE 14. **Mapping from the *s* plane to the *z* plane**

EXAMPLE 30: MAPPING FROM THE *S* PLANE TO THE *Z* PLANE

What is the *z* value corresponding to the *s* value 3+*j*2 assuming that *T* = 1? Express this value in both polar and Cartesian coordinates.

Solution: The *z* value $Ae^{j\theta}$ is given by $e^{sT} = e^{3+j2} = e^3 e^{j2} = 20.08e^{j2}$. Using Euler's formula, we can write this as $20.08(\cos(2) + j\sin(2)) = 20.08(-0.42 + j(0.91)) = -8.36 + j18.26$.

■

5.11.2 Properties of the Z transform

The Z transform shares many of the properties of the Fourier and Laplace transforms, namely linearity, time-shifting, time-scaling, differentiation, and the convolution property, though in a slightly different form. We summarize some important properties in Table 5. For simplicity, we implicitly assume that signal values are spaced a unit time step apart, i.e., *T* =1.

Property	**Pre-condition**	**Post condition**	**Notes**
Linearity	$x_1[k] \overset{Z}{\leftrightarrow} X_1(z) \quad \alpha_1 < \lvert z \rvert < \beta_1$ $x_2[k] \overset{Z}{\leftrightarrow} X_2(z) \quad \alpha_2 < \lvert z \rvert < \beta_2$	$ax_1[k] + bx_2[k] \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z)$ $max(\alpha_1, \alpha_2) < \lvert z \rvert < min(\beta_1, \beta_2)$	*a* and *b* are arbitrary constants and can be complex. This assumes that no pole-zero cancellations are involved.
Time shifting	$x[k] \overset{Z}{\leftrightarrow} X(z) \quad \alpha < \lvert z \rvert < \beta$	$x[k-a] \overset{Z}{\leftrightarrow} z^{-a}X(z) \quad \alpha < \lvert z \rvert < \beta$ except z=0 if k>0 and z=∞ $if\ k < 0$	Shifting a signal *a* steps forward multiplies its transform by z^{-a}.
Scaling in the *z* domain	$x[k] \overset{Z}{\leftrightarrow} X(z) \quad \alpha < \lvert z \rvert < \beta$	$a^k x[k] \overset{Z}{\leftrightarrow} X\left(\frac{z}{a}\right) \quad (\lvert a \rvert \alpha < \lvert z \rvert < \lvert a \rvert \beta)$	An expansion in the frequency scale compresses the time scale

Property	Pre-condition	Post condition	Notes
Time reversal	$x[k] \overset{Z}{\leftrightarrow} X(z) \qquad \alpha < \|z\| < \beta$	$x[-k] \overset{Z}{\leftrightarrow} X(z^{-1}) \qquad \frac{1}{\beta} < \|z\| < \frac{1}{\alpha}$	Reversing the direction of a signal is equivalent to using z^{-1} instead of z in the expression of the transform
Time-domain convolution	$x_1[k] \overset{Z}{\leftrightarrow} X_1(z) \qquad \alpha_1 < \|z\| < \beta_1$ $x_2[k] \overset{Z}{\leftrightarrow} X_2(z) \qquad \alpha_2 < \|z\| < \beta_2$	$x_1[k] \otimes x_2[k] \overset{Z}{\leftrightarrow} X_1(z)X_2(z)$ $max(\alpha_1, \alpha_2) < \|z\| < min(\beta_1, \beta_2)$	Convolution in the time domain corresponds to multiplication in the transform domain
First difference	$x[k] \overset{Z}{\leftrightarrow} X(z) \qquad \alpha < \|z\| < \beta$	$x[k] - x[k-1] \overset{Z}{\leftrightarrow} (1 - z^{-1})X(z)$ $max(\alpha, 0) < \|z\| < \beta$	The first difference is equivalent to differentiation in the time domain

TABLE 5. **Some properties of the Z transform**

These properties, along with the a table of common transforms (Table 6) allow us to derive the transform of most common signals without having to derive them from first principles.

No.	Signal	*x*[*k*]	*X*[*z*]	Region of convergence
1	Delta or unit impulse	$\delta[k]$	1	All z
2	Delayed delta	$\delta[k - k_0]$	z^{-k_0}	$z \neq 0$
3	Unit step	$u[k]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
4	Ramp	$ku[k]$	$\frac{z^{-1}}{(1 - z^{-1})^2}$	$\|z\| > 1$
5	Exponential	$a^k u[k]$	$\frac{1}{1 - az^{-1}}$	$\|z\| > a$
6	Sine	$\sin(\omega_0 k)u[k]$	$\frac{z^{-1}\sin(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\|z\| > 1$
7	Cosine	$\cos(\omega_0 k)u[k]$	$\frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\|z\| > 1$
8	Exponentially modulated sine	$a^k \sin(\omega_0 k)u[k]$	$\frac{az^{-1}\sin(\omega_0)}{1 - 2az^{-1}\cos(\omega_0) + a^2 z^{-2}}$	$\|z\| > a$
9	Exponentially modulate cosine	$a^k \cos(\omega_0 k)u[k]$	$\frac{1 - az^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + (a^2 z)^{-2}}$	$\|z\| > a$

TABLE 6. **Some standard Z transforms**

EXAMPLE 31: SOLVING A SYSTEM USING THE Z TRANSFORM

Consider a discrete-time system such that the Z transform $H(z)$ of its transfer function $h[k]$ is $\frac{1}{1-z^{-1}}$. What is its response to the input $2^{-k}u[k]$?

Solution: Since $x[k] = 2^{-k}u[k]$, from Row 5 of Table 6, $X(z) = \frac{1}{1-\frac{z^{-1}}{2}}$. Therefore, $Y(z) = \frac{1}{(1-z^{-1})\left(1-\frac{z^{-1}}{2}\right)}$. Using the method of partial fractions (i.e., writing the expression as $\frac{a}{1-z^{-1}} + \frac{b}{1-\frac{z^{-1}}{2}}$ and solving for a and b), we find that $Y(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{z^{-1}}{2}}$. To find the inverse transform, recall that the Z transform is linear, so we only need to find the inverse transform of each term in isolation. From Row 3 and Row 5 of Table 6, we get $y[k] = 2u[k] + \left(\frac{1}{2}\right)^k u[k]$, which is a discrete unit step of height 2 to which is added a decaying discrete exponential.

■

5.12 Further Reading

An excellent text that delves more deeply into the material covered in this chapter using a unique graphical approach is P. Kraniuskas, Transforms in Signals and Systems, Addison-Wesley, 1992. A more traditional approach can be found in B.P. Lathi, Linear Systems and Signals, Oxford, 2005. A lucid description of complex numbers and the Fourier transform along with historical anecdotes can be found in P.J. Nahin, Dr. Euler’s Fabulous Formula, Princeton, 2006.

5.13 Exercises

1 Complex arithmetic

Compute $e^{-j\frac{\pi}{2}} + e^{j\frac{\pi}{2}}$

2 Phase angle

What is the phase angle corresponding to the complex number $1+j$.

3 Discrete convolution

Let $z(t)$ be the convolution of $x(t)$, $0 \le t \le 9$ = 1, 3, 5, 2, 5, 8, 7, 3, 9, 4 and $y(t)$, $0 \le t \le 9$ = 3, 1, 7, 4, 5, 9, 7, 1, 3, 8. Compute $z(5)$

4 Signals

Given an example of a signal that is continuous, digital, aperiodic, and time-unlimited.

5 Complex exponential

Describe the Im-t projection of the curve $(e^{t}(\cos 3t + j\sin 3t))$.

6 Linearity

Is the system defined by the transfer function $H(x) = 5\frac{dx}{dt} + 1$ linear?

7 LTI system

What is the output of a LTI system when the input is a real sinusoid?.

8 Natural response

Compute the natural response of the LTI system given by $2\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 15y(t) = 32x(t)$.

9 Natural response

What is the exact natural response of the system described by $2\frac{d^2y(t)}{dt^2} + y(t) = 32x(t)$ $y(0) = 0$, $\dot{y}(0) = 1$?

What is its frequency?

10 Stability

Characterize the stability of the system in Problem 9.

11 Fourier series

Find the Fourier Series corresponding to the infinite series of isosceles triangular pulses of base width 2τ and height 1 spaced apart T_0 seconds. Use the fact that $\int xe^{ax}dx = \frac{axe^{ax} - e^{ax}}{a^2} + C$.

12 Fourier series

Compute the third coefficient of the Fourier series representing a periodic rectangular pulse of width $1s$ spaced apart by 10s.

13 Fourier transform

Find the Fourier transform of a single left-triangular pulse defined by1- t in [0,1].

14 Inverse Fourier transform

Compute the inverse Fourier transform of the function $\pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$ from first principles.

15 Computing the Fourier transform

Use the properties of the Fourier transform to compute the transform of the function $\cos(\omega_0(t + t_0)) + \sin(\omega_0(t - t_0))$.

16 Laplace transform

Compute the Laplace transform of the sinusoid $u(t)\sin\omega_0 t$ from first principles. Locate its pole(s) and zero(es).

17 Laplace transform

Find the Laplace transform of the signal $u(t)\sin\omega_0(t-t_0)$.

18 Solving a system using the Laplace transform

Consider a system whose impulse response $h(t)$ is given by $\cos(\omega_0 t)$. What is its response to the input $e^{-t}u(t)$?

19 Discrete-time Fourier transform

Compute the discrete-time Fourier transform of the signal $0.5^n u[nT]$ where $0 < a < 1$.

20 Discrete-time-and-frequency Fourier transform

Compute the fourth Fourier term for the signal shown in Figure 13.

21 Z transform

Compute the Z transform of the discrete ramp $nu[k]$ from first principles.

22 Z transform

Compute the Z transform of the signal $x[k] = e^{a(k-k_0)}$ where a is a complex constant and $k \geq 0, k_0 > 0$.

CHAPTER 6 Stochastic Processes and Queueing Theory

6.1 Overview

Queues arise naturally when entities demanding service or **customers** interact asynchronously with entities providing service or **servers**. **Service requests** may arrive at a server when it is either unavailable or busy serving other requests. In such cases, service demands must be either queued or dropped. A server that queues demands instead of dropping them can smooth over fluctuations in the rate of service and in the rate of request arrivals, leading to the formation of a queueing system. The study of the probabilistic behaviour of such systems is the subject of queueing theory.

EXAMPLE 1: EXAMPLES OF QUEUEING SYSTEMS

Here are some examples of queueing systems:

1. The arrival and service of packets at the output queue of a switch. Packets may arrive when the output link is busy, in which case the packets (which are implicit service requests) must be buffered (i.e., queued).
2. The arrival of and service of HTTP requests at a web server. If the web server is busy serving a request, incoming requests are queued.
3. The arrival of telephone calls to a switch control processor in the telephone network. The processor may be unable to service the call because the switch is busy. In this case, the call is queued, awaiting the release of network resources.

■

Given a queueing system, we would like to compute certain quantities of interest, such as:

- The expected waiting time for a service request (also called the **queueing delay**).
- The mean number of service requests that are awaiting service (the **backlog**).
- The expected fraction of time that the server is busy (the **utilization factor**).
- The expected fraction of service requests that must be dropped because there is no more space left in the queue (the **drop rate**).
- The mean length of a busy (or idle) period.

Queueing theory allows us to compute these quantities--both for a single queue and for interconnected networks of queues--as long as the incoming traffic and the servers obey certain simplifying conditions. Unfortunately, measurements show that traffic in real networks do *not* obey these conditions. Moreover, we cannot mathematically analyse most networks that are subjected to realistic traffic workloads. Nevertheless, it is worth studying queueing theory for two important reasons. First, it gives us fundamental insights into the behaviour of queueing systems. These insights apply even to systems that are mathematically intractable. Second, the solutions from queueing theory even from unrealistic traffic models are a reasonable first approximation to reality. Therefore, as long as we keep in mind that results from queueing theory are only approximate and are primarily meant to give an insight into a real system, we can derive considerable benefit from the mathematical theory of queues.

6.1.1 A general queueing system

We now introduce some standard notation. A **queue** is formed when **customers** present **service requests** or **jobs** to one or more **servers**. Customers arrive at a rate of λ customers/second at times t and the time between arrivals is described by the **interarrival time distribution** $A(t) = P(\text{time between arrivals} \leq t)$. We denote the service time by x which has a **service time distribution** $B(x) = P(\text{service time} \leq x)$ with a mean service rate of μ customers/second. Customers are assumed to wait in the queue with a **mean waiting time** T. Note that in this chapter we will primarily study a single queue in isolation.

6.1.2 Little's theorem

Little's theorem is a fundamental result that holds true for *all* arrival and service processes. It states that the mean number of customers in a queueing system is the product of their mean waiting time and their mean arrival rate.

Proof of Little's theorem

Suppose customers arrive to an empty queue[1] at a mean rate of λ customers/second. An average of λt customer arrive in t seconds. Let T denote the mean waiting time of a newly arriving customer in seconds. The total time spent waiting in the queue across all customers during the time interval is therefore λTt customer-seconds.

Let N denote the mean number of customers in the queue. In one second, these N customers accumulate N customer seconds of total waiting time. Thus, in t seconds, they accumulate a total of Nt customer-seconds of waiting time. This must equal λTt, which implies that $N=\lambda T$.

■

Note that this argument is independent of the length of the time interval t. Moreover, it does not depend on the order of service of customers, or the number of servers, or on the way in which customers arrive. Thus, it is a powerful and general law applicable to all queueing systems.

EXAMPLE 2: LITTLE'S THEOREM

Suppose you receive email at the average rate of one message every five minutes. If you read all your incoming mail instantaneously once an hour, what is the average time that a message remains unread?

Solution

The mean message arrival rate is 12 messages/hour. Because you read email once an hour, the mean number of unread messages is 6 (this is the expected number of messages received in half an hour). By Little's theorem, this is the product of the mean arrival rate and the mean waiting time, which immediately tells us that the mean time for a message to be unread is $\frac{6}{12}$ hours =30 minutes.

■

EXAMPLE 3: LITTLE'S THEOREM

Suppose that 10,800 HTTP requests arrive to a web server over the course of the busiest hour of the day. If the mean waiting time for service should be under 6 seconds, what should be the largest allowed queue length?

Solution

1. The same reasoning applies when customers arrive to a non-empty queue, but arrivals to an empty queue simplifies the analysis.

The arrival rate λ = 10,800/3600 = 3 requests/second. We want $T \le 6$. Now, $N = \lambda T$, so $T=N/\lambda$. This means that $N/\lambda \le 6$ or that $N \le 6*3 = 18$. So, if the mean queueing delay is to be no larger than 6 seconds, the mean queue length should not exceed 18. In practice, the web server could return a server busy response when the queue exceeds 18 requests. This is conservative because then 18 is the *maximum* queue length, rather than its mean.

■

6.2 Stochastic processes

The foundation of queueing theory is the mathematical study of a **stochastic process**. Such a process is used to model the arrival and service processes in a queue. We will both intuitively and mathematically define a stochastic process and then study some standard stochastic processes.

EXAMPLE 4: DETERMINISTIC AND STOCHASTIC PROCESSES

Consider a staircase with 10 steps numbered 1 through 10 and a person standing on the first step, which is numbered 1. Suppose that there is a clock next to the staircase that ticks once a second starting at time 1. Finally, assume that it takes zero time to climb each step.

If the person were to climb one step at each clock tick, then we can predict exactly where the person would be at each time step. At time 0, the person is on step 1, and would stay there until just before time 1. When the clock ticks and time increments to 1, the person would be at step 2 and would stay there until just before time 2. At time 2 the person would be on step 3, and so on. We therefore call the act of climbing the staircase in this fashion a **deterministic** process.

In contrast, suppose the person either climbs one step or goes down one step or stays on the same step with some (potentially zero) probability. With this change, we lose predictability. That is, we no longer know exactly where the person will be at any moment in time: we can only attach probabilities to the set of places where the person *could* be at that time. The process is no longer deterministic: it is **stochastic**.

We capture this by means of a random variable X_i that denotes the step the person is on immediately after the *i*th clock tick. The random variable is associated with the probability distribution $\pi(i)$ over the positions where the person could be immediately after that time. For instance, at time 0, the person is at step 1 with probability 1, so the distribution of X_0 over the domain {1, 2,...,10} is given by the discrete probability distribution $\pi(0)$ = {1.0, 0,..., 0}. Suppose that the probability that the person goes up is p, that the person goes down is q, and that the person stays on the same step is $1-p-q$, except at step 1, where the probability of going up is p, and the probability of staying on the same step is $1-p$. Then, immediately after time 1 (after the first clock tick) $\pi(1) = \{1-p, p, 0,...,0\}$. Similarly, $\pi(2) = \{(1-p)^2+pq, p(1-p-q) + (1-p)p, p^2, 0,...,0\}$, and so on.

To compute $\pi(i)$ given $\pi(i-1)$ we determine the different ways that we can reach each particular step summing the probability over all possible ways to reach that step.

Note that we distinguish between the distribution of the random variables at each time step and the actual trajectory taken by a person. For a given trajectory, at each time instant, the person is, of course, only on one step of the staircase. The trajectories are created by sampling from the distributions $\pi(i)$. A trajectory is also therefore called a **sample path**.

■

This example motivates the following definition of a **stochastic process**: it is a family of random variables X_i that are indexed by the time index *i*. The value of the random variable (in a particular trajectory) is also called the **state** of the stochastic process at that point in time. Without loss of generality, we can think of the states as being chosen from the integers from 1 to *N*. Thus, we can imagine the process as 'moving' from the state corresponding to the value taken by random variable X_i in a given trajectory to the state corresponding to the value taken by random variable X_{i+1} at time $i+1$, just like the person moves from one stair to another. As we have shown, given the probabilities of moving from one step to another, we can, in principle, compute the distribution of each X_i: this is the distribution of the stochastic process over the state space at that time.

Time is discrete in this example. In other situations, a system is better modelled when time is continuous. In this case, the family of random variables corresponding to the stochastic process consists of the variables $X(t_1)$, $X(t_2)$,... where the t_i represent the times at which the state transitions occur. Given the probability of moving from step to step, we can compute $\pi(t_{i+1})$, the distribution of $X(t_{i+1})$, from $\pi(t_i)$ the distribution of $X(t_i)$.

In the example, the person's movements were limited to moving up or down one step on each clock tick. We could, instead, allow the person to go from a given step to any other step (not just the steps above and below) with some probability. Indeed, this distribution of probabilities could differ at different steps and even differ at each clock tick! And, finally, the person could be on a ramp, so that the amount of movement could be a real positive or negative quantity, rather than an integer. These variations are all within the scope of definition of a stochastic process, but the analysis of the corresponding processes is progressively more difficult. We will first describe some standard types of stochastic processes and then focus on the simplest ones.

6.2.1 Discrete and continuous stochastic processes

A stochastic process can be classified as a discrete or continuous process in two different ways: the values assumed by the random variables (also called the **state space**) can be discrete or continuous, and the index variable, i.e., time, can also be discrete or continuous.

A **discrete-space process** is one where the random variables X_i take on discrete values. Without loss of generality, we can think of the state in a discrete-space process as being indexed by an integer in the range 1,2,...N.

EXAMPLE 5: DISCRETE-SPACE PROCESS

Continuing with Example 4, we see that the set of possible states is the set of stairs, which forms a discrete set.

■

A **continuous-space process** is one where the random variables take on values from a finite or infinite continuous interval (or a set of such intervals).

EXAMPLE 6: CONTINUOUS-STATE PROCESS

Continuing with Example 4, consider a person walking up and down a ramp, rather than a stair. This would allow movements by real amounts. Therefore, the random variable corresponding to the state of the process can take on real values, and the corresponding stochastic process would be a continuous-space process.

■

In a **discrete-time** process, the indices of the random variables are integers. We can think of the stochastic process in a particular trajectory as moving from one state to another at these points in time.

EXAMPLE 7: DISCRETE-TIME PROCESS

Continuing with Example 4, this corresponds to a person moving from one step to another exactly at each clock tick. Such a stochastic process is also called a **stochastic sequence**.

■

In a **continuous-time** process, the times when the process can move to a new state are chosen from a real interval.

EXAMPLE 8: CONTINUOUS-TIME PROCESS

Continuing with Example 4, this corresponds to a person moving from stair to stair at will, independent of the clock.

■

Stochastic processes corresponding to all four combinations of {discrete space, continuous space} and {discrete time, continuous time} are well-known.

6.2.2 Markov processes

An important aspect of a stochastic process is how the probability of transitioning from one state to another is influenced by past history. Continuing with our staircase example (a discrete time and discrete space stochastic process), consider a person who is allowed to go from any stair to any other stair. Moreover, we will ask they person to obey the following rules: if he or she arrives at stair 5 from stair 6, move to stair 3. If, however, he or she arrives at stair 5 from stair 3, move to stair 9. In all other cases, move to stair 1. Suppose at some point in time we see that the person is on stair 5. What happens next?

The answer is: we don't know. It depends on where the person was in the previous time step. Stated more precisely, the distribution of the random variable $X(n+1)$ (i.e., $\pi(n+1)$) when $X(n) = 5$ depends on the value of $X(n\text{-}1)$. Generalizing from this example, we can define more complex stochastic processes for whom $\pi(n+1)$ depends not only on $X(n)$, but also on $X(n\text{-}1)$, $X(n\text{-}2)$,..., $X(1)$. Such systems are inherently complex and there is little we can say about them.

In an attempt to curb this complexity, consider the following rule:

$\pi(n+1)$ depends only on the value of $X(n)$

This rule simplifies the situation: if the person is on step 5 at time n, then we know π_{n+1} *independent of the prior history.* As we will see, this allows us to easily compute many quantities of interest about the process. Moreover, many naturally occurring stochastic processes obey this rule. Due to these two facts, stochastic processes that obey this rule are given a special name: they are called **Markov processes**, in honour of A.N. Markov, who first studied them in 1907.

Formally, we state the **Markov property** as (for the case of discrete time stochastic processes):

$$P(X_{n+1} = j | X_n = i_n, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \ldots, X_1 = i_1) = P(X_{n+1} = j | X_n = i_n) \quad \textbf{(EQ 72)}$$

The conditional probability $P(X_{n+1} = j | X_n = i_n)$ is called the **transition probability** to go from state i_n to state j.

EXAMPLE 9: MARKOV PROCESS

Consider a discrete-time discrete-space Markov process whose state space is {1,2,3}. Let $P(X_3 = 1 \mid X_2 = 1) = 0.2$; $P(X_3 = 2 \mid X_2 = 1) = 0.4$; $P(X_3 = 3 \mid X_2 = 1) = 0.4$. Suppose that we know that $X_2 = 1$. Then, the Markov property allows us to compute $\pi(3)$ no matter which path was taken to reach the state 1 at time 2.

■

Note that for a discrete time stochastic process at time step n, we already know the past history, that is the sequence of prior states. At time n, we are usually interested in computing $\pi(n+1)$ given this past history. The Markov property allows us to forget everything about history except the value of the current random variable, which encapsulates all past history. This is similar in spirit to the memorylessness property of the exponential distribution (see Section 1.6.3 on page 23).

A similar property holds true for continuous time stochastic processes. For simplicity, we will first study discrete-time processes that obey the Markov property, also known as **discrete time Markov chains**, before considering continuous time Markov processes.

6.2.3 Homogeneity, state transition diagrams, and the Chapman-Kolmogorov equations

A stochastic process may satisfy the Markov property even if its transition probabilities vary over time. A process whose transition probabilities are time-dependent is called a **non-homogeneous Markov process**. Of course, this greatly complicates the analysis. We can simplify the analysis by decreeing that the transition probabilities should be time-independent. In our example, this means that when the person is on a particular step, say step 4, the probability of going to any other step is always the same, no matter *when* they got to step 4. Such a process is called a **homogeneous Markov process.** For a homogeneous Markov process, we define the *time-independent* transition probability between state i and state j as $p_{ij} = P(X_n = j|X_{n-1} = i)$ for any n.

EXAMPLE 10: HOMOGENEOUS MARKOV PROCESS

Consider the discrete-time discrete-space stochastic process in Example 9. If this process were homogeneous, we need not consider exactly one point in time, such as time 2. Instead, if $P(X_{n+1} = 1 \mid X_n = 1) = 0.2$; $P(X_{n+1} = 2 \mid X_n = 1) = 0.4$; $P(X_{n+1} = 3 \mid X_n = 1) = 0.4$, we can compute the distribution X_{n+1} given X_n for all values of n.

■

The state transition probabilities for a homogeneous Markov chain with N states have two equivalent representations. The first is in the form of an $N \times N$ transition matrix **A** whose elements are the probabilities p_{ij}. This representation has the attractive property that the probability of going from any state i to state j in two steps is given by the elements of $\mathbf{A}^2$. The second is as a graph (see Example 11). In this representation, vertices represent states and the annotation on an edge from vertex i to vertex j is p_{ij}. This visually represents a Markov chain. Note that a non-homogeneous Markov chain requires such a state transition diagram for each time step.

EXAMPLE 11: REPRESENTING A HOMOGENEOUS MARKOV PROCESS

Continuing with Example 10: we have already seen that $p_{11} = 0.2$, $p_{12} = 0.4$, $p_{13} = 0.4$. Suppose that $p_{21} = 1.0$, $p_{22} = 0$, $p_{23} = 0$ and $p_{31} = 0.5$, $p_{32} = 0.25$, $p_{33} = 0.25$. Then, we can represent it in two ways as shown below:

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 1.0 & 0 & 0 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$$

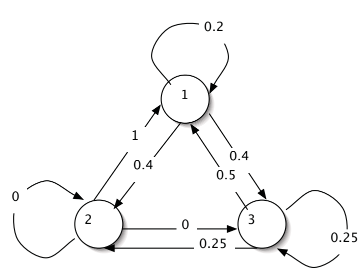


FIGURE 15. **State transition diagram for Example 11**

Note that ***A*** is a right stochastic matrix (see Section 3.6 on page 98).

■

Given the set of transition probabilities p_{ij} for a homogeneous Markov chain, we can define the m-step transition probability from state i to state j denoted $p_{ij}^{(m)}$ by

$$p_{ij}^{(m)} = P(X_{n+m}=j | X_n = i) = \sum_k p_{ij}^{(m-1)} p_{kj} \qquad \text{m = 2,3,...} \tag{EQ 73}$$

where the sum of products form comes from summing across independent events (that of going to some intermediate state in m-1 steps), and each term is a product, because it is the combination of two independent events (going from state i to state k and from state k to state m). These relations exist only because of the Markovian nature of the chain.They are important enough that they are given their own name: the **Chapman-Kolmogorov** equations. The Chapman-Kolmogorov equations can also be stated as:

$$p_{ij}^{(m)} = P(X_{n+m}=j | X_n = i) = \sum_k p_{ik} p_{kj}^{(m-1)} \qquad \text{m = 2,3,... .} \tag{EQ 74}$$

Comparing the two, we see that in the first formulation traces the trajectory of the process as it goes from state i to state k in m-1 steps, and from state k to state j in one step, and the second formulation traces the trajectory of the process as it goes from state i to state k in one step and from state k to state j in m-1 steps. Clearly, these are equivalent recursions.

6.2.4 Irreducibility

If every state of a stochastic process can be reached from every other state after a finite number of steps, then the process is called **irreducible**, otherwise it is **reducible**. Moreover, if there are states of a stochastic process can be separated into subsets that are mutually unreachable from each other, we call each such set a **separable sub-chain**.

EXAMPLE 12: A REDUCIBLE MARKOV CHAIN

Consider the Markov chain in Figure 16. Here, the transition probabilities p_{ij} for i even and j odd or i odd and j even are 0. Therefore, if the initial state of the process is an even-numbered state, the trajectory of process is confined to even-numbered states. Alternatively, if the process starts from an odd-numbered state, it will forever stay in odd-numbered states. The even-numbered states are unreachable from the odd-numbered steps states and the chain, therefore, is reducible. Indeed, we could separate the even-numbered and odd-numbered states into separate chains that would equivalently describe the process. We can generalize this idea to construct stochastic processes that can be decomposed into as many sub-chains as we wish.

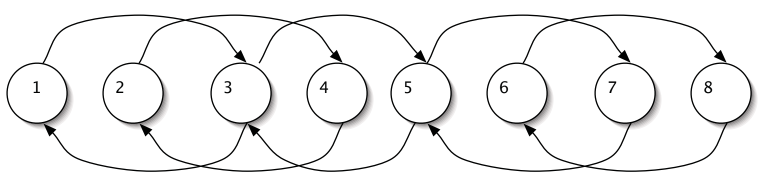


FIGURE 16. A reducible Markov chain

■

6.2.5 Recurrence

For every state j of a stochastic process, one of two conditions must hold: after entering state j, either the probability of re-entering state j after a finite number of steps is 1, or there is some non-zero probability that the state is not re-entered after a finite number of steps. In Example 4, this is equivalent to saying that after stepping on a stair, say stair 6, it either certain that the person will return to stair 6, or there is a non-zero probability that the person will not return to stair 6. If return to a state is certain, we call the state **recurrent**, otherwise we call it **transient**.

Let f_j^n denote the probability that the *first* return to state j is after n steps. State j is recurrent if $\sum_{n=1}^{\infty} f_j^n = 1$ and transient otherwise.

Although a state is recurrent, its expected recurrence period, defined as $\sum_{n=1}^{\infty} n f_j^n$, may be infinite. This sum may diverge if f_j^n is sufficiently large for large values of n. In such cases, the mean recurrence period is infinite, and the state is called **recurrent null**. Otherwise, it is called **recurrent non-null**.

6.2.6 Periodicity

Given a recurrent state j, suppose the only way to return to that state is to take r, $2r$, $3r$... steps, with $r \geq 2$. We then call the state j **periodic**, with a period r. Periodic states arise when the Markov chain has a cycle. A trivial way to check if a state is periodic is to see if it has a self-loop, that is $p_{jj} > 0$. If so, the state can be re-entered with any desired number of steps, which makes $r = 1$, and the state **aperiodic**. For an irreducible Markov chain, if *any* state has a self-loop, then all states are aperiodic.

EXAMPLE 13: PERIODIC AND APERIODIC MARKOV CHAINS

The Markov chain in Figure 16 is periodic with period 2 and the chain in Figure 15 is aperiodic.

■

6.2.7 Ergodicity

The sequence of states visited by a stochastic process is called its **trajectory**. For example, a valid trajectory for the chain in Figure 15 is $1 \to 2 \to 1 \to 3 \to \ldots$. Given a trajectory, we can compute a statistic on it, such as the fraction of the trajectory spent in a particular state. This statistic is the limiting ratio of the number of occurrences of that state to the length of the trajectory. Because trajectories can made as long as we desire, if the limiting statistic exists, it can be approximated as closely as we wish. We call such a limit a **time average**.

Now, consider a set of instances of the same stochastic process. At each time step, each instance changes state according to the same transition probabilities. Nevertheless, the trajectory associated with any pair of processes in the ensemble may differ due to their stochastic nature. Suppose we want to compute a statistic on the ensemble of trajectories at any time step. For instance, we may wish to compute the fraction of trajectories that are in state 1 at time step 10. If the limiting statistic exists, by making the ensemble sufficiently large, we can approximate this statistic as closely as we wish.We call such a limit a **ensemble average**.

An interesting question is whether a statistic computed as a time average is the same as a statistic computed as an ensemble average. As the next example shows, this need not necessarily be the case!

EXAMPLE 14: TIME AND SPACE AVERAGES

Consider the stochastic process shown in Figure 16. Suppose the initial state is 1 and the statistic we wish to compute is the fraction of time spent in an odd-numbered state. Clearly, no matter how long the length of the trajectory, this time average will be 1.0. Now, consider an ensemble of instances of this process where the initial state is chosen equally probably as 1 or 2. For this ensemble, the limiting value of the statistic at any time step will be 0.5 and is the ensemble average. For this stochastic process, the time average differs from the ensemble average.

■

Intuitively speaking, a stochastic process is **ergodic** if every statistic computed as a time average over a sufficiently long single trajectory can also be computed as a ensemble average over a sufficiently large number of trajectories. For this to be true, the sequence of states visited by the stochastic process over time should look statistically identical to the set of states occupied by an ensemble of processes at a single time step. We now consider the conditions under which a stochastic process is ergodic[2].

To begin with, we define a *state j* to be ergodic if it is recurrent non-null and aperiodic. Continuing with Example 4, it is a stair that the person will return to (recurrent), with a mean recurrence period that is finite (non-null), and such that the returning times do not have a least common divisor larger than 1 (aperiodic). If all the states in a Markov chain are ergodic, other than for a finite set of transient states, the chain itself is ergodic[3]. It can be shown that a finite aperiodic irreducible Markov chain is always ergodic (in other words, all states of a finite irreducible Markov chain are recurrent non-null).

EXAMPLE 15: ERGODIC MARKOV CHAIN

The Markov chain in Figure 15 is finite, aperiodic, and irreducible. Therefore, it is also ergodic.

■

If a chain is ergodic, then it is insensitive to its initial state $\pi(0)$. Independent of its initial state, $\pi(n)$, the distribution of X_n (for reasonably large values of *n*) is the same. Non-ergodic chains are either recurrent null (so that they may take a long time to return to some state), reducible (so some parts of the chain do not communicate with others), or periodic (so that quantities of interest also share the same period).

6.2.8 A fundamental theorem

We now have enough terminology to state (without proof) a fundamental theorem of queueing theory:

Theorem 1: The states of an irreducible Markov chain are either all transient, all recurrent null, or recurrent non-null. If any state is periodic, then all states are periodic with the same period *r*.

■

Intuitively, this categorizes all Markov chains into a few types. The first are those where the process goes from state to state but never returns to any state. In this case, all states are transient. In the second and third type of chain the process returns to at least one of the states. But the chain is irreducible, and so we can go from that state to all other states. Therefore, if the process can return to any one state, by definition it can return to all other states, which makes all states recurrent. In the second type, the transition probabilities are such that the expected recurrence period is infinite, so that all states are recurrent null. In the third type, the expected recurrence period for all states is finite. For this type, we have two sub-types: the periodic recurrent non-null chains, whose states all share the same period, and the aperiodic recurrent non-null (ergodic) chains, for whom no such period can be defined.

6.2.9 Stationary (equilibrium) probability of a Markov chain

Recall that for a homogeneous Markov chain the state transition probabilities are time-independent. For a homogeneous chain, we expect that the probability of *being* in any particular state to also be time-independent (if the probability of going from one state to another does not depend on time, the probability of being in any state shouldn't either).

Of course, the probability of being in a particular state may depend on the initial state, especially for non-ergodic chains, that are sensitive to their initial conditions. We define the **stationary probability distribution** of a Markov chain as follows: Suppose we start with the initial distribution $\pi(0) = \pi^*$. Then, π^* is also the stationary distribution of the chain, if for all *n*, $\pi(n) = \pi^*$. Intuitively, if we start with the probability of being in each state *j* as defined by the stationary distribution, then the transitions from each state according to the transition probabilities do not change the probability of being in each state.

EXAMPLE 16: STATIONARY DISTRIBUTION

Compute the stationary distribution of the Markov chain in Figure 17.

2. There are many mutually incompatible definitions of ergodicity in the literature. The definition presented here was chosen because it has a simple intuitive basis.
3. We allow a finite number of transient states in the chain because over sufficiently long trajectories or for a sufficiently large ensemble their contribution to any statistic is negligible.

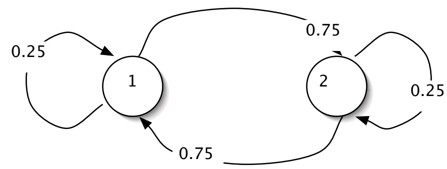


FIGURE 17. **A simple Markov chain**

Solution

Suppose that the initial probability of being in state 1 is 0.5 and of being in state 2 is 0.5, that is $\pi(0) = \begin{bmatrix}0.5 & 0.5\end{bmatrix}$. After one time step, the probability of being in state 1 is 0.25*0.5 + 0.75*0.5, where the first term is the probability of remaining in state 1, and the second term is the probability of going from state 2 to state 1, and we sum these probabilities because these are independent events. As expected, this sums to 0.5, so that the probability of being in state 1 in time step 2 is also 0.5. Symmetrically, if the probability of being in state 2 at time 1 is 0.5, the probability of being in state 2 at time 2 is also 0.5. Therefore, the stationary probability distribution of this chain is π^* = [0.5 0.5].

■

6.2.10 A second fundamental theorem

We now state a second fundamental theorem that allows us to compute stationary probabilities for any Markov chain.

Theorem 2: In an irreducible and aperiodic homogeneous Markov chain, the limiting probability distribution

$$\pi^* = \lim_{n \to \infty} \pi(n) \qquad \text{(EQ 75)}$$

always exists and is independent of the initial state probability distribution $\pi(0)$. Moreover, if all the states are ergodic (being recurrent non-null, in addition to being aperiodic), then π_j^*, the stationary probability of being in state *j*, is non-zero and can be uniquely determined by solving the following set of equations:

$$\begin{aligned} \sum_j \pi_j^* &= 1 \\ \pi_j^* &= \sum_i \pi_i^* p_{ij} \end{aligned} \qquad \text{(EQ 76)}$$

■

This theorem provides us with a simple set of equations to determine the stationary probability that the Markov chain is in any particular state. We only need verify that the set of states is finite, memoryless (i.e., satisfies the Markov property), irreducible (all states can be reached from each other), and aperiodic (for example, because of at least one self-loop). These properties can be verified through simple inspection. Then, we can solve the system of equations above to obtain the stationary probability of being in each state.

Example 15: (Stationary probability of a Markov chain

Compute the stationary probability for the Markov chain in Figure 15.

Solution

Note that this chain is ergodic, so we can use Theorem 2 to obtain the following equations:

$$\pi_1^* = 0.2\pi_1^* + 1\pi_2^* + 0.5\pi_3^*$$
$$\pi_2^* = 0.4\pi_1^* + 0\pi_2^* + 0.25\pi_3^*$$
$$\pi_3^* = 0.4\pi_1^* + 0\pi_2^* + 0.25\pi_3^*$$
$$1 = \pi_1^* + \pi_2^* + \pi_3^*$$

We solve this system of equations, using, for example, Gaussian elimination (see Section 3.4.2 on page 83) to obtain: π_1^* = 15/31; π_2^* = 8/31; π_3^* = 8/31, which is the stationary probability distribution of the chain (Verify this!).

■

6.2.11 Mean residence time in a state

Besides knowing the stationary probability of being in a particular state of a Markov chain, we would also like to know the expected duration that the process spends in each state. This can be computed by first obtaining the probability P(system stays in state *j* for *m* additional steps given that it just entered state *j*). The probability that the system stays in the same state after one time step is clearly p_{jj}. Moreover, after one time step, being Markovian, the process has no memory that it was in that state earlier. Therefore, the probability of staying in the state for *m* steps is given by $p_{jj}{}^m(1-p_{jj})$, which is a geometrically distributed random variable with parameter (1- p_{jj}) (see Section 1.5.3 on page 20). This allows us to compute the mean of the distribution, that is, the expected residence time in state *j*, as $1/(1-p_{jj})$.

EXAMPLE 17: RESIDENCE TIME

Compute the residence times in each state of the Markov chain shown in Figure 15.

Solution

p_{11} = 0.2, so E(residence time in state 1) = 1/(1-0.2) = 1/0.8 = 1.25.

p_{22} = 0, so E(residence time in state 1) = 1/(1-0) = 1.

p_{33} = 0.25, so E(residence time in state 1) = 1/(1-0.25) = 1/0.75 = 1.33.

■

6.3 Continuous-time Markov Chains

Our discussion so far has focused on discrete-time Markov chains, where state transitions happen every clock tick. We now turn our attention to continuous-time chains, where state transitions can happen independent of clock ticks. Most of the intuitions developed for discrete-time chains carry through to continuous-time chains, with a few modifications. The main point of difference is that we need to consider the time instants t_1, t_2,... when state transitions occur, rather than assuming a state transition occurs at every clock tick. We will briefly state the main results for a continuous-time stochastic process, then focus on a specific type of continuous-time process: the birth-death process.

6.3.1 Markov property for continuous-time stochastic processes

We first state the Markov property for continuous-time stochastic processes. The stochastic process $X(t)$ forms a continuous-time Markov chain if for all integers *n* and for any sequence of times t_1, t_2,...,t_{n+1} such that $t_1 < t_2 < ... < t_{n+1}$

$$P(X(t_{n+1}) = j \mid X(t_1)= i_1, X(t_2)= i_2,...,X(t_n)= i_n) = \mathrm{P}(X(t_{n+1}=j) \mid X(t_n)= i_n) \quad \textbf{(EQ 77)}$$

Intuitively, this means that the future ($X(t_{n+1})$) depends on the past only through the current state i_n.

The definitions of homogeneity, irreducibility, recurrence, periodicity, and ergodicity introduced for discrete-time Markov chains in Section 6.2.3 on page 174 continue to hold for continuous-time chains with essentially no change, so we will not restate them here.

6.3.2 Residence time in a continuous-time Markov chain

Analogous to the geometric distribution of residence times in a discrete-time chain, residence times are exponentially distributed for a continuous-time Markov chain for essentially the same reasons. If we denote the residence time in state j by R_j, the exponential distribution gives us the memorylessness property:

$$P(R_j > s + t | R_j > s) = P(R_j > t) \quad \textbf{(EQ 78)}$$

6.3.3 Stationary probability distribution for a continuous-time Markov chain

The definition of the stationary probability of a continuous-time Markov chain closely follows that of a discrete-time Markov chain. Therefore, we omit the intermediate details and directly present the set of equations necessary to compute the stationary probability of a continuous-time homogeneous Markov chain.

Define the transition probability of going from state i to state j by

$$p_{ij}(t) = P(X(s + t) = j | X(s) = i) \quad \textbf{(EQ 79)}$$

Intuitively, this means that if the process is at state i at any time s, then the probability that it will get to state j after a time interval t, is given by $p_{ij}(t)$. This is independent of the value of s because the process is homogeneous.

Define the quantity q_{ij}, which denotes the *rate* at which the process departs from state i to state j (where j and i differ) when it is in state i:

$$q_{ij} = \lim_{\Delta t \to 0} p_{ij}(\Delta t) / \Delta t \quad \textbf{(EQ 80)}$$

That is, the probability that the process transitions from i to j during any interval of length Δt time units, conditional on it already being at state i, is $q_{ij}\Delta t$. We also define the negative quantity q_{ii} by:

$$q_{ii} = -\sum_{j \neq i} q_{ij} \quad \textbf{(EQ 81)}$$

Then, $-q_{ii}$ is the rate at which the process does *not* stay in state i (i.e. departs to some other state) during an interval of length Δt time units. Because $\sum_j p_{ij}(t) = 1$, (at any time t, the chain transitions to *some* state, including the current state) we see that

$$\sum_j q_{ij}(t) = 0 \quad \textbf{(EQ 82)}$$

With these quantities in hand, we can define the time evolution of the probability of being in state j at time t, defined as $\pi_j(t)$, by:

$$\frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{k \neq j} q_{kj}\pi_k(t) \quad \textbf{(EQ 83)}$$

For ergodic continuous-time Markov chains, as $t \to \infty$, these probabilities tend to the stationary probability distributions π_j^* which are implicitly defined by:

$$q_{jj}\pi_j^* + \sum_{k \neq j} q_{kj}\pi_k^* = 0$$
$$\sum_j \pi_j^* = 1 \qquad \text{(EQ 84)}$$

Notice that this is Equation 83 with the rate of change of the probability set to 0--which is what one would expect for a stationary probability-- and with the time-dependent probabilities replaced by their time-independent limiting values.

This ends our brief summary of continuous-time Markov processes. Instead of studying general continuous-time processes, we will instead focus on a smaller but very important sub-class: that of continuous-time birth-death processes.

6.4 Birth-Death processes

This section discusses a special class of continuous-time homogenous Markov chains that have the property that state transitions are permitted from state *j* only to states *j*-1 and *j*+1 (if these states exist). This is well-suited to describe processes like the arrival and departure of customers from a queue (the subject of queueing theory, after all!) where the state index corresponds to the number of customers awaiting service. More precisely, if the number of customers in the system is *j*, then the Markov chain is considered to be in state *j*. Customer arrivals cause the number of customers in the system to increase by one, which moves the process to state *j*+1 and this happens at a rate $q_{j,j+1}$. Similarly, customer departures (due to service) cause the process to move from state *j* to state *j-1*, and this happens at the rate $q_{j,j-1}$. In keeping with standard terminology, we denote:

$$\lambda_j = q_{j,\,j+1}$$
$$\mu_j = q_{j,\,j-1} \qquad \text{(EQ 85)}$$

and these are also called the **birth** and **death rates** respectively. Note that these rates can be state-dependent (but cannot be time-dependent due to homogeneity). Also, by definition, at state *i*, the transition rates q_{ij} are 0 for all *j* other than *i*, *i-1*, and *i+1*. Given this fact, Equation 82, and Equation 85, we find that:

$$q_{jj} = -(\lambda_j + \mu_j) \qquad \text{(EQ 86)}$$

For a birth-death process, being in state *j* has the intuitive meaning that the population size is *j*, that is, there are *j* customers in the queueing system. Note that when $j = 1$, we have one customer in the system, the customer that is receiving service, and there are *none* in the queue. Generalizing, in state *j*, we have one customer receiving service and *j*-1 in the queue awaiting service.

6.4.1 Time-evolution of a birth-death process

Because a birth-death process is a continuous-time Markov chain, the time evolution of $\pi_j(t)$ is given by Equation 83. We substitute Equation 85 and Equation 86 to find:

$$\frac{d\pi_j(t)}{dt} = -(\lambda_j + \mu_j)\pi_j(t) + \lambda_{j-1}\pi_{j-1}(t) + \mu_{j+1}\pi_{j+1}(t) \qquad j \geq 1$$
$$\frac{d\pi_0(t)}{dt} = -\lambda_0\pi_0(t) + \mu_1\pi_1(t) \qquad j = 0 \qquad \text{(EQ 87)}$$

This describes the time-evolution of a birth-death system. In practice, solving these equations is complex, and does not give too many insights into the structure of the system. These are better obtained from the stationary probability distribution, which we study next.

6.4.2 Stationary probability distribution of a birth-death process

Because a birth-death process is an ergodic continuous-time Markov chain, its stationary probability distribution is given by Equation 84. Denoting the stationary probability of being in state j π_j^* by P_j and substituting Equation 85 and Equation 86 into Equation 84, we obtain the following equations:

$$
\begin{aligned}
0 &= -(\lambda_j + \mu_j)\pi_j^* + \lambda_{j-1}\pi_{j-1}^* + \mu_{j+1}\pi_{j+1}^* \qquad j \geq 1 \\
0 &= -\lambda_0\pi_0^* + \mu_1\pi_1^* \qquad j = 0 \\
&\sum_j \pi_j = 1
\end{aligned}
\tag{EQ 88}
$$

In matrix form, we can write the first two equations as

$$\boldsymbol{PQ} = \mathbf{0} \tag{EQ 89}$$

where

$$\boldsymbol{P} = \begin{bmatrix} \pi_0^* & \pi_1^* & \pi_2^* & \dots \end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \dots & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{EQ 90}$$

The two matrices are infinite-dimensional if the population size is unbounded. By defining $\boldsymbol{P}(t)$ as

$$\boldsymbol{P}(t) = \begin{bmatrix} \pi_0(t) & \pi_1(t) & \pi_2(t) & \dots \end{bmatrix}$$

we can rewrite Equation 87 as

$$d\boldsymbol{P}(t)/dt = \boldsymbol{P}(t)\boldsymbol{Q} \tag{EQ 91}$$

6.4.3 Finding the transition-rate matrix

The $\boldsymbol{Q}$ matrix defined by Equation 90 is also called the **transition rate matrix**. It is important because it allows us to derive both the time-dependent evolution of the system (i.e., $P_j(t)$), through Equation 91, and the long-term probability of being in state j, through Equation 89. Thus, in practice, the first step in studying a birth-death process is to write down its $\boldsymbol{Q}$ matrix.

Consider the following representation of a generic birth-death process:

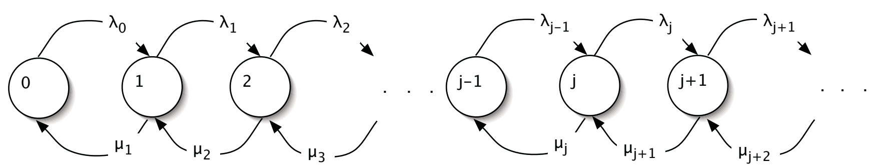


FIGURE 18. State-transition-rate diagram for a birth-death process

Here, we represent each state j by a circle and we label the arc from state j to state k with the transition rate q_{jk}. From this figure, we can determine $\boldsymbol{Q}$ for a birth-death process as follows: notice that the diagonal elements of $\boldsymbol{Q}$, i.e., q_{jj} are the negative of the quantities *leaving* state j. Focussing on the jth column, the $q_{j-1,j}$th elements, immediately above the diagonal (such as element q_{01}), are the rates entering state j from state $j-1$, i.e., λ_{j-1} and the $q_{j+1,j}$th elements, immediately below the diagonal (such as element q_{32}), are the rates entering state j from state $j+1$. All other elements are 0. In each row, the quantities sum to zero, due to Equation 82.

Thus, given the state-transition-rate diagram, it is possible to quickly construct $\boldsymbol{Q}$ and use this to obtain the time-dependent and time-independent (stationary) probabilities of being in each state. We will now use this approach to study some standard birth-death systems.

Note that this inspection approach also applies to all continuous time Markov chains, where we can determine the elements of the $\boldsymbol{Q}$ matrix by inspecting the corresponding state-transition-rate diagram, then solving for $\boldsymbol{P}$ and $\boldsymbol{P}(t)$ using Equation 89 and Equation 91.

EXAMPLE 18: TRANSITION-RATE MATRIX FOR A BIRTH-DEATH PROCESS

Consider the state-rate-transition diagram in Figure 19. What are the $\boldsymbol{P}$ and $\boldsymbol{Q}$ matrices for this system? What are the equations for its time-evolution and the long-term probability of being in each state?

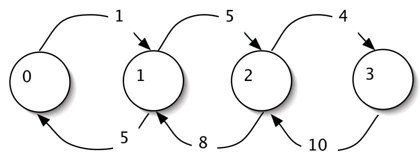


FIGURE 19. **A simple birth-death process**

The $\boldsymbol{P}$ matrix is $\begin{bmatrix}\pi_0^* & \pi_1^* & \pi_2^* & \pi_3^*\end{bmatrix}$ and $\boldsymbol{P}(t) = \begin{bmatrix}\pi_0(t) & \pi_1(t) & \pi_2(t) & \pi_3(t)\end{bmatrix}$. By inspection, we can write the $\boldsymbol{Q}$ matrix as

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 5 & -10 & 5 & 0 \\ 0 & 8 & -12 & 4 \\ 0 & 0 & 10 & -10 \end{bmatrix}$$

Therefore, the time-evolution of state probabilities is given by:

$$\frac{d}{dt}\left(\begin{bmatrix}\pi_0(t) & \pi_1(t) & \pi_2(t) & \pi_3(t)\end{bmatrix}\right) = \begin{bmatrix}\pi_0(t) & \pi_1(t) & \pi_2(t) & \pi_3(t)\end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 5 & -10 & 5 & 0 \\ 0 & 8 & -12 & 4 \\ 0 & 0 & 10 & -10 \end{bmatrix}$$

$$\begin{bmatrix}\dot{\pi}_0(t) & \dot{\pi}_1(t) & \dot{\pi}_2(t) & \dot{\pi}_3(t)\end{bmatrix} = \begin{bmatrix}\pi_0(t) & \pi_1(t) & \pi_2(t) & \pi_3(t)\end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 5 & -10 & 5 & 0 \\ 0 & 8 & -12 & 4 \\ 0 & 0 & 10 & -10 \end{bmatrix}$$

which is a system of differential equations that can be solved to give the evolution of the time-varying probabilities $\pi_i(t)$.

The long-term probability of being in each state is given by:

$$\begin{bmatrix}\pi_0^* & \pi_1^* & \pi_2^* & \pi_3^*\end{bmatrix}\begin{bmatrix}-1 & 1 & 0 & 0\\ 5 & -10 & 5 & 0\\ 0 & 8 & -12 & 4\\ 0 & 0 & 10 & -10\end{bmatrix} = 0$$

which is a system of linear equations in four variables that can be solved to obtain the stationary probability distribution of the chain.

■

6.4.4 A pure-birth (Poisson) process

Consider a system where $\lambda_j = \lambda$ for all j (the departure rate from all states is the same), and $\mu_j = 0$ for all j (the death rate is 0). This represents a **Poisson** process whose population grows without bound and whose rate of growth is λ independent of the population size (that is, we expect the population to grow by 1 every $1/\lambda$ seconds independent of the current population size).

We study two properties of this process: the probability of being in state j at time t, which corresponds to having j arrivals in time t, and the distribution of inter-arrival times, that is, the expected time between going from any state to the adjacent state.

We can derive the probability of being in any state directly from Equation 87. Substituting the values for λ and μ in this equation, we get:

$$\frac{d\pi_j(t)}{dt} = -\lambda\pi_j(t) + \lambda\pi_{j-1}(t) \qquad j \geq 1$$
$$\frac{d\pi_0(t)}{dt} = -\lambda\pi_0(t) \tag{EQ 92}$$

The second equation is a trivial differential equation whose solution is given by

$$\pi_0(t) = e^{-\lambda t} \tag{EQ 93}$$

We substitute this into the first equation to get

$$\frac{d\pi_1(t)}{dt} = -\lambda\pi_1(t) + \lambda e^{-\lambda t} \tag{EQ 94}$$

whose solution is

$$\pi_1(t) = \lambda e^{-\lambda t} \tag{EQ 95}$$

By repeatedly substituting this into the first equation, we obtain

$$\pi_j(t) = \frac{(\lambda t)^j}{j!}e^{-\lambda t} \tag{EQ 96}$$

This is the density function for the Poisson distribution (see Section 1.5.4 on page 20) with parameter λt. Thus, for a Poisson process with parameter λ, the probability of j arrivals in time t, which is also the probability of being in state j at time t, is given by a Poisson distribution with parameter λt. Because the mean of the Poisson distribution is also its parameter, the expected number of arrivals in time t is λt. This is intuitively pleasing: the arrival rate is λ so in time t we should see, on average, λt arrivals.

EXAMPLE 19: POISSON PROCESS

Consider students arriving to a class as a Poisson process at a mean rate of 5 students/second. What is the probability that the classroom has (a) 2 students after 2 seconds (b) 10 students after 2 seconds?

Solution

We have $\lambda = 5$ and $t = 2$, so the Poisson parameter is 10. The probability of having 2 students in the classroom after 2 seconds is $\pi_2(2) = (10^2/2!)\, e^{-10} = 50*e^{-10} = 2.26*10^{-3}$. This is a rather unlikely event.

The probability of having 10 students in the classroom after 2 seconds is $\pi_{10}(2) = (10^{10}/10!)\, e^{-10} = 0.125$. Note that the expected number of students after 2 seconds is 10, yet the probability that the expected number of students is actually achieved is only one in eight!

■

We now derive the interarrival time distribution for a Poisson process. Let *a* denote the continuous random variable that represents the time between any two arrivals: we seek the distribution for *a*. Consider the cumulative density function of *a*, given by the probability $P(a \le t) = 1 - P(a > t)$. But $P(a > t)$ is just P(0 customer arrivals in time $(0, t)$) = 1- $\pi_0(t)$ = $1 - e^{-\lambda t}$, $t \ge 0$. The density function is given by differentiating this expression to get:

$$a(t) = \lambda e^{-\lambda t} \qquad \text{(EQ 97)}$$

We recognize this as an exponential distribution (see Section 1.6.3 on page 23). This gives us the following important result:

> The interarrival times for a Poisson process are drawn from an exponential distribution.

We note that the exponential distribution is memoryless. Thus, for a Poisson process, not only is the rate of transitioning to the next state (the birth rate) independent of the current population size, the *time* at which this transition occurs does not depend on how long the process has been at the current population size.

6.4.5 Stationary probability distribution for a birth-death process

We now return to computing the stationary probability distribution for a general birth-death process using Equation 88. From the second equation, we immediately obtain

$$\pi_1^* = \frac{\lambda_0}{\mu_1}\pi_0^* \qquad \text{(EQ 98)}$$

Substituting this into the first equation, we find that

$$\pi_2^* = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0^* \qquad \text{(EQ 99)}$$

Repeating this substitution, we find that P_j is given by

$$\pi_j^* = \frac{\lambda_0\lambda_1\ldots\lambda_{j-1}}{\mu_1\mu_2\ldots\mu_j}\pi_0^* = \pi_0^*\prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}} \qquad \text{(EQ 100)}$$

We therefore obtain the long-term probabilities of being in any state *j* as a function of the probability of being in state 0 and the system parameters. Knowing that these probabilities sum to 1, we can determine

$$\pi_0^* = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad \text{(EQ 101)}$$

This can be substituted back into Equation 100 to obtain the long-term probability of being in any state *j*. Of course, we need to ensure that the series in the denominator of Equation 101 actually converges! Otherwise, π_0^* is undefined, and so are all the other π_i^* s. It turns out that the condition for convergence (as well as for the chain to be ergodic) is the existence of a value j_0 such that for all values of $j > j_0$, $\lambda_j < \mu_j$. We interpret this to mean that after the population reaches some threshold j_0, the rate of departures must exceed the rate of arrivals. This makes intuitive sense: otherwise, the population size will grow (in expectation) without bound and the probability of any particular population size will be 0.

EXAMPLE 20: GENERAL EQUILIBRIUM SOLUTION

Find the equilibrium probabilities of being in each state for the birth-death process shown in Figure 19.

Solution

From Equation 101, we get

π_0^* = 1/[1 + 1/5 + (1*5)/(5*8) + (1*5*4)/5*8*10)] = 0.73.

This can be substituted into Equation 100 obtain

π_1^* = 1/5 π_0^* = 0.2 * 0.73 =0.146.

π_1^* = 1/8 π_0^* = 0.125 * 0.73 =0.09.

π_2^* = 1/20 π_0^* = 0.05 * 0.73 = 0.0365.

As a check, 0.73 + 0.146 + 0.09 + 0.0365 = 1.0025, which is within the rounding error.

■

6.5 The M/M/1 queue

This section present the M/M/1 queue, the simplest non-trivial queueing system. Here, the a/b/c notation, also called **Kendall notation**, denotes that:

1. (the ‘a’ portion in the notation): the arrival process is ‘Markovian’, i.e, it is a Poisson process with exponentially distributed inter-arrival times
2. (the ‘b’ portion in the notation): the departure process is ‘Markovian’, i.e., it is a Poisson process with exponentially-distributed inter-departure times
3. (the ‘c’ portion in the notation): there is a single server

Extended forms of the notation describe the size of the buffers available (we assume an infinite number), the service discipline (we assume first-come-first-served) and other queueing parameters. However, the three-parameter version of the notation is the one that is commonly used.

The M/M/1 queueing system is a birth-death Markov process with a state-independent arrival rate λ and a state-independent departure rate μ. These rates are therefore also independent of the population size. This is counter-intuitive, in that the rate of departure from a small population is the same as the rate of departure from a large population.

We study the long-term behaviour of the M/M/1 queue by removing state-dependence (the subscript *j*) in the transition rates in the analysis of Section 6.4.5. From Equation 100, we find that:

$$\pi_j^* = \pi_0^* \prod_{i=0}^{j-1} \frac{\lambda}{\mu} = \pi_0^* \left(\frac{\lambda}{\mu}\right)^j \qquad j \geq 0 \tag{EQ 102}$$

To obtain π_0^*, we use Equation 101 to obtain

$$\pi_0^* = \frac{1}{\left(1 + \sum_{j=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^j\right)} \tag{EQ 103}$$

When $\lambda < \mu$, the infinite sum in the denominator converges, and the denominator reduces to $\left(\frac{1}{1-\frac{\lambda}{\mu}}\right)$, so that

$$\pi_0^* = 1 - \frac{\lambda}{\mu} \tag{EQ 104}$$

The ratio λ/μ represents the intensity of the arrival rate, as a fraction of the service rate and can be viewed as the utilization of the system. The value is important enough that it deserves its own symbol, ρ, which allows us to write Equation 104 as

$$\pi_0^* = 1 - \rho \tag{EQ 105}$$

This equation has the intuitive meaning that the probability that the system is idle (i.e., π_0^*) is (1 - utilization). It can be shown that this relationship is true for *all* queueing systems whose population size is unbounded.

We now use Equation 102 to obtain the stationary probability of being in any state *j* as:

$$\pi_j^* = \rho^j(1-\rho) \tag{EQ 106}$$

Note that this is a geometric distribution.

EXAMPLE 21: M/M/1 QUEUE

Consider a link to which packets arrive as a Poisson process at a rate of 300 packets/sec such that the time taken to service a packet is exponentially distributed. Suppose that the mean packet length is 500 bytes, and that the link capacity is 1.5 Mbps. What is the probability that the link's queue has 1, 2 and 10 packets respectively?

Solution

The packet length is 500 bytes = 4000 bits, so that the link service rate of 1,500,000 bits/sec = 375 packets/sec. Therefore, the utilization is 300/375 = 0.8. When the link queue has 1 packet, it is in state $j = 2$, because one packet is being served at that time. Thus, we need $\pi_2^* = 0.8^2 * 0.2 = 0.128$. For the queue having two packets, we compute $\pi_3^* = 0.8^3 * 0.2 = 0.1$. For 10 packets in the queue, we compute $\pi_{11}^* = 0.8^{11} * 0.2 = 0.0067$, an fairly small quantity. Thus, even when the utilization is high (80%), the queue size is quite small, rarely exceeding 10 packets.

■

Note that the long-term probability that the population size is *j* depends only on the utilization of the system. As the utilization increases, the probability of reaching larger population sizes increases. To see this analytically, consider the mean number of customers in the system (which is also the mean population size), denoted $\bar{N}$ defined by

$$\bar{N} = \sum_{j=0}^{\infty} j\pi_j^* \qquad \text{(EQ 107)}$$

It can be shown that this sum converges (i.e., when $\lambda < \mu$) and that

$$\text{Mean number of customers in the system} = \bar{N} = \frac{\rho}{(1-\rho)} \qquad \text{(EQ 108)}$$

EXAMPLE 22: MEAN NUMBER OF CUSTOMERS IN THE QUEUE

Compute the mean number of packets in the system of Example 20.

Solution

The utilization is 0.8, so the mean number of packets in the system is 0.8/(1-0.8) = 0.8/0.2 = 4. Of these, we expect three to be in the queue, and one will be in service.

■

It is obvious from Equation 108 that as $\rho \to 1$, $\bar{N} \to \infty$. That is, as the arrival rate approaches the service rate, the expected number of customers in the system grows without bound. This is somewhat unexpected: after all, the arrival rate is smaller than the service rate: why then should the number of customers grow? The reason is that we are dealing with stochastic processes. Even though the arrival rate, on average, is lower than the service rate, there will be time periods when the short-term arrival rate exceeds the service rate. For instance, even if the mean arrival rate is 1 customer per second, there will be short intervals during which two or even three customers may arrive in one second. During this time, the queue builds up, and is drained when the service rate exceeds the arrival rate. In fact, there is an interesting asymmetry in the system: when the (short-term) arrival rate exceeds the (short-term) service rate, the queue builds up, but when the service rate exceeds the arrival rate, if the queue is empty, the system does not build up 'service credits.' The server is merely idle. Thus, the system tends to build up queues that are only drained over time. This is reflected in the fact that as the utilization of the system increases, the mean number of customers in the system increases sharply.

It is remarkable that this fundamental insight into the behaviour of a real queueing system can be derived with only elementary queueing theory. Moreover, this insight carries over to all other queueing systems: as the utilization approaches 1, the system becomes **congested**. The behaviour of the mean queue length (which also corresponds to the waiting time, through Little's theorem), is shown in Figure 20.

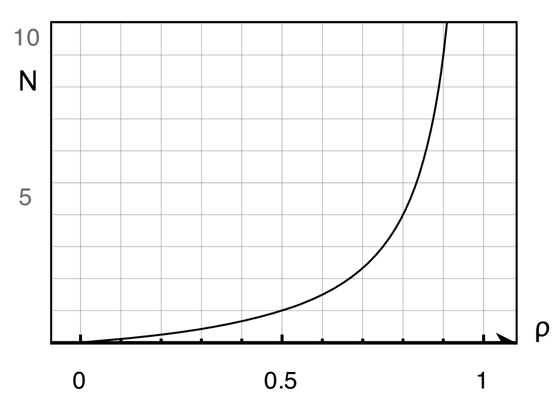


FIGURE 20. Mean queue length as a function of utilization

It is clear that the queue length asymptotes to infinity as the utilization approaches 1. In networking terms, this means that as the arrival rate approaches a link's capacity, the queue at the immediately preceding router or switch will grow without bound, causing packet loss. This analysis allows us to derive a practical guideline: we should provision enough service capacity so that the system utilization never exceeds a threshold of around 70%. Alternatively, if this threshold is exceeded, either service requests should be dropped, or new service capacity should be made available so that the utilization decreases.

Another related quantity of interest for this queue is the mean waiting time in the queue. From Little's theorem, the mean number of customers in the system is the product of their mean waiting time and their mean arrival rate, so $\frac{\rho}{(1-\rho)}c$ = mean waiting time *λ, which means that

$$\text{Mean waiting time} = \frac{\frac{\rho}{\lambda}}{(1-\rho)} = \frac{\frac{1}{\mu}}{(1-\rho)} \qquad \text{(EQ 109)}$$

This quantity also grows without bound as the utilization approaches 1.

EXAMPLE 23: WAITING TIME OF A M/M/1 QUEUE

What is the mean waiting time for a packet in the queue described in Example 20?

Solution

For this queue, μ = 375 and ρ = 0.8. So, the mean waiting time is (1/375)/(1-0.8) = 5/375 seconds = 13.3 ms.

■

6.6 Two variations on the M/M/1 queue

We now briefly consider two variations on the M/M/1 queue, essentially to give insight into how one proceeds with the analysis of a queueing system.

6.6.1 The M/M/∞ queue: a responsive server

Suppose that a provider of service capacity brings on a new server to serve every arriving customer. This is like a private bank where new agents are brought on to provide individual attention to each customer when she or he arrives. This system can be modelled as a queue with an infinite number of servers (though, at any time, the number of servers is finite).

We can model and analyse this queue using the same techniques as an M/M/1 queue. We start with the state-transition-rate diagram show in Figure 21. Note that μ_j, the rate of departure from the jth queue, is $j\mu$

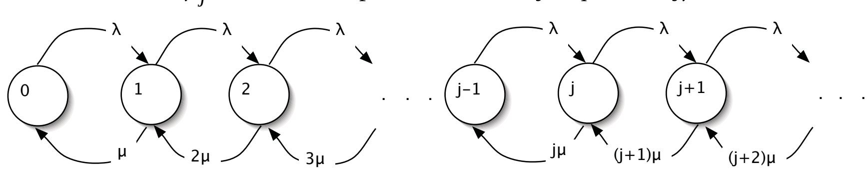


FIGURE 21. State-transition-rate diagram for an M/M/∞ queues

which models the fact that when there j customers, there are j servers. From the diagram, we can directly invoke Equation 100 to write down π_j^*, the stationary probability of being in state j, as

$$\pi_j^* = \pi_0^* \prod_{i=0}^{j-1} \frac{\lambda}{(i+1)\mu} = \pi_0^* \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} \quad \text{(EQ 110)}$$

We solve for π_0^* using Equation 101 as

$$\pi_0^* = \frac{1}{\left[1 + \sum_{j=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!}\right]} \quad \text{(EQ 111)}$$

Recalling the standard expansion $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, we see that

$$\pi_0^* = e^{-\frac{\lambda}{\mu}} \quad \text{(EQ 112)}$$

$$\pi_j^* = \pi_0^* \prod_{i=0}^{j-1} \frac{\lambda}{(i+1)\mu} = e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} \quad \text{(EQ 113)}$$

Equation 113 shows that the stationary probability of being in state j is given by the Poisson distribution with parameter λ/μ. Thus, with ‘infinite’ servers, the number of customers in the queue follows the Poisson distribution. This allows us to compute the expected number of customers in the queue as the mean of the Poisson, which is its parameter, i.e., λ/μ. All other parameters of interest for this queueing system can be derived from Equation 113.

EXAMPLE 24: RESPONSIVE SERVER

Suppose customers arrive at a private bank, modelled as a responsive server, as a Poisson process at the rate of 10 customers/hour. Suppose that a customer’s needs can be met on average in 20 minutes, and that the service time distribution is exponentially distributed. What is the probability that there are five customers in the bank at any point in time?

Solution

We have $\lambda = 10$ and $\mu = 3$ (i.e. 3 customers can be served an hour, on average, by a server). Thus $\pi_0^* = e^{-10/3} = 0.036$. We need to find $\pi_5^* = 0.036 * (10/3)^5 * 1/5! = 0.123$. Thus, there is a nearly one in eight chance that there will be 5 customers in the bank at any given time.

■

6.6.2 M/M/1/K: bounded buffers

Suppose that the queueing system has only K-1 buffers. In this case, the population size (including the customer in service) cannot grow beyond K, and arrivals when the system is in state K are lost (similar to packet loss when arriving to a full queue). To model this, we can simply ignore arrivals to state K, which means that we will never enter states K+1, $K+2$,... This results in a state-transition-rate diagram shown in Figure 22.

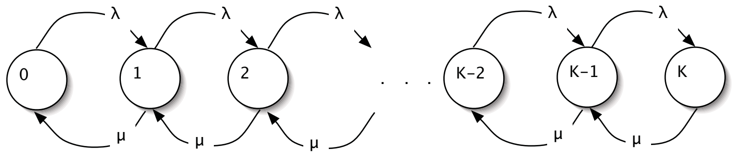


FIGURE 22. **State-transition-rate diagram for an M/M/1/K queue**

The state transition rates are therefore

$$\lambda_j = \begin{cases} \lambda & j < K \\ 0 & j \geq K \end{cases} \qquad \text{(EQ 114)}$$

$$\mu_j = \mu \qquad j=1,2,...,K$$

We can therefore use Equation 100 to write π_j^* as

$$\pi_j^* = \begin{cases} \pi_0^* \left(\frac{\lambda}{\mu}\right)^j & j \leq K \\ 0 & j > K \end{cases} \qquad \text{(EQ 115)}$$

We use Equation 101 to obtain

$$\pi_0^* = \frac{1}{\left[1 + \sum_{j=1}^{K} \left(\frac{\lambda}{\mu}\right)^j\right]} \qquad \text{(EQ 116)}$$

Given the standard result $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$, we can simplify this to

$$\pi_0^* = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \qquad \text{(EQ 117)}$$

So, we can now write Equation 115 as

$$\pi_j^* = \begin{cases} \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \left(\frac{\lambda}{\mu}\right)^j = \dfrac{1-\rho}{1-\rho^{K+1}} \rho^j & j \leq K \\ 0 & j > K \end{cases} \qquad \text{(EQ 118)}$$

As before, given these probabilities, we can compute all quantities of interest about the queueing system, such as the distribution of the queue length, the mean number of customers in the queue, and the mean waiting time. In particular, the intuitive meaning of π_K^* is the probability that the system is 'full' when it has a buffer of size *K*-1. So, π_K^* can be interpreted as the **blocking probability** of a M/M/1/K queue. We can then choose *K* as a sizing parameter to make π_K^* as small as desired.

Note that in this system, $\pi_0^* \neq 1 - \rho$ because the system size is bounded. Therefore, the number of customers served in a chosen time period may be lower than what the utilization indicates because customer arrivals when the queue is full are lost. Moreover, the system is stable by definition, independent of the utilization, because excess arrivals are automatically dropped.

EXAMPLE 25: M/M/1/K QUEUE

Consider the system of Example 20, but with the restriction that the queue only has four buffers. What is the probability that three of these are in use? How many buffers should we provision to ensure that the blocking probability is no more than 10^{-6}?

Solution

We have $K = 5$, and $\frac{\lambda}{\mu}$ =0.8. From Equation 117, π_0^* = (1-0.8)/(1-0.8^6) = 0.27. If three buffers are in use, then the system is in state j = 4. From Equation 115, we get π_4^* = 0.27(0.8)^4=0.11.

To size the buffer, we have to choose K such that $\pi_K^* < 10^{-6}$. We solve for K^* using the inequality $10^{-6} > ((.2)(0.8)^K)/(1-0.8^{K+1})$, to obtain $K^* = 55$. Thus, we need 54 buffers to satisfy this blocking probability.

■

6.7 Other queueing systems

We now turn our attention to queueing systems that go beyond the Markovian and exponential framework. The queueing systems become much more difficult to analyze, so we will merely state the results for two important queueing systems.

6.7.1 M/D/1: deterministic service times

Consider a queueing system where arrivals are from a Poisson process, but service times are deterministic. That is, as long as the queue is non-empty, the inter-departure time is deterministic (rather than exponentially distributed). Representing the inter-departure time (a constant) by μ, and the utilization by $\rho = \lambda/\mu$, it can be shown that the system is stable (i.e., the queue length is finite) as long as $\lambda < \mu$. Moreover, the long-term probability that the number of customers in the system is j, i.e., P_j is given by

$$\pi_j^* = \begin{cases} 1-\rho & j=0 \\ (1-\rho)(e^{\rho}-1) & j=1 \\ (1-\rho)\left(\sum_{i=0}^{j} \frac{(-1)^{j-i}(i\rho)^{j-i-1}(i\rho+j-i)e^{i\rho}}{(j-1)!}\right) & j>1 \end{cases} \quad \text{(EQ 119)}$$

This allows us to derive the mean number of customers in the system as:

$$\text{Mean customers in the system} = \rho + \frac{\rho^2}{2(1-\rho)} \quad \text{(EQ 120)}$$

and the mean response time as:

$$\text{Mean response time} = \frac{1}{\mu} + \frac{\rho}{2\mu(1-\rho)} \tag{EQ 121}$$

Other quantities of interest regardinwg the M/D/1 queue can be found in standard texts on queueing theory.

6.7.2 G/G/1

Once the arrival and service processes become non-Poisson, the analysis of even a single queue becomes challenging. For such systems few results are available other than Little's theorem, and also that, if the queue size is unbounded, $\pi_0^* = 1\text{-}\rho$. A detailed study of such queues is beyond the scope of this text.

6.7.3 Networks of queues

So far, we have only studied the behaviour of a single queue. This is like studying a network with a single router - not very interesting! What happens when we link the output of a queue to the input of another queue, as we do in any computer network? Intuitively, what we are doing is to make the inter-departure process of one queue the inter-arrival process for the second queue. Moreover, we may have more than one inter-departure process mix to form the inter-arrival process. Can this be analysed?

We represent this composite system, also called a 'tandem of queues' as shown in Figure 23.

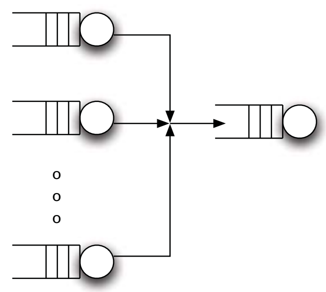


FIGURE 23. A network of queues

Here, each queue is shown by a buffer (with customers or jobs in it) and a server (represented by a circle). Jobs served by the servers on the left enter the queue of the server on the right. Each queue and associated server is also called a *node* (drawing on the obvious graph analogy).

If all the queues on the left are M/M/1 queues, recall that their inter-departure processes are Poisson. Moreover, it can be shown that the mixture of Poisson processes is also a Poisson process whose parameter is the sum of the individual processes. Therefore, the input to the queue on the right is a Poisson process that can be analysed as an M/M/1 queue. This leads to the fundamental insight that a tandem of M/M/1 queues is analytically tractable. Because the departure process of a M/M/*m* queue (i.e. a queue with *m* servers) is also Poisson, this result holds true for tandems of M/M/*m* queues.

We can make things a bit more complicated: we can allow customers to enter *any* queue (node) as a Poisson process and we can also allow customers that leave a node to exit the system altogether with some probability or join any other node in the system with some probability. Note that this can potentially lead to cycles, where customers go through some set of nodes more than once. Nevertheless, Jackson was able to show that these networks behave as if each M/M/m queue was being fed by a single Poisson stream. Such networks are also called **Jacksonian** networks in his honour. For a Jacksonian network, we have a strong result: let $\pi^*_{k_1 k_2 \ldots k_n}$ denote the long-term probability that there are k_1 customers at the first node, k_2 customers at the second node, and so on. Then:

$$\pi^*_{k_1 k_2 \ldots k_n} = \pi^*_{k_1}\pi^*_{k_2}\ldots\pi^*_{k_n} \tag{EQ 122}$$

That is, the joint probability of having a certain number of customers in each queue is the product of the individual probabilities. We interpret this to mean that each queue in the system acts as if it was independent of the others. This **product-form** of the probability distribution greatly simplifies analysis.

Despite the elegance and power of Jacksonian network analysis, it can be rarely applied to study practical computer networks. This is because customers (packets) in real networks rarely arrive as a Poisson process. Thus, the output process is also non-Poisson, which makes subsequent analysis complex. In recent years, the development of network calculus and stochastic network calculus has allowed significant inroads into the study of non-Jacksonian networks.

6.1 Further reading

The definitive introductory text on queueing theory is the two volume text: L. Kleinrock, Queueing Systems, Wiley Interscience, 1975. A modern and through treatment of Markov chains can be found in P. Bremaud, Markov Chains, Springer, 1999. Further details on network calculus can be found in J.-Y. Le Boudec and P. Thiran, Network Calculus, Springer, 2001.

6.8 Exercises

1 Little's theorem

Patients arriving to the emergency room at the Grand River Hospital have a mean waiting time of three hours. It has been found that, averaged over the period of a day, that patients arrive at the rate of one every five minutes. (a) How many patients are awaiting treatment on average at any given point in time? (b) What should be the size of the waiting room so that it can accommodate everyone?

2 A stochastic process

Consider that in Example 4, a person is on an infinite staircase on stair number 10 at time 0 and potentially moves once every clock tick. Suppose that he moves from stair i to stair $i+1$ with probability 0.2, and from stair i to stair $i-1$ with probability 0.2 (the probability of staying on stair i is 0.6). Compute the probability that the person is on each stair at time 1 (after the first move), time 2, and time 3.

3 Discrete and continuous state and time processes

Come with your own examples for all four combinations of discrete state/discrete time/continuous state/continuous time processes.

4 Markov process

Is the process in Exercise 2 a Markov process? Why or why not?

5 Homogeneity

Is the process in Exercise 2 homogeneous? Why or why not?

6 Representation

(a) Represent the process in Exercise 2 using a transition matrix and a state transition diagram. (b) Do the rows in this matrix have to sum to 1? Do the columns in this matrix have to sum to 1? Why or why not? (c) Now, assume that the staircase has only 4 steps. Make appropriate assumptions (what are these?) to represent this finite process as a transition matrix and a state transition diagram.

7 Reducibility

Is the chain in Exercise 2 reducible? Why or why not?

8 Recurrence

Is state 1 in the chain in Exercise 6(c) recurrent? Compute f_1^1, f_1^2 and f_1^3.

9 Periodicity

Is the chain in Exercise 2 periodic? If not, give an example of a chain with period N for arbitrary $N > 1$.

10 Ergodicity

Is any state in the chain of Exercise 6(c) non-ergodic? Why or why not?

11 Stationary probability

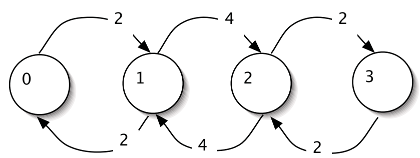
Compute the stationary probability distribution of the chain in Exercise 6(c).

12 Residence times

Compute the residence time in each state of the Markov chain in Exercise 6(c).

13 Stationary probability of a birth-death-process

Consider the state-rate-transition diagram shown below.



(a) Compare this with the state transition probability diagram in Exercise 6(c). What features are the same, and what differ?

(b) Write down the $\boldsymbol{Q}$ matrix for this system.

(c) Use the $\boldsymbol{Q}$ matrix to compute the stationary probability distribution of this chain.

14 Poisson process

Prove that the inter-departure time of a pure-death process is exponentially distributed.

15 Stationary probabilities of a birth-death process

Use Equation 30 to compute the stationary probability of the birth-death process in Exercise 13.

16 M/M/1 queue

Is the birth-death-process in Exercise 13 M/M/1? Why or why not?

17 M/M/1 queue

Consider a link to which packets arrive as a Poisson process at a rate of 450 packets/sec such that the time taken to service a packet is exponentially distributed. Suppose that the mean packet length is 250 bytes, and that the link capacity is 1 Mbps.

(a) What is the probability that the link's queue has 1, 2 and 10 packets respectively?

(b) What is the mean number of packets in the system? What is the mean number in the queue?

(c) What is the mean waiting time?

18 Responsive (M/M/∞) server

Compute the ratio of P_j for a responsive server to the same value for an M/M/1 queue. How does this ratio behave as a function of j?

19 M/M/1/K server

Assume that the queueing system in Exercise 17 has 10 buffers. Compute an upper bound on the probability of packet loss.

20 M/D/1 queue

Compute the mean number of customers in an M/D/1 system that has a utilization of 0.9. (a) How does this compare with a similarly loaded M/M/1 system? (b) Compute the ratio of the mean number of customers as a function of ρ. (c) Use this to compare the behavior of an M/D/1 queue with that of an M/M/1 queue under heavy load.

CHAPTER 7 Game Theory

Mathematical game theory is the study of the behaviour of decision-makers who are conscious that their actions affect each other and who may have imperfect knowledge both of each other and of the future. It originated with the mathematical study of traditional games such as bridge and chess. In these games, each player makes moves in response to, and in active consideration of, the moves of the other players. The concepts that arise in the study of such parlour games are widely applicable. In particular, they are relevant in situations where a scarce resource has to be shared amongst many entities and each entity tries to maximize its own share: a situation that frequently occurs in computer networks.

The difference between a game and classical mathematical optimization can be intuited from an example. Suppose you are looking for a particular store in an unfamiliar mall. This is an optimization problem, where you could choose a search strategy that minimizes the time taken to find the store, given the set of actions available to you (for example, walk past every store, look for a directory, or ask for help). In contrast, suppose you are looking for your lost friend in a mall. Should you stay in a central location so that your friend can find you? Or should you move around, with some risk that you may never meet? Introducing a second locus of decision-making completely changes the problem!

The origin of game theory, due almost entirely to von Neumann and Morgenstern in the first half of the twentieth century, was part of a larger project on the mathematicization of sociology which came from the world view that mathematics could be used to solve problems of society that had eluded centuries of past effort by qualitative sociologists. This world view, especially the use of game theory to model policies for global nuclear warfare, gave the theory a (perhaps deservedly so) bad reputation, though it was routinely used to study microeconomic problems. In recent years, game theory has given deep insights into the operation of the Internet and in the design of decentralized algorithms for resource sharing, in particular, the theory known as 'mechanism design.' Hence, there has been a resurgence of interest in these topics. This chapter describes the terminology of game theory, focuses on algorithmic aspects of mechanism design, and concludes with a sketch of the limitations of this approach.

7.1 Concepts and terminology

7.1.1 Preferences and preference ordering

The ultimate basis of game theory is **utility theory**, which in turn is grounded in the axiomatization of the preference relationships of an agent[1]. Note that the axioms of preferences refer to '**goods**.' By a good, we mean concrete objects such as a

1. The term 'agent' is widely used in the area of Artificial Intelligence to denote either a human being or software acting on a human's behalf.

bar of soap or a meal at a restaurant, as well as more abstract quantities such as the mean end-to-end delay, measured over intervals of one second and over the course of a given hour, between a particular browser and a web server.

Here are the **axioms of preferences**:

1. **Orderability***:* Given two goods, an agent must prefer one to the other or view them both as being equally preferable. There is no option to 'pass.' Therefore, in any set of goods, there must exist a set of equivalent most-preferred and least-preferred goods.
2. **Transitivity**: If an agent prefers good A to good B and good B to good C, then it prefers good A to good C.
3. **Continuity**: We assume that if an agent prefers good B more than good A and less than good C, it is always possible to define a lottery where with probability p the agent would win prize A and with the remaining probability the agent would win prize C, such that the agent equally prefers B and the lottery. We say that the agent is indifferent between B and a lottery with outcome pA + (1-p)C (see Example 1 below).
4. **Substitutability**: If an agent prefers good A and B equally, then we should be able to replace one with the other in any lottery.
5. **Monotonicity**: Given two lotteries with the same outcomes A and C, defined by pA + (1-p)C, qA + (1-q)C respectively, if an agent prefers A to C and $p>q$, then it prefers the first lottery to the second.
6. **Decomposability**: A **compound lottery** is a lottery that is run in two stages, where the winners of the first stage enter a second lottery and may subsequently either win again or lose. Decomposability means that such a compound lottery is equivalent to an appropriately-defined single-stage lottery: if the outcome of the first lottery is pA+ (1-p)B, and of the second lottery is qC+(1-q)D, and outcome B of the first lottery is participation in the second, then the outcome of the compound lottery is pA +(1-p)qC + (1-p)(1-q)D.

EXAMPLE 1: CONTINUITY OF PREFERENCES

Consider an agent that prefers an apple (A) to a banana (B) and a banana to a carrot (C). Suppose the agent participates in the following lottery: we divide the circumference of a circle into two sections, marked A and C, where the fraction of the circumference that is marked A is denoted p. A pointer pinned to the centre of the circle is spun. If the pointer stops spinning at a part of the circle marked A (which happens with probability p), the agent wins A. Otherwise, it wins C. Continuity of preferences implies that there exists a value p such that the agent is equally happy with B and the results of this lottery. Intuitively, when p is 1, then the agent always wins A, so it should prefer the lottery to the B. Conversely, when p is 0, then the agent always wins C, so it should prefer a B to the lottery. Therefore, it seems plausible that there is some intermediate point where the agent equally prefers the lottery and B.

■

These axioms of preference allows us to express the preference an agent may have for any element of a set of goods as a lottery over the preferences for the least and most preferred element. We can do more: suppose that we assign numerical values to the least and most preferred element, say 0 and 1. Then, we can assign the numerical value p to the preference of a good G, where the agent equally prefers G and a lottery where they win the most preferred element with probability p and the least preferred element with probability 1-p. We can therefore think of p as being a numerical value for the preference for G. Note that the preference for G is a **linear combination** of the preferences for the least and most preferred goods[2].

More generally, a **utility function** that assigns numerical values to preferences over goods allows us to numerically compare the preference expressed by an agent among these goods. Denote the least preferred good by L and the most preferred by M, then we can define a utility function U by: $U(L) = 0$, $U(M) = 1$, and for all other goods G, $U(G) = p$, where the agent equally prefers G and a lottery amongst L and M with odds p. With a slight change of perspective, we can imagine that U assigns **utilities** to goods, such that higher utilities correspond to more preferred goods. This assumes, of course, that the agent's preferences are consistent[3].

Utilities are useful for modelling competing objectives in **multi-objective optimization**. Suppose we wish to optimize a system where there is an inherent conflict amongst the objectives. For example, in most systems, increased performance comes

2. For a mathematical definition of a linear combination, please refer to Section 3.3.1 on page 80.
3. In the real world, sadly, consistency of preferences rarely holds. We discuss this in more detail in Section 7.4 on page 220.

at increased cost. We desire both better performance and lower cost. These competing objectives can be modelled with a utility function that increases with performance and decreases with cost (where the cost, itself, may depend on the performance). This naturally models the preferences we have over the goods of performance and cost. By maximizing this utility function, we can find the choice of system parameters that makes the most desirable trade-off between performance and cost.

EXAMPLE 2: UTILITY FUNCTION

Consider an ISP that charges $1.25/GB for transfers that exceed a monthly quota and a customer of this ISP who would like to transfer unlimited amounts of data, but also would not like to pay a large monthly bill. We can model the customer's preferences using a utility function as follows. The customer prefers more data transfer to less data transfer and smaller monthly payments to larger monthly payments. Let $d(x)$ denote the utility of x GB of transfer, where d is an increasing function of x, and let $p(y)$ denote the (dis-) utility of y dollars of payment, where p is an increasing function of y. Of course, y itself is a function of x, so we can write it as $y(x)$. The overall utility, U increases with d and decreases with p, modelling the conflict in the underlying objectives.

A typical form assumed for U is the linear function $U = ad(x) - p(y(x))$, where a is a tuning parameter that is chosen to balance the relative utilities of data transfer and money. This is true for all values of U. So, by setting U to 0, we find that $a = p(y(x))/d(x)$. That is, we can determine a by finding the amount of data transfer at which the cost of transfer just balances the gain. Of course, U can be far more complicated. Note also that U is linear function of d and p: this is another sense in which U can be linear.

Linear utility functions are rather unrealistic: most people experience diminishing returns with increases in the quantity of a good, which is better modelled by a non-linear curve of the form $1-e^{-x}$, where x denotes the quantity of the good.

■

Before we leave the topic of utility functions, we remark on two important properties. First, **utilities are unique only up to an affine transform**. Utility functions only establish preference *orderings* so the utility functions U_1 and aU_1+b, where a and b are real constants, are identical. A consequence is that utility functions are **personal** and the utility functions of two individuals are incomparable in a deep sense. So, if an agent were to assign a utility of 15 to some good, and another were to assign a utility of 7 to the same good, it does *not* mean that the first agent prefers it more. The second agent could easily have assigned the good a utility of 5000, for example, with an affine translation. Therefore, any scheme that assumes the ability to perform interpersonal utility comparison is simply wrong. Unfortunately, the game theory literature has many examples of published papers that make this fundamental error.

7.1.2 Terminology

We now examine some elementary concepts and terms used in game theory.

We model the interaction of decision-makers in the form of a **game**. Each decision-maker or **player** takes **actions** chosen from a finite or infinite set in response to prior actions by other players, as well as expectations about how their action will 'play out' in the other player's minds. These actions may be **simultaneous** or **sequential**.

When the game ends, that is, all the actions are carried out, each player is given a reward or **outcome** that depends on their actions. We assume that each player has a utility function that establishes a preference ordering among the outcomes: the utility of an outcome is called the **payoff**.

Players are assumed to be **rational**, that is, they make actions that will assure them the best possible payoff. The **Principle of Maximum Expected Utility** states that a rational player takes actions that maximize its expected utility from the game. Moreover, player rationality is assumed to be **common knowledge**: each player knows that every other player is rational, and this fact itself is known by everyone, with infinite recursion. (This is perhaps one of the most controversial assumptions made by game theory.)

This chapter assumes that players are **non-cooperative**, that is, they do not collude or form coalitions. Note that non-cooperation is not the same as adversarial or malicious behaviour. The assumption of non-cooperation holds true for game-theoretic models commonly used in computer networking problems: it is relaxed when studying cooperative game theory, a topic that we will, however, not touch upon.

A critical factor in distinguishing between different types of games is the degree of information each player has about other players and about the game. In some games, players may be able to observe every action and may also precisely know what every player's possible actions and objectives are (these are games of **perfect information**). A canonical example is the game of chess, where all possible actions of each player are codified by the rules of chess, and all actions are visible to the players. In other games, players may not be able to see other players' actions, not know other players' objectives, and, in some cases, may not even know how many other players are playing the game. For example, in the game of bridge, each player's hand is private, so a player cannot know the set of potential next actions of the other players. These limitations, naturally, limit the degree to which we can model the games and predict their outcomes. In some cases, such as when a player does not know the initial state of the game (for instance, any game of cards where the deck has been shuffled so no player knows what the other players' hands contain), we can think of there being a special player called 'Nature' that makes a randomizing initial move, after which the players play the actual game.

7.1.3 Strategies

A player's actions typically depend not only on prior actions of the other players (the **history** of the game) but also expectations on the responses this action may elicit from the other players. A player's **strategy** is a full description of the player's actions during the game. A strategy should be detailed enough that a player can give it to a disinterested third party and walk away (or give it to a computer for execution). It describes the player's actions taking into account every possible action by every other player. Clearly, a full description of a strategy is impossible other than for the simplest games. Nevertheless, the concept of a strategy is both useful (in that it allows us to precisely model a player's actions) and necessary (to determine the expected outcome of a game, which is expressed in the form of each player's preferred strategy). In every game-theoretic analysis, a critical modelling step is to enumerate the set of strategies available to each player.

There are two types of strategies. **Pure strategies** deterministically describe which actions a player makes at each stage of the game. In contrast, **mixed strategies** associate probabilities with two or more pure strategies. The actual strategy that is played is chosen according to these probabilities. In this sense, a pure strategy is a mixed strategy with the entire probability mass at a single point in the domain. Note that with a mixed strategy, once a specific pure strategy is chosen, the player cannot introduce any additional randomness--every action must be made deterministically in accordance with the pure strategy. In a repeated game, where a game is played repeatedly, a different probabilistic choice can be made for each underlying game instance.

EXAMPLE 3: PURE AND MIXED STRATEGIES

Consider the simple game of **Matching Pennies**. In this game, two players each have a coin that they simultaneously place on a table. Subsequently, each observes both coins. One player wins if the coins are both heads or both tails and the other player wins otherwise. A player has two pure strategies: play heads or play tails. However, we can also define an infinite number of mixed strategies, parametrized by a real number $p \in [0, 1]$, each of which corresponds to the strategy: play heads with probability p and tails with probability 1-p. We will show later that the optimal strategy for both players is to choose p = 0.5.

■

We denote the strategy adopted by player i by s_i. Thus, we can denote an entire game played by n players by the tuple of adopted strategies $(s_1, s_2,...,s_n)$, called the **strategy profile**. The payoff to player i as a result of the game is denoted $\pi_i(s_1, s_2,...,s_n)$ and represents the utility of player i when the game is played with that particular strategy profile.

EXAMPLE 4: STRATEGY PROFILE

For the game of Matching Pennies, suppose we denote the action 'play head' by H and the action 'play tail' by T. Then, the set of possible pure strategies is {HH, HT, TH, TT}, where each element is a strategy profile. Suppose that for both players, winning has utility 1 and losing has utility -1. Also, let player 1 win if the coins match and player 2 win otherwise. Then, we can write down the payoffs as:

$\pi_1(HH) = 1$, $\pi_1(HT) = -1$, $\pi_1(TH) = -1$, $\pi_1(TT) = 1$;

$\pi_2(HH) = -1$, $\pi_2(HT) = 1$, $\pi_2(TH) = 1$, $\pi_2(TT) = -1$;

In this example, the utility of player 2, for each strategy, is exactly the opposite of player 1. In other words, player 1 wins if and only if player 2 loses. Such a game is called a **zero-sum** game.

■

7.1.4 Normal- and extensive-form games

Games can be represented in two standard forms: **normal form** and **extensive form**. In normal form, a game with n players is represented by an n-dimensional array, where each dimension corresponds to the possible pure strategies of each player and each matrix element is an n-tuple that corresponds to the outcome for each player; the ith element of each tuple is the payoff (utility) to the ith player. This array is also called the **payoff matrix**. Note that all the strategies are assumed to be played simultaneously.

EXAMPLE 5: NORMAL FORM: MATCHING PENNIES

Here is the Matching Pennies game in normal form, where player 1's pure strategies are along the rows, and player 2's pure strategies are along the columns:

	H	**T**
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

TABLE 1. **The payoff matrix for Matching Pennies**

■

EXAMPLE 6: NORMAL FORM: WIFI GAME

In the 802.11 (WiFi) protocol, each station with data to send contends for airtime. For simplicity, assume that time is slotted and that each packet transmission takes exactly one time slot. If a station does not send data in a slot, airtime is wasted. If only one station sends data, it succeeds. If both send, both fail, and both must try again.

Consider an 802.11 wireless LAN with two active stations. We can model this as a game with two players. Consider actions over a single time slot. For each player, the possible actions are 'send' (S) and 'don't send' (D). We need to model the payoffs to each player as well. We assume that a station prefers success to a wasted time slot, and prefers a wasted time slot to a collision, because on a collision, not only is no progress achieved, but energy is also wasted. We can express these preferences by assigning a utility of -1 to a collision, 0 to a wasted time slot, and 1 to success. This allows us to represent the game in normal form as shown below:

	S	**D**
S	(-1,-1)	(1,0)
D	(0,1)	(0,0)

TABLE 2. **The payoff matrix for the WiFi game.**

■

There is a quirk with representing games in normal form that the following example demonstrates. Consider a two-person game where the row player can play pure strategies A or B, and if it plays A, it wins and the game is over. On the other hand, if it plays B, the column player can play Y or N, and the row player can then play C or D. What is the row player's strategy space? By convention, it is not {A, BC, BD}, but {AC, AD, BC, BD}. That is, even though the game ends after the row player plays A, we represent the alternatives AC and AD explicitly in the normal form.

In extensive form, the game is represented by a **game tree** where each node corresponds to the player whose turn it is to move and each departing edge represents a possible action that can be taken at that stage of the game. We identify each node in the tree with the past history of the game. Leaves correspond to outcomes of the game and are associated with the payoff to the players for that outcome. Games with sequential actions by the players are more naturally represented in extensive form.

EXAMPLE 7: EXTENSIVE FORM: THE PRICING GAME

Suppose an Internet Service Provider wants to roll out a new service with a price that could be low (L), medium (M) or high (H). Suppose the prices are 1, 2, and 3 respectively and the utility that the customer derives from the service is *a*. For each price, the customer can say yes (Y) or no (N). If the customer says yes, the ISP gets a payoff corresponding to the price and the customer gets the payoff *a - price* for some constant value *a*. If the customer says no, the payoff to both parties is 0. In extensive form, we represent it as shown in Figure 1.

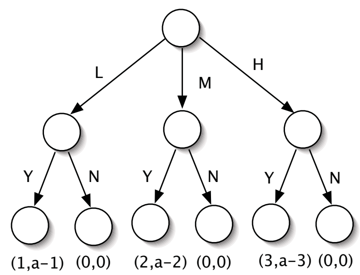


FIGURE 1. **Extensive form of the pricing game**

■

The extensive-form representation has an inherent problem with representing simultaneity. Consider a game where two players make simultaneous actions, such as Matching Pennies. If we arbitrarily choose nodes corresponding to player 1's actions as the first level of nodes, then, when player 2 takes an action, it can see what player 1 did. This, of course, makes the game pointless. Making the alternative choice does not remedy the situation. What we need is a way to hide a player's actions from the other player, that is, make the information available to it less than 'perfect.' We can do so by placing a subset of nodes in the game tree in the same **information set**. From the perspective of a player, all nodes in the set are equivalent (the player cannot distinguish between them). Graphically, we draw a dashed line between equivalent nodes. This allows us to easily represent simultaneous actions, as the next example shows.

EXAMPLE 8: REPRESENTING SIMULTANEOUS ACTIONS IN EXTENSIVE FORM

Figure 2 shows the extensive-form representation of the Matching Pennies game. Note that Player 2 cannot determine whether Player 1 played H or T.

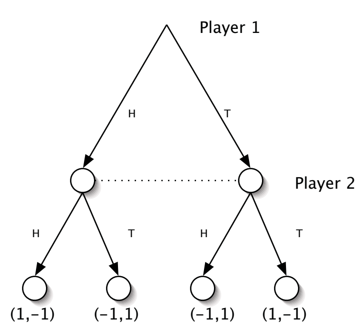


FIGURE 2. **Representing simultaneity in an extensive-form game**

■

It can be shown that, with the use of information sets, the normal and extensive form representations are equivalent. Therefore, from now on, we will not distinguish between the two.

Both normal and extensive forms grow exponentially in size with the number of actions and number of players. If there is only limited interaction between the players, we can use a compact **graphical game** representation instead.

A graphical game with *n* players is represented by an undirected graph with *n* vertices and a set of *n* payoff matrices, with one vertex and one matrix corresponding to each player. If node *i* has neighbours *j* and *k*, then player *i*'s payoffs are only a function of *i, j*, and *k*'s actions, and therefore the payoff matrix at node *i* only has entries for *i, j*, and *k*'s actions. The overall game is therefore composed from the individual sub-games. Note that we are not ruling out global impact of a player's actions. However, this occurs only through the propagation of local influences. This is similar in principle to Bayesian networks, where the joint probability distribution over all the underlying random variables is a product of the local conditional probability distributions at each node, and where the absence of edges between two vertices denotes independence. Kearns et al proposed this model in 2001 and it is rapidly gaining popularity because it can represent games that cannot be compactly represented in normal or extensive form. Note that a graphical game is equivalent to a normal form game: it is compact because it does not represent outcomes corresponding to simultaneous actions taken by non-interacting players.

7.1.5 Response and best response

Consider a two-player game with players labelled A and B. Recall that A's strategy s_A encapsulates this player's actions for the entire game. The strategy chosen by B *conditional on A playing* s_A is called B's **response** to s_A. Given the set of possible strategies that B could play, at least one of them will have a payoff as good as or higher than all the others. We call this B's **best response** to s_A.

We can generalize this to *n* players as follows. Let $(s_1, s_2,...,s_n)$ be a strategy profile. We denote by s_{-i} the tuple $(s_1, s_2,...,s_{i-1}, s_{i+1},...,s_n)$, that is, the strategy profile excluding *i*'s strategy. Then, the best response of player *i* to s_{-i} denoted s_i^* are the strategies (there may be more than one) that give *i* the highest possible payoff. That is:

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i^j, s_{-i}) \qquad \forall s_i^j \neq s_i^* \tag{EQ 123}$$

EXAMPLE 9: BEST RESPONSE

Consider the one-step WiFi game in Example 6. If the row player (station 1) plays S, then the column player's (station 2's) best response is D, because the payoff for SD is 0, and for SS is -1. On the other hand, if the row player plays D, then the best response is S, because the payoff for DS is 1, and for DD is 0.

■

7.1.6 Dominant and dominated strategy

In some games, it is possible to determine that a rational player should play a particular strategy no matter what the other players do. This is called a **dominant strategy** for that player. Mathematically, we say that a strategy s_i^* is a dominant strategy for player *i* if

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i^j, s_{-i}) \qquad \forall s_{-i}, \forall s_i^j \neq s_i^* \tag{EQ 1}$$

Compared to the best response, there is an additional universal quantifier over all s_{-i}, which indicates that the dominant strategy is the best response no matter what strategies the other players pick. If the inequality is strict, then the strategy is **strongly dominant**, else, if it is strict for at least one s_{-i} but not for all of them, it is **weakly dominant**.

Symmetrically, a strategy whose payoff is lower (strictly lower) than another strategy is weakly (strictly) dominated by the other strategy.

EXAMPLE 10: DOMINANT STRATEGY

In the WiFi game of Example 6, the column player's best response is S or D, depending on what the row player does. Therefore, this game does not have a dominant strategy for the column player. Symmetrically, the game also does not have a dominant strategy for the row player.

Wireless networks can exhibit the *capture effect*, where a transmission from a station succeeds even if there are competing transmissions, because the signal strength from that station is so strong that it overpowers the competition. In this case, the payoff matrix is:

	S	**D**
S	(-1,1)	(1,0)
D	(0,1)	(0,0)

TABLE 3. Payoff matrix for WiFi game with capture effect.

Transmissions from the column player (station) always succeed due to the capture effect. In this case, the dominant strategy for the column player is S: no matter what the row player does, the column player is better off doing S than D.

■

7.1.7 Bayesian games

We stated earlier that a critical factor in any game is the amount of information available to each player. In a game with perfect information, each player knows the other players, their past history of actions, the actions available to the other players, and the utility to the other players from each outcome. Consider a card game which starts with the deck being shuffled and dealt. No player knows the other players' hands, so they cannot know their possible actions. This is, therefore, a game with **imperfect information**.

A **Bayesian** game is a form of game with imperfect information, where each player can be from a set of possible **types**. If all players know each others' types, then the game becomes one of perfect information. Therefore, all the uncertainty is encapsulated in the selection of player types. In a card game, each player's hand corresponds to that person's type. The **state** of the game is the set of player types for that game.

A Bayesian game starts with a move by Nature that results in each player being randomly assigned a type, according to some probability mass function. We can also view this as Nature selecting some state of the game. At this point, it is possible that all players receive a **signal** that may give information about each other. Specifically, the signal eliminates some possible states. Thus, the conditional probability that a player has a particular type, given a signal, is greater than the unconditional probability of that type.

In an extensive-form game, imperfectness can be represented by putting more than one node in an equivalence class, where a player is unable to distinguish between nodes in the same equivalence class (they are in the same information set). The effect of a signal is to potentially reduce the size of an equivalence class.

EXAMPLE 11: BAYESIAN WIFI GAME

Consider the 802.11 game of Example 6, where the row player does not know the column player's signal strength. If the column player's signal strength is low, then the capture effect is absent, and the payoff matrix is the one shown in Table 2 on page 201. Otherwise, with the capture effect, the payoff matrix is shown in Table 3 on page 204. We can model this as a Bayesian game where the column player has one of two types: 'strong signal' and 'weak signal'. In the first move by Nature, the column player is assigned one of the two types according to some probability distribution (say, with probability 0.1 the player is strong, and with probability 0.9 the player is weak).

To incorporate the notion of a signal, consider that the row player can measure the received signal strength of the column player's transmission. Suppose that, given that the signal strength is high, the column player has 0.95 probability of being strong and 0.05 probability of being weak. Similarly, assume that the signal strength is low, the column player has a 0.1 probability of being strong and 0.9 probability of being weak. We can see that the signal allows the row player to better judge the type of the column player, potentially allowing it to improve its chances in the game.

■

7.1.8 Repeated games

Nearly any game can be repeated. Each repetition of a game is called a **stage**. With a finite number of repetitions, the overall game can be thought of as a single game with a much larger strategy profile space because each player can change strategies at each repetition, perhaps taking into account the outcome of the previous stage. Analysis of finite repeated games is thus more complex and does not necessarily give much more insight into the problem.

Paradoxically, the analysis of repeated games is somewhat simpler when games are repeated an infinite number of times. The reason is that in such a case every stage is equivalent: we do not have a 'final' stage that needs to be specially handled. In a game with infinite repetitions, both the normal and extensive forms are undefined. The payoff matrix in the normal form is infinite-dimensional, which means that we cannot represent the payoffs. Similarly, in the extensive form, there are no leaves, so we cannot assign payoffs to leaves, as we normally do. Moreover, if the player were to get a small positive payoff with two strategies in each stage, then *both* of them will result in infinite payoffs with infinite stages, so that they are equivalent. This doesn't make intuitive sense! To get around the problem, with infinitely repeated games, we represent the payoff of a (potentially infinitely long) strategy as the average payoff per stage, assuming the limit exists. Alternatively, we can **discount** future expected payoffs at each stage by a factor b that lies in (0,1). That is, the payoff at the ith stage is multiplied by b^i. This often results in a payoff that is finitely bounded.

EXAMPLE 12: REPEATED GAME

Consider the two-stage version of the WiFi game in Example 6. With one stage, there are only two strategies for each player; that is, S and D, and four strategy profiles SS, SD, DS, and DD. With two stages, they have four strategies each (SS, SD, DS, and DD) and therefore the normal form payoff matrix has 16 entries. The number of strategies for each player grows as s^i, where s is the number of strategies available to each player, and i is the number of stages. The payoff matrix has s^{2i} entries. Thus, repeated games are cumbersome to represent even with a few stages.

The infinitely-repeated WiFi game can be similarly defined, with each player's strategy being an infinite string chosen from the alphabet {S, D}. Assume that the discount factor is 0.8. Then, if the expected payoffs for the row player are r_0, r_1, r_2,...

then the discounted net payoff for that player is $\sum_{i=0}^{\infty} r_i 0.8^i$.

■

There are many possible strategies for infinite-repeated games. A simple one is to always play the same pure strategy. A more interesting strategy is **tit-for-tat**, where each player plays what the other player played in the previous stage.

7.2 Solving a game

7.2.1 Solution concept and equilibrium

Intuitively, the **solution** of a game is the set of strategies we expect rational players to adopt, given the payoff matrix. The solution of a game is also called its **equilibrium** point. The solution of a game *need not* be the strategy profile whose payoff to each player is greater than the payoff from any other strategy profile: as we will see shortly, rational players may sometimes be forced to play a strategy that actually gives them less payoff than some other strategy does! In common parlance, this is a 'lose-lose' situation.

The **solution concept** is the line of reasoning adopted in determining a solution or equilibrium. For the same game, different solution concepts can yield different equilibria. Game theorists usually make the simplifying assumption that all the players implicitly agree on using the same solution concept. If different players use different solution concepts the outcome is unpredictable.

7.2.2 Dominant strategy equilibria

The dominant strategy solution concept is used in games where each player has a strongly or weakly dominant strategy. The idea is simple: if each player has a strategy it should play irrespective of actions by other players, then it is reasonable to assume that the players will play this strategy. Note that the dominant strategy for a player may be a mixed strategy, i.e., a random variable defined over the domain of the pure strategies. When such an equilibrium exists, it is the preferred solution concept, because it makes the fewest demands on assumptions of player rationality.

EXAMPLE 13: DOMINANT STRATEGY SOLUTION FOR WIFI WITH CAPTURE

It is easy to solve a game when there is a dominant strategy for one of the players. Consider the capture effect WiFi game of Example 10. We saw that the dominant strategy for the column player was S. This is known both to the column player and to the row player. Therefore, the row player can assume that, as a rational player, the column player will play S. Given this, the row player's best response is D. Therefore, the solution to the game is DS, i.e., row player plays D and the column player plays S.

■

EXAMPLE 14: DOMINANT STRATEGY SOLUTION FOR MATCHING PENNIES

Consider the game of Matching Pennies introduced in Example 3. We stated without proof that the best strategy for both players was a mixed strategy that equally randomized between H and T. This is also the dominant strategy solution of the game. Intuitively, this makes sense because any deviation of either player from a 50% choice of H or T gives the other player an edge that can allow them to win in the long run, assuming that the game is repeated sufficiently often. The only way to counteract this is to play randomly, with an equal chance of playing H or T.

As an interesting aside, the Matching Pennies game also models penalty kicks in a soccer game. If the kicker and the goalkeeper both go left (L) or right (R), then the goalkeeper wins, otherwise the kicker wins. The best strategy for both, therefore, is to choose L or R with equal probability.

■

EXAMPLE 15: DOMINANT STRATEGY SOLUTION FOR DELAY-SENSITIVE WIFI

Consider the WiFi game with delay-sensitive stations, for whom the cost of waiting one slot (-2) is worse than the cost of a collision (-1).This game is represented below in the normal form:

	S	**D**
S	(-1,-1)	(1,-2)
D	(-2,1)	(-2,-2)

TABLE 4. **The payoff matrix for the WiFi game with delay-sensitive stations**

Note that for the row player, strategy S always returns higher payoffs than strategy D, no matter whether the column player plays S or D. So, S is a dominant strategy for this player. Symmetrically, S is also the dominant strategy for the column player. Therefore, the dominant strategy solution for this game is SS. This is a bad outcome because no progress will be made due to repeated collisions. System designers must be careful that their systems do not admit such unproductive outcomes. It is also interesting to note that a slight change in the payoff matrix completely the changes the outcome of the game!

■

EXAMPLE 16: DOMINANT STRATEGY SOLUTION FOR PRISONER'S DILEMMA

In the game known as the **Prisoner's Dilemma**, two prisoners in two isolated cells are offered a bargain by the warden. If they inform on the other prisoner (defect or D), they are set free and the other prisoner is given four more years of prison. If neither informs on the other (showing solidarity or S), they only get a year of prison each. If both defect, they both get three years of prison each. What should each prisoner do?

Solution:

The the payoff matrix as shown below:

	S	D
S	(-1,-1)	(-4,0)
D	(0,-4)	(-3,-3)

TABLE 5. The payoff matrix for the Prisoner's Dilemma game

Note that the dominant strategy for the row player is to play D (because 0 > -1 and -3 > -4). By symmetry, D is also the dominant strategy for the column player. Therefore, the dominant strategy equilibrium is DD. This game is called a 'dilemma' because the best possible outcome is SS. Nevertheless, the inexorable logic of game theory dictates that both prisoners will defect, a 'lose-lose' situation. We will see later that, using the solution concept of correlated equilibria, this sad outcome can be averted (as long as both prisoners agree to change their solution concept from dominant strategy to correlated equilibria).

■

7.2.3 Iterated removal of dominated strategies

In some games, not all players have dominant strategies, so it is not possible to find a dominant-strategy equilibrium. However, even if only one player has one or more dominant strategies (strategies whose payoffs are the same, and whose payoffs are greater than the player's other strategies), it may be possible to find a plausible equilibrium by deletion of *dominated* strategies. Specifically, if player *i* has a set of dominant strategies say $\{s_i^*\}$, then the other players can reason that player *i* will certainly play one of these strategies. Therefore, all of their own strategies incompatible with this set can be eliminated, which may then yield a dominant strategy set for one of the other players, say *j*. This, in turn, allows us to remove the dominated strategies for *j* and so on. In the end, we are hopefully left with a single strategy for each player, which is the equilibrium.

EXAMPLE 17: ITERATED REMOVAL OF DOMINANT STRATEGIES

Consider the WiFi game with capture (Table 3 on page 204). Recall that here the column player has a dominant strategy S. The row player can reason as follows: "Column will certainly play S. If Column plays S and I play S, I get -1 and if I play D, I get 0, so I should play D." Thus, with this solution concept, we can find the equilibrium DS.

■

As with dominant strategies, not all games are guaranteed to have a dominant strategy equilibrium even with the iterated removal of dominated strategies. Also, applying this concept is tricky, because a pure strategy may be dominated by a mixture of other pure strategies, although none of the strategies in the mixture dominate the pure strategy (but none of the strategies in the mixture can be dominated by the pure strategy, either).

7.2.4 Maximin equilibrium

The **maximin equilibrium** and its dual, the **minimax equilibrium**, are amongst the earliest and best-studied solution concepts. They both arise from the concept of **security level**, which is the guaranteed payoff to a particular player even if the other players coordinate their actions to minimize that player's payoff. The idea is for each player to choose a strategy that maximizes this guaranteed payoff, thus maximizing its security level. It is a pessimistic or perhaps evan a paranoid way to play a game, in the sense that it does not even assume the opponents are rational (they may even be willing to reduce their own payoff in order to harm other players). Nevertheless, no matter what the other players do, the maximin payoff is guaranteed.

Mathematically, define $s_{-i}^{\min}(s_i)$ to be the tuple of other player strategies that minimize the payoff to i when playing s_i. Then, the maximin strategy for i is s_i^* where $\pi(s_{-i}^{\min}(s_i^*)) > \pi(s_{-i}^{\min}(s_i^j)) \ \forall s_i \neq s_i^*$. Note that the maximin strategy for a player can be pure or mixed.

EXAMPLE 18: MAXIMIN STRATEGY

Consider the WiFi game of Example 6. If Row plays S, then the Column player can play S to give it a payoff of -1, but if Row plays D, the worst payoff is only 0. So, the maximin strategy for Row is D. Similarly, the maximin strategy for Column is also D. Therefore, the maximin equilibrium for this game is DD. Note that there is no dominant strategy equilibrium for this game.

■

EXAMPLE 19: GRAPHICAL REPRESENTATION OF A MAXIMIN STRATEGY

Consider the following payoff matrix for a two-player zero-sum game:

	C1	**C2**
R1	(3,-3)	(1,-1)
R2	(2,-2)	(4,-4)

TABLE 6. **A zero-sum game**

We would like to compute the maximin strategy for both players. In general, this strategy is a mixed strategy, so assume that Row plays strategy R1with probability p and R2 with probability (1-p). If Column plays C1, then Row gets a payoff of 3p + 2 (1-p) = p+2. If Column plays C2, then Row gets p + 4(1-p) = 4-3p. This is shown graphically in Figure 3, where the X axis is the choice of p and the Y axis is Row's payoff. It can be shown that if Row uses any value of p other than 0.5, then it may obtain a payoff lower than 2.5 (see Exercise 14). It can also be shown that even if the column player plays a mixed strategy, Row can guarantee itself a payoff of 2.5 by playing 0.5R1+0.5R2. Similarly, it can also be shown that the column player can hold the row player down to 2.5 by playing 0.75C1+0.25C2. If the column player plays any other strategy, the row player can obtain a greater payoff (and, because this is a zero sum game, the column player will get a lower payoff). Therefore, the maximin strategy for Column is 0.75C1+0.25C2 and the mixed strategy profile (0.5R1+0.5R2, 0.75C1+0.25C2) is the maximin equilibrium for this game.

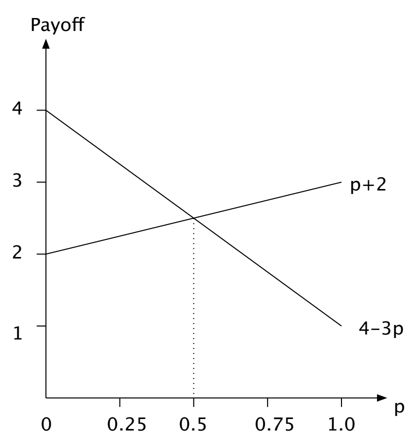


FIGURE 3. **Maximin strategy for the Row player in Example 19.**

■

In a two-person game, the equilibrium point is also called a **saddle** point. Any deviation from this point by the Row or Column player will decrease its guaranteed minimum payoff (though they move away from this point in orthogonal dimensions).

The dual of the maximin strategy is the minimax strategy. In the two-player version, a player acts so as to minimize the best possible payoff that the other player receives. In this sense, the player acts to maximally punish the other player without regard to their own payoff. The *n*-player version of this solution concept is somewhat more tricky to state and requires the players to act as if they form a coalition to punish a given player. This solution concept is therefore of only theoretical interest, in that it is the dual of the maximin strategy, so we will not consider it any further.

One of the earliest theorems in game theory was stated and proved by von Neumann in 1928 and is called the **Minimax theorem**. It states that in every finite, two-person, zero-sum game, player 1 is guaranteed a payoff of at least v independent of player 2's strategy, and, symmetrically, player 2 can restrict player 1 to a value of at most v, independent of player 1's strategy. This value may require players to play mixed strategies. The strategy corresponding to this guaranteed payoff is the maximin strategy. Because player 1 is guaranteed to get at least v, and player 2, being an opponent in a zero-sum game, and thus receiving a payoff of $-v$, would never want player 1 to get any payoff higher than v, this is an equilibrium. Thus, another way to view this theorem is that it asserts that every finite, two-person, zero-sum game has a equilibrium that results from both players playing the maximin strategy.

7.2.5 Nash equilibria

Although dominant strategy equilibria are preferred, many games do not allow such a solution. For instance, there is no dominant strategy solution for the Matching Pennies game or the WiFi game of Example 6. In such games, we turn to a different solution concept, called the **Nash equilibrium** concept.

The key idea in a Nash equilibrium is that players have no incentive to deviate from it. That is, *assuming every other player* is playing the strategy corresponding to the Nash equilibrium strategy profile, no player's payoffs are better by choosing any other strategy. Mathematically, a strategy profile $(s^*_1, s^*_2, ..., s^*_n)$ is a (weak[4]) Nash equilibrium if:

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*) \qquad \forall s_i \neq s_i^* \tag{EQ 2}$$

In a Nash equilibrium, each player plays its best response assuming all other players play their Nash strategy. This exposes both the power and weakness of the concept. The concept is powerful because it intuitively matches our expectations of a rational player who plays the best response to the other players' actions. Moreover, we only need to identify a potential Nash strategy profile (it doesn't matter how) and check whether any player has an incentive to deviate, which is straightforward. However, a Nash equilibrium pre-supposes that all players are going to play according to this solution concept. Worse, it is not unique: a game may have more than one Nash equilibrium. In this case, players need to either guess which Nash equilibrium will be chosen by the others, or coordinate their actions using an external signal.

Every dominant strategy equilibrium, by definition, is also a Nash equilibrium (though the converse is not true). For instance, in Prisoner's Dilemma, DD is both a dominant strategy and a Nash equilibrium. On the other hand, SS is not a Nash equilibrium because if a player assumes that the other player is going to play S, it pays for it to defect. Similarly, every maximin equilibrium is also a Nash equilibrium.

Note that a Nash equilibrium may involve mixed strategies, as the next example shows.

EXAMPLE 20: NASH EQUILIBRIUM FOR MATCHING PENNIES

4. The equilibrium is strict if the inequality is strict.

Recall the Matching Pennies game (shown in Table 1 on page 201). We will prove that the Nash equilibrium is for both players to play a mixed strategy with the probability of H (or T) = 0.5 (represented as 0.5H + 0.5T).

Consider the situation for the row player. Assume that the column player plays 0.5H + 0.5T. Let the row player play pH + (1-p)T. Row plays H with probability p. We have two cases: Column plays H or Column plays T. (a) Column plays H with probability 0.5, giving Row a payoff of 1, so that the expected payoff for Row in this case is p. (b) Column plays T with probability 0.5, giving Row a payoff of -1, so that the expected payoff for Row in this case is -p. Thus, the expected payoff when Row plays H is 0. Arguing along the same lines, the expected payoff for Row when it plays T is also 0. So, we have the interesting situation that, if Column plays 0.5H + 0.5 T, then no matter what Row does, its expected payoff is 0. By symmetry, if Row plays 0.5H+0.5T, then the expected utility of Column is 0, no matter what it does. Therefore, Equation 2 holds (albeit with equality, rather than inequality) and this is therefore a (weak) Nash equilibrium.

■

EXAMPLE 21: FINDING NASH EQUILIBRIA: BATTLE OF THE SEXES

Consider the following **coordination game**, popularly called the 'Battle of the Sexes.' A couple want to choose between going to a prizefight F (which the row player wants) and going to a ballet B (which the column player wants)[5]. Both would rather be with each other than by themselves.

	F	**B**
F	(2,1)	(0,0)
B	(0,0)	(1,2)

TABLE 7. The payoff matrix for the 'Battle of the Sexes' coordination game

It is obvious that the two pure-strategy Nash equilibria are FF and BB. There is, however, also a mixed-strategy Nash equilibrium where Row plays 2/3F + 1/3 B and Column plays 1/3F + 2/3B. To see this, assume that Column plays 1/3F + 2/3B and Row plays pF + (1-p)B. Then, Row's expected payoff is p(1/3*2) + (1-p)(2/3*1) = 2p/3 + 2/3 -2p/3 = 2/3, independent of p. Symmetrically, it can be shown that when Row plays 2/3F + 1/3B, Column always gets 2/3 independent of its strategy. Therefore, Equation 2 holds with equality, and this is a weak Nash equilibrium. It is worth noting that this mixed equilibrium gives a lower payoff than either of the pure equilibria.

■

In 1951, J. Nash proved a famous theorem that earned him the Nobel prize in economics. It states that every game with a finite number of players and actions has at least one Nash equilibrium. Thus, we can always find at least one Nash equilibrium for finite normal-form games. The Nash theorem, however, only proves existence of an equilibrium (by relying on a fixed point property of iterated maps described by the Brouwer fixed-point theorem) rather than telling us how to find this equilibrium. Therefore, in practical cases, finding the actual equilibrium can be challenging. In doing so, however, it is useful to rely on the following fact: if a mixed strategy uses strategy s, then s cannot be strongly dominated by any other strategy (otherwise, you could remove this strategy from the mixture and increase the payoff).

So far, we have considered Nash equilibria in games with discrete actions. However, a player's actions may be chosen from a subset of the real line. Finding Nash equilibria in such games is essentially the same as finding *local* maxima of a vector-valued function of a vector of real variables. More precisely, consider the mapping G from a vector $\mathbf{x}$ in $\mathbf{R}^n$, corresponding to the action chosen by each player, to the payoff vector π which determines the corresponding payoff to each player. We say that a payoff vector π^i dominates payoff vector π^j if every element of π^i is greater than or equal to the corresponding members of π^j. At each local maximum $\mathbf{x}^m$ of G, $\pi^m = G(\mathbf{x}^m)$ dominates $\pi_j = G(\mathbf{x}^j)$ for all $\mathbf{x}^j$ in the neighbourhood of $\mathbf{x}^m$. We can use any standard optimization technique, even a heuristic such as hill climbing, to find these local maxima. If G is globally convex, that is, there is a single global maximum, then *any* hill climbing technique will find the global maximum. In many

5. This being the twenty-first century, we'll leave the sex of the row and column players unspecified.

papers on game theory, G is assumed to be convex, so that this approach can be used. It is an contentions point, however, whether this convexity assumption holds in the real world.

7.2.6 Correlated equilibria

A fundamental problem with the Nash equilibrium concept arises when a game has more than one Nash equilibrium, when each player has to somehow guess which of the game's equilibria the others pick. Even if there is a unique equilibrium, each player has to know exactly which strategy every other player is going to play. If the other players are playing mixed strategies, each player also has to know exactly how every other player is going to mix each of their pure strategies. This isn't very realistic!

A hint to a possible solution lies in the formulation of a Bayesian game, where a move by Nature decides player types and each player has the same subjective probability distribution over the types chosen by Nature. (Once the player types are chosen, however, the rest of the game is a standard game and is solved using a standard solution concept.) The key idea of a correlated equilibrium is to extend the subjective probability distribution of each player not just to player types but also to player *strategies*, given a *shared* view on the current state of the world. That is, each player has a subjective probability distribution over possible strategies of the other players, conditional on a random variable called the 'state of the world' and tries to maximize their own payoff conditional on this distribution. In the original formulation of this solution concept, the shared common view on the state of the world was expressed in the form of an external, globally-trusted, correlating agent who tells each player what to do. Such an agent is not really necessary. Nevertheless, it is easier to discuss a correlated equilibrium assuming such an agent, and we shall do so as well.

More precisely, we assume the existence of an external agent that, based on some probably distribution, chooses a strategy profile for the game. Each player is assumed to know the probability distribution used to choose the strategy profile, but only its own pure strategy corresponding to this strategy profile. The external agent does not tell the player, however, what it told the other agents (this models the fact that each player has their own subjective probability distribution on their pure strategies as a function of the state of the world). The player then plays this strategy. We say that the resulting strategy profile is a **correlated equilibrium** if the players do not have any incentive to deviate from this strategy.

EXAMPLE 22: SIMPLE CORRELATED EQUILIBRIA

Recall from Example 15 that in the Prisoner's Dilemma (Table 5 on page 207) both the Dominant Strategy and Nash equilibrium is DD, with payoff (-3,-3). Suppose we introduce correlation through a coin toss. If the coin lands heads up, then the players are told to play DS, and if tails up, they are told to play SD. Note that when told to play S, Row would gain by playing D instead! So, it has an incentive to deviate, and this is *not* a correlated equilibrium.

In contrast, consider the Battle of the Sexes game (Table 7 on page 210). Suppose an outside agent told each player to play FF or BB based on the results of a coin toss. Consider the Row player's point of view. If it is told to play F, then it does pay for it to deviate and play B. Symmetrically, neither does Column gain from deviation. Therefore, this is a correlated equilibrium. If the coin is fair, both can achieve an expected gain of 0.5*2 + 0.5*1 = 1.5, which is more than they can gain from a mixed Nash strategy, which gives them only 2/3 each.

■

Correlated equilibria can be complex if the correlating device does not allow a player to determine exactly what the other players would do, as was the case with the Battle of the Sexes. This is illustrated by the following game.

EXAMPLE 23: CORRELATED EQUILIBRIUM FOR 'CHICKEN'

The game below models the game of 'chicken' where two racers rush towards each other at full speed. The first person to pull away is the chicken and loses. Of course, if neither pulls away, both lose. The game matrix is given below, where D stands for 'dare' and C for 'chicken.'

	D	**C**
D	(0,0)	(7,2)
C	(2,7)	(6,6)

TABLE 8. Payoff matrix for the Chicken game

We assume the existence of an external agent, and we assume that both players know that the external agent says CC with probability 1/3, DC with probability 1/3, and CD with probability 1/3 (and never says DD). Suppose Row is told to play D. Then, it knows that Column was asked to play C. Row goes from 7 to 6 by deviating, so it will not deviate. Suppose Row is told to play C. Then, there is probability 1/2 that the other player was told to play D and probability 1/2 the other player was told to play C. Assuming the other player does not deviate from the correlated equilibrium, the expected utility of deviating and playing D is 0(1/2) + 7(1/2) = 3.5 and the expected utility of listening to the agent and playing C is 2(1/2) + 6(1/2) = 4. So, the player would prefer to play C, i.e. not deviate. From the symmetry of the game, the same argument holds for the Column player. Therefore, neither player will deviate from the suggestion of the agent, and this is a correlated equilibrium.

■

Note that in a correlated equilibrium, we have a free variable, which is the probability with which the advisor asks each player to play each pure strategy (from which none of the players will deviate). By choosing different values for this distribution, we can achieve a range of payoffs to each player.

The correlated equilibrium concept is more appealing than a Nash equilibrium, in that it does not require players to know the exact strategy for every player, just that they will condition on the same 'state of the world.' Moreover, every Nash equilibrium (whether using pure or mixed strategies) can be shown to be a correlated equilibrium that only advises pure strategies. Therefore, correlated equilibria are the more powerful solution concept, especially if an external correlating agent can be naturally found in the problem domain.

7.2.7 Other solution concepts

Our treatment of solution concepts is necessarily limited. Several concepts such as rationalizability, sub-game perfectness, trembling-hand perfectness[6], ε-Nash equilibria, and evolutionary stability have been studied in the literature. More detail on these can be found in a standard texts on game theory (see Section 7.5 on page 221)

7.3 Mechanism design

Traditionally, game theory studies the behaviour of players when the game has already been specified. In contrast, **mechanism design** sets up the rules of a game such that rational utility maximizing players, in equilibrium, will behave as the designers intended. The key idea is to choose the rules of the game so that each player's attempt to maximize their utility also achieves the desired outcome *without* the designer knowing each player's utility function. Intuitively, the game is set up so that the players do what the designers want them to do because the players themselves want to do it!

7.3.1 Examples of practical mechanisms

Mechanism design arises naturally in some common situations. Consider the owner of a good who wants to sell it to the buyer who will pay the most for it, who is also the buyer who values the good most highly. This can be achieved by an **English auction**, where the good is simultaneously presented to all buyers and the price of the good is gradually raised until all but one buyer drops out. If we treat the bidders as players in a game, it is easy to see that in equilibrium, this auction mechanism ensures that a player's bid reflects their true valuation.

As another example, consider an election officer who wants to choose one of the candidates running for office as the winner. Presumably the winner should, in some way, reflect the wishes of 'society.' The electoral officer implements what is known, in the literature, as a **social choice function**. Again, we desire a mechanism such that, assuming that such a social choice exists in the first place, each voter reveals their true preferences (instead of voting 'strategically,' that is, voting other than for their choice in an attempt to influence the final outcome).

6. This intriguing term refers to a game where a player, due to a 'trembling hand' may occasionally make a sub-optimal move.

It turns out that similar considerations arise in several networking problems. For instance, if several users want to download a file using BitTorrent, how can we ensure that every user does their fair share in downloading and sharing the torrent? Similarly, if we have a group of mobile phone users sharing content using ad hoc phone-to-phone data propagation, how can we make sure that every user has an incentive to participate in the scheme despite using their scarce battery resources? The general area of mechanism design addresses these issues. Due to considerations of space, we will merely touch upon the main concepts.

7.3.2 Three negative results

In using mechanism design to achieve social choices, it is useful to keep three negative results in mind: the Condorcet Paradox, Arrow's theorem, and the Gibbard-Satterthwaite theorem

The **Condorcet Paradox** demonstrates even an election with just three voters and three candidates does not have a self-consistent majority or 'social' choice. Consider an election with three candidates *a*, *b*, and *c*. Let voter 1 prefer *a* to *b* and *b* to *c*. Let voter 2 prefer *b* to *c* and *c* to *a*. Finally, let voter 3 prefer *c* to *a*, and *a* to *b*. Now, note that two voters (1 and 3) prefer *a* to *b*. So, the majority prefer *a* to *b*, and in the social choice, it must certainly be the case the *a* is preferred to *b*. However, the voter preferences are rotationally symmetric, so a majority also prefers *b* to *c* and *c* to *a*. If preferences are transitive, which is necessary for consistency, then we find that a majority prefers *a* to *b* to *c* to *a*! Thus, in reflecting a society's choice of candidates, we have to give up either majority (i.e. have a dictatorship), or consistency, or transitivity. None of these are appealing choices. One may think that the problem here is that we are using a simple majority rule. What if this were replaced with something more sophisticated, such as proportional representation? Indeed, several schemes, called **voting methods**, have been proposed in the literature that are more sophisticated than majority and remedy some of its problems. Although they each have their strengths, they all run afoul of a fundamental theorem of social choice called Arrow's theorem.

Arrow's theorem states that under some very general conditions, we cannot reasonably compose individual *strict* orderings of alternatives (i.e., orderings where every alternative has a definite rank, though multiple alternatives may have the same rank) to form a global strict ordering. The theorem assumes that individual preferences are arbitrary: one individual may rank alternative 1, for example, as its most-preferred alternative, but another may rank the same alternative as its least-preferred alternative. Define a **social welfare function** as a function that maps from a set of individual strict orderings of a finite set of alternatives to a global ordering. We say that a social welfare function satisfies **unanimity** if it orders alternative *a* higher than *b* iff every individual ranks *a* higher than *b*. A social welfare function satisfies **independence of irrelevant alternatives** if the aggregate ranking of two alternatives *a* and *b* depends only on how each individual ranks *a* and *b*. That is, the other alternatives could be arbitrarily ranked, but as long as every individual ranks *a* higher than *b*, so should the social welfare function. Finally, we call a social welfare function a **dictatorship** if there is an individual *i* such that the social welfare function's choice of the top-ranked alternative is that individual's choice of the top-ranked alternative, no matter what the other individuals desire[7]. Arrow's theorem, for which he won the Nobel prize, states that every social welfare function over a set of more than two alternatives that satisfies unanimity and independence of irrelevant choices is a dictatorship! This is troublesome, in that we have to give up either unanimity or independence of irrelevant choices to avoid dictatorship.

The third minefield in the design of social choice functions is called the **Gibbard-Satterthwaite theorem** (which can also be formulated as an extension of Arrow's theorem). Recall that social choice functions decide on a choice of a specific alternative rather than an ordering on alternatives. We call a social choice function **strategically manipulable** by an individual if that individual can influence the outcome of the social choice function in their favour by misrepresenting their preferences (i.e., lying about their preference ordering). However, if a social choice function *cannot* be manipulated by an individual in this manner, it is called **incentive compatible**. We will now show that if we have an election with two candidates, standard majority voting is incentive compatible. To see this, note that a voter can manipulate the outcome if and only if there are an odd number of voters and this voter is casting the deciding vote. If the voter lies when they cast the deciding vote, they do not influence the election in their favour: the outcome is the opposite of what they want. Therefore, the outcome is non-manipulable, that is, incentive compatible. The Gibbard-Satterthwaite theorem states that if *f* is any incentive-compatible social choice function that decides between more then two alternatives, then *f* is a dictatorship, that is, there is some individual who can control the outcome of the election. The consequence of this theorem is that even the most sophisticated scheme for aggregating individual choices can either be manipulated or be dictated to! This is a strong negative result.

7. This is a weak form of dictatorship, in that if the other individuals were to change their preference orderings, the identity of the dictator could change. The idea is that any voting method that meets Arrow's criteria necessarily transfers the power to decide the social choice, i.e, cast the 'deciding vote,' one of the (perhaps unwitting) individuals.

There are several ways out of this quandary. One of them, which turns out the basis of most approaches to mechanism design, is to introduce the notion of money. Specifically, in Arrow's framework, the utility of each individual is expressed only through its preferred ordering, and, in fact, such a simplistic notion of utility is necessary for the Gibbard-Satterthwaite theorem. If, however, we assume that an individual's utility is *quasi-linear*, where the utility depends not only on the alternative selected, but also on an additional monetary side payment, then preferences for alternatives cannot be arbitrary and both Arrow's and the Gibbard-Satterthwaite theorem can be avoided. In the context of voting, what this means is that an individual whose choice did not affect the final outcome would be compensated by a certain amount of money. So, the greater the attempt by a voter to change other voters' choices, the more it will cost. Therefore, assuming that everyone has the same amount of money, we can avoid manipulations (or at least, buying elections will only be for the rich).

7.3.3 Two examples

To fix ideas, we first study two simple examples of mechanism design, where a **principal** designs a game so that the players or **agents** carry out the principal's wishes.

EXAMPLE 24: (PRICE DISCRIMINATION)

Suppose a manufacturer of a communication device called the uPhone (the principal) sells to one of two types of customers (the agents). Chatters (C) need only one uPhone, because most of their friends already have one, and Antediluvians (A) need at least two, one for each party making a call. What price should be set for them?

A naive solution would be to price the uPhone at, say, \$100, so that C pays \$100 and A pays \$200. Suppose the internal value that C ascribes to a uPhone is \$50, and the internal value that A ascribes to two uPhones is \$300. By pricing it at \$100, no C will buy it, and every A who buys it would have been willing to pay \$150 per uPhone, so that manufacturer is leaving money on the table. Can we do better?

If the manufacturer knew that the internal valuation of C for a uPhone was c and of A was a and if $2a > c$, it could price the uPhones as follows:

One uPhone costs c but two cost $\min(2a, 2c)$

This way, chatters would pay c, so manufacturer would not lose sales to them. Moreover, because $2a > c$, they are never tempted to get two when they don't need it. If $2a > 2c$, and the price of two uPhones were greater than $2c$, then Antediluvians would just buy two uPhones individually, so there would be no point in setting the price for two uPhones any higher than $2c$. On the other hand, if the price for two uPhone were more than $2a$, then no As would buy uPhones. So, the price for two should be the smaller of $2a$ and $2c$. This **discriminative pricing scheme** gives a seller the most possible profit and the largest possible customer base.

■

Note that for this scheme to work, we need to know the internal valuation of each type of customer. But this is private information: how can a seller determine it? A hint to a solution can be found in the **Vickrey** or **second price** auction.

Example 24 (Vickrey auction)

Consider the sale of a good by auction. Unlike the previous example, assume that the seller does not know the internal valuations of the buyers (in which case the solution is trivial). The Vickrey auction awards the good to the highest bidder, but charges the winner the second-highest bid. We now prove that this results in each buyer telling the seller its true internal valuation[8].

Suppose the valuation of a bidder for a good is v, the price it bids is b and the highest price bid by any other bidder is p. The utility of a bid b is given by v-b if $b>p$ (and the bidder wins) and 0 if $b<p$ (and the bidder loses). If a bidder is going to tell a lie, then either (A) $b>v$ or (B) $b<v$. Suppose $b>v$. Now we have two more cases: either (A.1) the bidder wins the auction, so that $b>p$ or (A.2) the bidder loses so $b<p$. Suppose case A.1 holds:

8. It does not, unfortunately, result in the seller getting the best price. If the second-highest bid is very low, the seller may end up with essentially nothing, although there was a willing buyer at a higher price.

- (A.1.a) If $p<v$ the bidder gains a utility of v-p. But it would have obtained the same gain by telling the truth, so telling a lie doesn't help.
- (A.1.b) If $p>v$, the bidder loses utility of p-v, so lying hurts.

With case A.2, telling a lie does not help, since the utility from the transaction is zero[9].

Now, suppose the bidder bids a lower valuation (i.e., $b<v$). Again, we have two sub-cases: either the bidder wins (B.1) or it loses (B.2).

- (B.1) If it wins, it pays the second price, and by reasoning along the same lines as cases A.1.a and A.1.b, we can see that telling a lie either hurts or has the same payoff as telling the truth.
- (B.2) If it loses, the bidder loses utility of p-v, so lying hurts.

We have shown, therefore, that in all cases, telling a lie is either as good as or worse than telling the truth. Therefore, a rational player would tell the truth and reveal its true internal valuation to the seller. This is called **truth revelation.**

Note that the price obtained by the seller is not as high as in the previous example (where the seller would have obtained the internal valuation of the highest bidder). Nevertheless, this simple scheme has the remarkable property that the design of the mechanism makes it incentive compatible for a rational player to reveal the truth. This is at the heart of all mechanism design.

Importantly, we require the players to care about how much money they bid, that is, their utilities are quasi-linear. If this was not the case, due to the Gibbard-Satterthwaite theorem, we would end up with a dictatorship (where only player would decide who gets the good) or would have the price of the good strategically manipulable by one or more players.

■

7.3.4 Formalization

We now formalize the intuitions we have developed so far. Assume that principal P is designing a mechanism with n agents, indexed by i. Assume that the mechanism is associated with a set O of one of $|O|$ outcomes (chosen by the principal), with each action called o. For instance, in the uPhone example, at any given time, there is one agent playing the game (the customer in the store), so n = 1. The set of possible outcomes O is {(0,.), (1,c), (2,min(2a, 2c))}, where the first tuple represents "don't purchase," the second outcome represents "purchase 1 for c" and the third outcome represents "purchase 2 for min(2a, 2c)." We assume that each agent has a type t_i which captures all the private information relevant to their decision-making. For instance, in the uPhone example, the type was A or C. Each t_i is drawn from a set of all possible types for the ith player, T_i. We assume that each agent has a preference ordering over the outcomes, which are represented in the form of private, quasi-linear utility functions $U_i(o, t_i)$. For example, the utility function of a chatter is $U_c((1,c), c) = c\text{-}c = 0$; $U_c((2, \min(2a, 2c)), c) = \min(2a, 2c) - c < 0$.

We define the possible 'states of the world' as the product of all possible agent types, and denote it $T = T_1 \times T_2 \times ... \times T_n$, where '$\times$' is the cross product. A **social choice function** is a mapping f from T to O, that describes, for each possible state of the world, the desired outcome. This is outcome that the principal would have chosen *assuming that it knew the true state of the world*. In the uPhone example, there is a single agent (customer) at any given time, who has type A or C. The social choice function f maps C to (1, c) and A to (2, min(2c, 2a)). Of course, a principal does not know the true type of an agent. So, we seek a mechanism that results in the right outcome without knowledge of the state of the world.

A **mechanism** is an n-tuple of strategy spaces (also called message or action spaces) $S = S_1 \times S_2 \times ... \times S_n$ and an **outcome function** g that maps from S to O. Each S_i represents the possible strategies (or actions) allowed to an agent in the mechanism, that is, the rules of the corresponding game. In the uPhone example, $S_A = S_C$ = {buy nothing, buy one uPhone, buy two uPhones}. The function g represents the outcome as a function of the agent's strategies. In the uPhone example, this is the

9. Of course, a player who knows the internal valuation of other players could artificially boost up their price to just below that of the (known) winning bid, to hurt the winner. But, this violates the assumption that internal valuations are secret.

pricing schedule, that maps from 'buy one' to price c and from 'buy two' to price min($2a$, $2c$). Recall that each player has utilities over these outcomes.

A mechanism $M = (S_1,...,S_n, g(.))$ is said to **implement** social choice function $f(T)$ if there is an equilibrium strategy profile $s^* = (s_1^*(t_1), s_2^*(t_2), \ldots, s_n^*(t_n))$ of the game induced by M such that:

$$g(s_1^*(t_1), s_2^*(t_2), \ldots, s_n^*(t_n)) = f(t_1, \ldots, t_n) \qquad \textbf{(EQ 3)}$$

The equilibrium of the game depends on the underlying solution concept. The most widely used concept is dominant strategy. However, in some cases, a Nash equilibrium (no agent will deviate from the equilibrium) or a Bayes-Nash equilibrium (the expected gain to each agent at the Nash equilibrium exceeds the expected utility from deviation) is also used. We will only study dominant strategy solutions here because they are usually thought to be more plausible than the other solution concepts. In this solution concept, letting s_{-i} represent the strategies of players other than i:

$$u_i(g(s_i^*(t_i), s_{-i}^*(t_{-i})), t_i) \geq u_i(g(\tilde{s}_i(t_i), \tilde{s}_{-i}(t_{-i})), t_i) \qquad \forall i, \forall t_i, \forall \tilde{s}_i \neq s_i^*, \forall \tilde{s}_{-i} \qquad \textbf{(EQ 4)}$$

This implies that for player i, no matter what the other players play, the utility from the dominant strategy is as much as or greater than any other strategy.

7.3.5 Desirable properties of a mechanism

We now define certain desirable properties of any mechanism. Note that these are not mutually compatible: we need to balance between them in any practical mechanism.

Individual rationality: No agent should be forced to participate in the mechanism: every agent should receive a greater utility from participation than non-participation. (For Bayesian agents, these would be expectations rather than actual utilities, conditioned on each agent's prior knowledge of (potentially) their own type and the types of the other agents.)

Incentive compatibility mechanism is incentive compatible if it is in the best interests of each agent to cooperate. More precisely, if the designer of the mechanism would like agent i to play strategy s_i^* in a dominant-strategy equilibrium, then the mechanism is incentive compatible if, in such an equilibrium, the payoff to agent i when it plays s_i^* is as good as or better than the payoff with any other strategy.

Strategy-proofness: A mechanism is strategy-proof if it is incentive-compatible and the equilibrium is a dominant-strategy equilibrium.

Efficiency: A mechanism is efficient if the selected outcome maximizes the total utility. At first glance, this seems impossible: after all, a player's utility is private! However, recall our assumption that the principal knows the form of the utility function of each player and all it does not know is a single type parameter. As we will see below, this permits the desired maximization. A second objection is whether summing utilities is meaningful given that utilities are only defined up to an affine transform. The introduction of a common 'money' parameter that all agents value allows us to plausibly maximize the sum of the utilities.

Budget-balance: In general, a mechanism may require transfers of money between agents. A mechanism is budget-balanced if these net transfers across agents are zero, so that the principal does not have to inject money into the system. (When dealing with Bayesian agents, we need to distinguish between *ex ante* budget balance, which means that the budget is balanced only in expectation. With *ex post* budget balance, the budget is always balanced.)

Fairness: In some cases, we would like the mechanism to select the outcome that minimizes the variance in the utilities of the agents. This is defined as 'fairness.'

Revenue maximization: Obviously, the designer of the mechanism would like to get the most possible revenue from the mechanism.

Pareto optimality: A mechanism is Pareto optimal if it implements outcomes where increasing the utility of any agent would necessarily decrease the utility of some other agent. That is, there is no ‘slack’ in the system.

7.3.6 Revelation principle

Mechanism design permits players to adopt arbitrarily complex strategies. Yet, consider the following particularly simple strategy called **direct revelation**: agent i tells the principal its type t_i. Of course, the agent could lie. Nevertheless, it should be clear that revelation greatly restricts the strategy spaces and simplifies the mechanism.

Formally, a direct-revelation mechanism $M = (T_1,...,T_n, g(.))$ is a mechanism where $S_i = T_i$, i.e., the strategy space for agent i is its set of valid types. A direct-revelation mechanism is incentive compatible if, in equilibrium, the chosen strategy is to tell the truth, i.e., $s_i(t_i) = t_i$ for all t_i in T_i. Note that in a direct-revelation mechanism, the outcome function g is the same as the social choice function f, because they both operate on the same space of agent types.

Suppose we restrict mechanisms to only those where the only strategy allowed to an agent is direct revelation. Are there mechanisms that are more complex and can therefore achieve outcomes that this simple mechanism cannot? The surprising answer is that there are not! Every mechanism, no matter how complex, that achieves its goals through a dominant strategy equilibrium can be reduced to a mechanism where the only strategy for an agent is direct revelation and the solution concept is dominant strategy equilibrium.

To see this, note that the complex mechanism must require the player to play *some* strategy $s_i^*(t_i)$ in equilibrium. Not only are the strategies that are allowed each agent under the control of the principal but also these strategies can only depend on t_i. Therefore, the principal could always simulate s_i if it were told t_i. Thus, no matter how complex s_i, all that the agent i needs to tell the principal is t_i and the principal would compute the same outcome in equilibrium as would the complex mechanism. The preceding reasoning, with a modicum of mathematical formalism and is known as the **Revelation Principle**.

Given this principle, we need only study mechanisms of the direct-revelation type. Moreover, we would like to design mechanisms where truth-telling is the dominant strategy, or, in short, direct-revelation incentive-compatible mechanisms. But, do any such mechanisms exist? The answer is affirmative, as the next section shows.

7.3.7 Vickrey-Clarke-Groves mechanism[10]

The Vickrey-Clarke-Groves (**VCG**) mechanism is a direct-revelation mechanism that makes truth-telling incentive compatible. Because all mechanisms that use dominant-strategy as a solution concept can be reduced to an equivalent direct-revelation mechanism, the VCG mechanism is a widely-used building block in the design of dominant strategy mechanisms.

In the VCG mechanism, each agent tells the principal its (purported) type. Based on these types, the principal computes an outcome x (i.e., the social choice). It also asks each agent to make a payment p_i. The agent would not like to pay the principal any money, so its utility declines with increasing payments. By choosing payments carefully, the principal can make truth telling the dominant strategy equilibrium of the game, so that the social choice that is computed based on reported types is the true social choice.

Specifically, given an outcome x, the VCG mechanism assumes that agents have quasi-linear utility functions of the form

$$(u_i(x, p_i, t_i) = v_i(x, t_i) - p_i) \qquad \textbf{(EQ 5)}$$

10.The Clarke, Groves, and Vickrey mechanisms differ slightly in their generality. For simplicity, we will refer to all three interchangeably as VCG mechanisms.

where the principal knows the form of v_i but not the parameter t_i. v is the **valuation function**, and it describes how highly each agent values the outcome ('public good') x, based on its type. Agent i of true type t_i tells the principal that its type is $\hat{t}_i$. The principal computes the social choice or outcome x^* as:

$$x^* = g(\hat{t}_1, \ldots, \hat{t}_n) = \underset{x}{\arg\max} \sum_i v_i(x, \hat{t}_i) \qquad \text{(EQ 6)}$$

Thus, x^* is chosen to maximize the sum of individual valuations as a function of the reported types. Note that this potentially makes x^* strategically manipulable, in that an agent may report a type that would make $v_i(x^*, \hat{t}_i)$ be more in line with i's wishes (making i a dictator). To avoid this, the VCG mechanism asks each player to make a payment p_i, where

$$p_i = h_i(\hat{t}_{-i}) - \sum_{j \neq i} v_j(x^*, \hat{t}_j) \qquad \text{(EQ 7)}$$

where $h(.)$ is any function that is independent of t_i. The key idea is to pay agent i an amount equal to the sum of the other player's valuations. So, given that the social choice is x^*, the agent i's utility becomes:

$$\begin{aligned} u_i(x^*, p_i, t_i) &= v_i(x^*, t_i) - p_i \\ &= v_i(x^*, t_i) - \left(h_i(\hat{t}_{-i}) - \sum_{j \neq i} v_j(x^*, \hat{t}_j)\right) \\ &= -h_i(\hat{t}_{-i}) + v_i(x^*, t_i) + \sum_{j \neq i} v_j(x^*, \hat{t}_j) \end{aligned} \qquad \text{(EQ 8)}$$

Of these three terms, agent i has no control over the first term, because h does not depend on i. Similarly, it has no control over the third term, which sums over the valuations of the other agents. It can only control the second term by its choice of reporting its type. It should therefore report a type that maximizes the value of $v_i(x^*, t_i)$. How can it do that? Recall that the mechanism finds x as the value that maximizes $\sum_i v_i(x, \hat{t}_i) = v_i(x^*, \hat{t}_i) + \sum_{j \neq i} v_j(x^*, \hat{t}_j)$. Comparing this with Equation 8, we see that agent i can maximize its utility by making $v_i(x^*, t_i)$ the same as $v_i(x^*, \hat{t}_i)$, and this will happen only if $\hat{t}_i = t_i$, that is, it tells the truth.

Essentially, the VCG mechanism forces each agent's utility function to be the sum of the reported valuations of all the users. Thus, every agent reports its type truthfully so that the overall maximization is in its own favour. This makes truth-telling incentive compatible, so VCG is not strategically manipulable. Moreover, this is the only known mechanism that provides both individual rationality and Pareto efficiency.

We have thus far left $h(.)$ undefined. Different choices of $h(.)$ can achieve different outcomes. For example, it can be used to achieve **weak budget balance** (the principal may net revenue, but will never have to pay money out), or individual rationality (no agent will be worse off participating than not participating). There is a particularly well-chosen value of h, called the **Clarke Pivot** value that guarantees individual rationality while also maximizing revenue for the principal (but not necessarily budget balance), that we describe next.

First, define x^{-i} as the social choice computed without taking agent i's input into account:

$$x^{-i} = \underset{x}{\arg\max} \sum_{j \neq i} v_j(x, \hat{t}_j) \qquad \text{(EQ 9)}$$

Then, the Clarke Pivot price that i pays, p_i is given by

$$p_i = \sum_{j \neq i} v_j(x^{-i}, \hat{t}_j) - \sum_{j \neq i} v_j(x^*, \hat{t}_j) \qquad \text{(EQ 10)}$$

With this definition of p_i, we find that agent i's utility is given by:

$$\begin{aligned} u_i(x^*, p_i, t_i) &= v_i(x^*, t_i) - p_i = v_i(x^*, t_i) - \left(\sum_{j \neq i} v_j(x^{-i}, \hat{t}_j) - \sum_{j \neq i} v_j(x^*, \hat{t}_j)\right) \\ &= \left(\sum_{j \neq i} v_j(x^*, \hat{t}_j) + v_i(x^*, t_i)\right) - \sum_{j \neq i} v_j(x^{-i}, \hat{t}_j) \end{aligned} \qquad \text{(EQ 11)}$$

The first term (in the parentheses) can be viewed as the overall utility from social choice $x^*$and the second term as the utility due to the social choice made considering everyone but i. Therefore, the VCG mechanism gives agent i a utility that corresponds to its own contribution to the overall utility or social welfare.

EXAMPLE 25: VCG MECHANISM

Consider a company where three departments would like to purchase and share a single enterprise router that costs $3000. The department IT heads get together with the CIO who wants to know whether they really value the router enough to justify having the company spend $3000 on it. If the CIO simply asked the department IT heads (the agents) how much they value the router, they have no incentive to tell the truth, so they would all insist that they needed it. The CIO could, instead, implement a VCG mechanism as follows. Suppose that agent 1 thinks their department's share of the router is worth $500, agent 2 thinks their department's share is also worth $500, and agent 3 thinks their department's share of the router is worth $2500. We represent this as $v_1 = v_2 = 500$; $v_3 = 2500$. Since they sum to more than $3000, the router should be bought, assuming the agents tell the truth. That is, x^* = 'purchase.'

To ensure truthfulness, the CIO demands payments from each agent (this could be from the departmental IT budget). Assume that the CIO uses the Clarke Pivot payment rule described in Equation 10. We see that x^{-1} = 'purchase', x^{-2} = 'purchase', and x^{-3} = 'do not purchase.' Obviously, v_i is 0 if the decision is 'do not purchase' and the valuation described above if the decision is 'purchase.' This allows us to compute p_1 = (500 + 2500) - (500 + 2500) = 0, which is also the same as p_2. However, p_3 = (0) - (500 + 500) = -1000. In other words, p_3 receives a net payment of $1000 from the CIO! We see that with the Clarke Pivot value, the VCG mechanism does not achieve budget balance. We do achieve individual rationality: everyone is better off participating in the mechanism than not.

In general, if the non-participation of a single agent can affect the outcome (as it can here), we cannot achieve budget balance with a VCG mechanism. Nevertheless, it is important to note that the VCG mechanism makes truth-telling a dominant strategy, so the CIO can expect that each department head will tell the truth.

■

Despite its lack of budget balance, the VCG mechanism can be proved to be individually rational, efficient, and strategy proof. Moreover, under some weak conditions (including the assumption that no single agent can affect the outcome), the VCG mechanism can also achieve weak budget balance. Therefore, it is the most widely used dominant-strategy mechanism.

7.3.8 Problems with VCG mechanisms

The VCG mechanism has two important properties. First, it allows us to design and analyze practical network protocols and algorithms using game theory. It is, therefore, the most engineering-oriented aspect of game theory, thus appealing to computer scientists and engineers. Second, it is remarkable in that it makes agents reveal their true types. This is intimately connected to Byzantine agreement, a classic problem in distributed computer algorithm design. Nevertheless, there are many drawbacks of the VCG approach that we briefly discuss next.

Information requirements: The VCG mechanism assumes that the principal knows the form of the utility function of each agent so that revelation of the agent's type is sufficient to compute its utility. This is a strong assumption. It may not always be possible for principals to know agent utilities, which, after all, reflect their complex inner motivations.

Complexity of valuation function: The computation of the optimal social choice requires the principal to compute the valuation for each agent for each possible alternative. Consider a principal that wants to sell m different goods, where players can buy any subset of the goods and value each subset differently. This is called a *combinatorial auction* and may reflect the fact that agents may benefit only from purchasing two or three specific goods, rather than from each good individually. Then, each agent needs to specify up to 2^m different values and the principal would need to compute sums over all possible partitions of the m goods and their allocation to the agents, an enormous task.

Centralization: The social choice function in a VCG is computed by a single centralized principal who receives inputs from all the agents. Imagine a resource allocation problem with hundreds or thousands of agents: this would require the principal to perform a very large optimization, which is computationally expensive. It would be preferable to have this computation broken up into smaller, distributed computations.

Non-approximability: The VCG mechanism (in dominant strategies, at least) requires that the principal compute the exact optimal value of the sum of the agent valuations. If this is not the optimal value, then agents lose incentive compatibility. However, finding the optimal point of the sum of valuations is complex, and may only be approximable, leaving the mechanism potentially open to manipulation.

Fairness: The VCG scheme assumes that all agents have the same value for money. If this is not true (if, for example, richer agents care less for money than poorer agents), then fairness is not assured.

Budget-balance: We would like any mechanism to be net budget-balanced, so that there are no net payments made to the principal or to the agents. At least, it should not cost the principal money to run the mechanism. However, if a single player can affect the outcome, VCG is not even weakly budget-balanced. It turns out that a different solution concept called the Bayes-Nash equilibrium can guarantee budget balance. The corresponding mechanism, called **d'Aspremont-Gerard-Varet** (dAGVA) after its inventors, lets each agent compute expected utilities as a function of their prior subjective probabilities on the types of the other agents. However, this budget balance comes at the expense of individual rationality (some agents would be better off not participating in the mechanism).

7.4 Limitations of game theory

Having studied some aspects of game theory, we now outline some of its limitations.

- Perhaps the biggest problem with using game theory in real life is ensuring that all players are aware of each others' utilities from each outcome. In real life, players often do not know what actions other players are permitted, their payoffs for each outcome, and the utility they gain from these payoffs.
- A second problem has to do with modelling time. A normal form game is played simultaneously by all players and an extensive form game is played sequentially. In neither case, however, do we model the timing of the underlying events. Time and delay are critical factors in most networking problems. For example, in a wireless LAN, a station's transmission is known to others only after a non-trivial delay. Therefore, each player may see a different view of the world at each point in time. This affects the outcome of the game but is not modelled by classical game models.
- Almost all games assume that players are rational. However, there is considerable experimental evidence that people are not rational and sometimes do not even have consistent preferences, undermining utility functions as valid descriptions of user satisfaction.
- Most game models assume that the number of players does not change over time. However, in most typical networks, the number of players (i.e., endpoints sharing a resource) may vary over time. An endpoint usually does not know who else is sharing a given resource, let alone their utilities and payoffs.
- Any social welfare function that maximizes sums of utilities is implicitly performing inter-personal utility comparisons. This is fundamentally invalid. The standard justification is that all players have the same value for money, but this is certainly an unrealistic assumption.

- As we have seen with mechanism design, games may require massive communication amongst players or between agents and the principal. For instance, to form Bayesian expectations on the other players' types, a player may need to observe their past behaviour in great detail. This is both an invasion of privacy and expensive to communicate.

We conclude that it is unlikely that the results of game theory can be used directly in practical situations. However, it provides deep insights into modelling the behaviour of selfish, rational agents and into the design of communication protocols that cannot be manipulated to subvert the designer's intentions. These make it well worth our study.

7.5 Further Reading

The classic original text on game theory, J. Von Neumann and O. Morgenstern, The Theory of Games and Economic Behaviour, Princeton University Press, 1944 (or the 60th anniversary edition released in 2004) is still a wonderful and insightful read. Another classic is R.D. Luce and H. Raiffa, Games and Decisions, John Wiley and Sons, 1957. A more recent introductory text is M.J. Osborne and A. Rubinstein, A Course in Game Theory, MIT Press, 1994.

A brief overview of contemporary game theory can be found in K. Leyton-Brown and Y. Shoham, Essentials of Game Theory, Morgan and Claypool, 2008. A computer science perspective on game theory, and in particular mechanism design, can be found in N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani, Algorithmic Game Theory, Cambridge University Press, 2007.

Graphical games were introduced and discussed in more detail in M. Kearns, M. Littman, and S. Singh, Graphical Models for Game Theory, Proceedings of the Conference on Uncertainty in Artificial Intelligence, 2001. A critique of VCG mechanisms can be found in M.H. Rothkopf, "Thirteen reasons the Vickrey-Clarke-Groves process is not practical," Operations Research, 55:2, pp191-197, 2007.

7.6 Exercises

1 Preferences

Suppose that you equally like a banana and a lottery that gives you an apple 30% of the time and a carrot 70% of the time. Also, you equally like a peach and a lottery that gives you an apple 10% of the time and a carrot 90% of the time. (a) What can you say about your relative preferences for bananas and peaches? If you had a lottery whose payoffs were bananas and carrots, what probability of winning a banana or a carrot would be equally preferred to a peach?

2 Utility functions

Your cable company gives you 10GB of free data transfer a month, and charges $5/GB thereafter. Suppose that your utility from transferring x GB of data is $100(1-e^{-0.25x})$ and that your disutility from paying $1 is 1. How much data should you transfer in a month to maximize your utility?

3 Pure and mixed strategies

Consider the game of tic-tac-toe. What are the possible actions for the first move of the first player (ignore symmetries)? What would constitute a pure strategy? What would constitute a mixed strategy? Would you ever play a mixed strategy for this game? Why or why not?

4 Zero-sum game

If the payoffs $(a, -a)$ of every outcome of a zero sum game were changed so that the new payoffs were $(a+5, -5a)$, the game would no longer be zero sum. But, would the structure of the game change?

5 Representation

Represent the Pricing game of Example 7 in Normal form.

6 Representation

Prove that normal and extensive form are equivalent if information sets are permitted.

7 Best response

What is the best response for the customer in the Pricing game (Example 7)?

8 Dominant strategy

Suppose that you are not well prepared for a final, and you think you might fail it. If you miss the exam, you will certainly fail it. What is your dominant strategy: attend or miss? Why?

9 Bayesian game

Does the Bayesian game in Example 11 have a dominant strategy for the Row player? If so, what is it?

10 Repeated game

Suppose that both players in Prisoner's Dilemma (Example 15) play their dominant strategy in an infinitely repeated game with a discount factor of 0.6. What is their payoff for the repeated game?

11 Dominant strategy equilibrium

Interpret the meaning of the dominant strategy equilibrium of Example 14. Look up how the 802.11e EDCA protocol solves this problem.

12 Iterated deletion

Show an example of a pure strategy that is dominated by a mixture of other pure strategies, although none of the strategies in the mixture dominate the pure strategy.

13 Maximin

What are the maximin equilibria in Examples 10 and 14?

14 Maximin in a zero-sum game

Show that in Example 18 if Row uses any value of p other than 0.5, then it may get a payoff lower than 2.5 if Column plays either pure or mixed strategies.

15 Nash equilibrium

Referring to Example 19, assume that if Column plays a mixed strategy with probability qH + (1-q)T instead of its Nash equilibrium strategy. What is Row's mixed strategy best response?

16 Correlated equilibrium

Does the WiFi game of Example 6 have a correlated equilibrium? If so, describe it.

17 Price discrimination

Outline the design of a price-discrimination mechanism with n player types (whose valuations are known).

18 VCG mechanism

The CIO of a company wants to decide how much capacity to buy from its ISP. The cost of capacity is \$20/Mbps/month. There are three departments in the company, who value capacity as follows: department 1 (D1) values capacity x Mbps/month at \$20(*1-$e^{-0.5x}$), D2 values it at \$40(*1-$e^{-0.5x}$), D3 values it at \$80(*1-$e^{-0.5x}$). (a) Assuming the disutility of ISP payment is linear in the amount of payment, what is the overall function that the CIO should maximize? (b) What is type of each department? (c) What is the optimal social choice? (d) What are the Clarke Pivot payments for each department? (e) Is this budget balanced?

CHAPTER 8

Elements of Control Theory

8.1 Introduction

A computer network is an engineered system: a system that has been designed to meet certain design goals. Two such goals are **responsiveness** and **stability**. Responsiveness allows a system to continue to meet its performance goals despite changes in its operating environment such as workload fluctuations or component failures. Stability prevents a system from failing or from being sent into uncontrollable oscillations due to certain inputs. Control theory provides powerful tools for building responsive and stable systems. For example, by mathematically modelling the behaviour of a controlled system, it shows that there is a fundamental contradiction between responsiveness and stability: the more responsive a system, the greater the chance that it will be unstable.

This chapter presents the elements of control theory. For simplicity, we primarily consider continuous-time, linear control systems and classical control strategies. The concepts here, however, should provide a strong foundation for the interested student to delve further into this subject.

Note that Chapter 5 on Signals, Systems, and Transforms is an essential prerequisite for understanding the material in this chapter. We first describe a generic control system. We then characterize the behaviour of generic first- and second- order systems, culminating with the design and analysis of classical control systems. We end with an introduction to 'modern' control systems, i.e., state-space representation and the principles of observability and controllability, and digital control.

8.2 Overview of a Controlled System

A generic feedback control system is shown in Figure 1. The system to be controlled is usually referred to as the **plant**. We assume that the plant and all control elements are continuous, linear and time-invariant. Given a control **input signal** u, the plant responds with an **output signal** y that is also additively influenced by an uncontrollable **disturbance signal** w. The control goal is to maintain the output of the plant at some **reference value** r despite disturbances. This is done using a **controller**. The controller compares the reference signal value with the **measured output signal** b, where this measurement itself may be subject to error. The controller chooses a **control signal** *u* such that the future output of the plant eventually matches the reference input, despite disturbances and measurement errors.

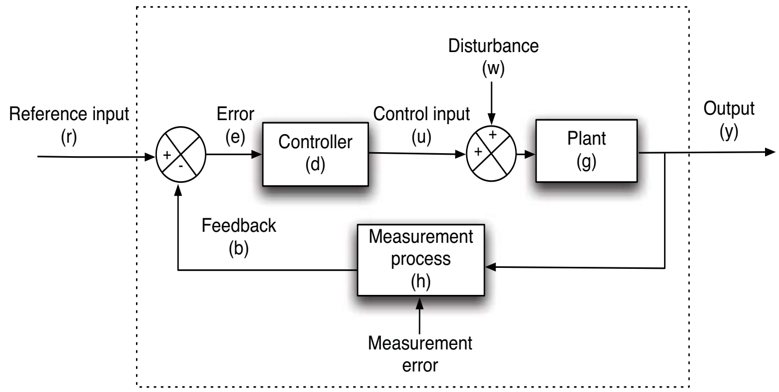


FIGURE 1. **General model of a controlled system**

EXAMPLE 1: A SIMPLE CONTROL SYSTEM

A thermostat-controlled home heating furnace is an example of a simple control system. A user can specify a desired home temperature *r* using the control interface. The controller compares this temperature with the actual temperature *b* measured using a sensor. If the difference (error *e*) is positive, it activates a heating furnace by sending it a control signal *u*, eventually raising the home's air temperature *y* to the desired reference value. At this point, the error drops to zero and the furnace is turned off[1]. A disturbance *w*, such as the inrush of cold outside air due to opening a door or window, results in a drop of the measured temperature *y*. The controller compensates for this disturbance by turning on the furnace and eventually causing the output to return to the reference temperature.

Even this simple example demonstrates three lessons that apply to all control systems. First, immediately after the disturbance, a home's temperature could be far from the reference temperature. The presence of the controller does not guarantee that the home temperature will be always at the reference value, only that, in the absence of disturbances, this reference value will eventually be reached. If there are a series of disturbances in quick succession, the system may never reach equilibrium.

Second, given that a system is currently not in equilibrium, it is up to the controller to decide how fast to restore the system to this state. A system is said to have a high **gain** or be **responsive**, if the controller tries to restore the system 'rapidly.' Although this is usually desirable, a system with too high a gain can become unstable. This is because in a system with high gain, even a short delay in transferring measurements from the plant to the controller causes the controller to over-react to stale information, causing oscillations. In this example, a controller with too high a gain could result in a home whose temperature persistently oscillates from being too hot to being too cold. Choosing the degree of gain in the controller that balances responsiveness and stability is a critical issue in any control system.

Third, note that the controller simply cannot compensate for measurement errors. For example, if the temperature sensor always reads two degrees higher than the true temperature, the home's temperature will always be two degrees lower than the reference value. Therefore, in any feedback control system, it is important to use the best sensors possible.

■

EXAMPLE 2: CONTROLLING WEB SERVER LOAD

1. This example is highly simplified. Modern furnaces also have more sophisticated control elements, such as a controller that turns the furnace turns on slightly before it is needed, so that it heats air instead of cooling it, and a controller that uses measurements to predict how long a furnace needs to be on to heat the home by a certain amount.

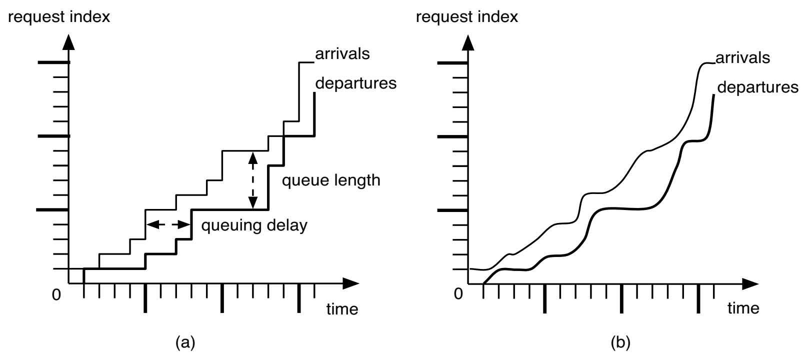


FIGURE 2. **(a) Discrete arrivals and departures (b) Fluid approximation**

Consider a web server that responds to GET and POST queries from web browsers.The server uses a buffer to hold pending requests before they are served so that requests arriving to a busy server are not lost. Figure 2(a) shows the times at which requests arrive and depart from the buffer. Requests arrive at discrete time instants (shown on the X axis) and are served after spending some time in the buffer. For example, request 4 arrives at time 5 and is served at time 7. The system being causal, the departure curve is always at or below the arrival curve.

Although arrivals and departures are discrete events, it is convenient to model them using a continuous 'fluid' approximation as shown in Figure 2(b). Intuitively, this approximation is obtained by smoothing the arrival and departure curves, which removes abrupt transitions without introducing too much distortion. If queries arrive fast enough, the fluid approximation has been found in practice to be reasonably accurate and, in any case, greatly simplifies mathematical analysis.

Note the vertical distance between the arrival and departure curves at any point in time represents the number of queries in the buffer (the queue length) at that time. The control goal is to ensure that this quantity neither exceeds the buffer capacity --causing requests to be lost due to buffer overflow--nor drops to zero--causing a buffer underflow and idling the web server--unless, of course, there is no work to be done. Instead, we would like to keep the number of queries in the buffer at some reference value r that is small enough to not greatly impact the server response time but large enough that if the web server were to suddenly obtain additional capacity, the buffer would not underflow. The greater the value of r, the higher the expected query service rate and the higher the query-processing delay.

The problem is that a client does not know the web server's service rate: this depends on how many other clients are being served and how long it take to satisfy a client request. We model this uncertainty by representing the web server's service rate as a time-varying 'disturbance' $w(t)$. The control goal is to manage the client request rate $u(t)$ to prevent buffer underflow and overflow. Specifically, we assume that the controller is given a buffer occupancy reference value $r(t)$ and that it is instantaneously informed of the current number of bits in the buffer, denoted $y(t)$. It uses this feedback information to choose $u(t)$ such that $y(t)$ tracks $r(t)$ despite disturbance $w(t)$.

Note the similarity between this control system and a home thermostat. In both cases, the controller tries to keep output y at the reference level r despite disturbance w by comparing the measured state of the system with the desired reference value.

■

8.3 Modelling a System

Motivated by Example 2, we turn our attention to the process of mathematically modelling a system. By a 'system,' we mean both the plant in Figure 1and the controlled system shown in dotted box. Both the plant and the controlled system have an input and output, where the output depends on the input. By using identical approaches to model both the *open* uncontrolled system and the *closed* controlled system, we can recursively control a complex system. That is, we can model and control one aspect of a plant, then represent this controlled system as black box when modelling the remainder of the system. This approach is necessary to control a complex system. We will study some practical problems using this approach in Section 8.8 on page 241.

Choosing which aspects of a system should be modelled and which aspects can be ignored is an art rather than a science. Developing a system model that is mathematically tractable and yet represents all the relevant aspects of the system will usually require several rounds of refinement. We first outline an approach that has been found to work well in practice, then focus on three mathematically equivalent representations of a system.

8.3.1 Modelling approach

Start by understanding the working of the system and its component parts. Determine the physical capabilities of each component, its limits of operation, and the operating regimes where it is linear or non-linear. Note that components in a control system can usually be described in terms of the concepts of **effort** and **flow**. Effort represents an input into the system which causes a flow. Their product is **power** and the total power expended over some time duration is called **energy**. For example, in a mechanical system, forces correspond to effort and velocities correspond to flow. The product of force and velocity is mechanical power and the integral of power over time is the amount of mechanical energy used. In an electrical system, voltages correspond to effort and currents to flow. Their product is electrical power, whose integral is electrical energy. Similarly, in a computer network, the quantity of data at a source that needs to be sent to a destination represents effort and the rate at which data is sent corresponds to flow. Although they can be analogously defined, the concepts of power and energy are not commonly used in computer networks.

System components can be either active or passive. Active elements generate power (not energy!) that flows through the system. The dynamical operation of the system is essentially an attempt to redistribute this power over time. In contrast, passive elements store and dissipate energy, but do not generate power. In an electrical system, a voltage source such as a battery is an active element that drives the system, whereas capacitors and inductors are passive elements that store and dissipate energy. Similarly, a data source is an active component that generates effort that causes a flow on links and into routers. In contrast, a buffer is a passive component that stores and releases energy, like a capacitor.

The next step is to capture the laws (such as conservation) that constrain the operation of the system. For example, in a mechanical system, Newton's laws form the basic equilibrium constraints. In a computer network, the rate at which a buffer stores data is the difference between the data arrival and departure rates. Similarly, during the slow-start phase of a TCP connection, the rate of transmission grows exponentially. These constitute the mathematical laws of operation of the corresponding components.

Finally, state the intended goal of the control system in terms of the ideal equilibrium operating point (by convention, this operating point is defined to be 'zero'). If possible, quantify the permissible operating error. This makes it possible to determine the degree to which a chosen control law meets requirements.

At the end of the modelling process, it should be possible to capture the operation of the system in a handful of equations.We will see some concrete examples of the modelling process shortly.

8.3.2 Mathematical representation

There are three common ways to mathematically represent a system(i.e., either the plant or the dotted box in Figure 1). These are the **state variable**, **impulse response**, and **transfer function** representations. Our discussion assumes that the system is **linear** and **time-invariant** (**LTI**). Recall that a time-invariant system is one whose parameters do not change with time: if the system has an output $y(t)$ at time t for an input $u(t)$, then it has the output $y(t-T)$ for an input $u(t-T)$. That is, if $y(t) = G(u(t))$, then $y(t-T) = G(u(t-T))$. Recall also that a linear system is one that exhibits **superposition**: if the input u_1 leads to output y_1 and input u_2 leads to output y_2, then for all constants k_1 and k_2, the input $(k_1u_1 + k_2u_2)$ leads to the output $(k_1y_1+k_2y_2)$. That is, if $y_1 = G(u_1)$ and $y_2 = G(u_2)$ then $k_1y_1 + k_2y_2 = G(k_1u_1 + k_2u_2)$. For an LTI system, it is always possible to choose the initial time to be $t = 0$. We assume the system to be causal, so the input signal is zero for $t < 0$.

State variable representation

In the state variable representation, we first choose a set of state variables that completely characterize the state of the system. Note that by choosing the set of state variables, we are implicitly deciding which aspects of a system to model and which to ignore: a subtle and far-reaching decision.

The number of state variables in the system typically corresponds to the number of energy storage elements in the system because each state variable tracks the amount of energy currently in that storage element. In an electrical network with, say, three capacitors and four inductors, we need at least seven state variables. For the web server in Example 2, there is only one energy storage element, so we need only one state variable: for example, the buffer occupancy level $y(t)$ (which, in this simple example, is also its output variable). In general, the set of state variables is not unique: the system could equally well be characterized by a state variable corresponding to $\dot{y}(t)$, which is the buffer fill or drain rate, rather than its occupancy level.

We denote the set of state variables by the column vector $\boldsymbol{x}(t)$, whose dimension is the **order** of the system. The system is represented by the equations that govern the evolution of $\boldsymbol{x}(t)$. We first consider a **single-input single-output system**, with scalar input u and a scalar output r. To model the complete system (i.e., including the disturbance input), we also consider a vector disturbance $\boldsymbol{w}$. Let

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{bmatrix}$$

Then, any linear time-invariant (LTI) system can be represented using the state update equations below (where the subscript t has been left out for clarity):

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_1u + f_1w_1 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_2u + f_2w_2 \\ &\dots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_nu + f_nw_n \end{aligned} \tag{EQ 1}$$

Moreover, the output can be written as a function of the state, the disturbances, and the input as:

$$y(t) = c_1x_1 + c_2x_2 + \dots + c_nx_n + du + e_1w_1 + \dots + e_nw_n$$

where the a_is, b_is, c_is, e_is, and the f_is are scalar constants. Using matrix notation, we can compactly write this as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{F}\boldsymbol{w} \\ y &= \boldsymbol{c}^T\boldsymbol{x} + du + \boldsymbol{e}^T\boldsymbol{w} \end{aligned} \tag{EQ 2}$$

where the matrix $\boldsymbol{A}$ the diagonal matrix $\boldsymbol{F}$, and the column vectors $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{e}$ are appropriately defined.

Note that the a_{ii} terms represent how the state element x_i evolves independent of the other state elements, whereas the a_{ij} terms represent the degree of coupling between the ith and jth state elements. If these coupling terms are zero, the matrix $\boldsymbol{A}$ is diagonal, and the system can be decomposed into a set of n non-interacting systems. It is often possible to transform the basis set of state elements into an alternative set of elements that either diagonalizes $\boldsymbol{A}$ or at least puts it in the Jordan normal form (i.e., with elements only on the diagonal and one row above). This maximally decouples the system, making it easier to analyze. This **similarity transformation** is discussed further in Section 3.5.7 on page 96.

EXAMPLE 3: STATE REPRESENTATION OF A WEB SERVER

We use the state space representation to model the web server of Example 2. We wish to model the entire system, so the model incorporates the disturbance input w. We choose the state variable $x(t)$ to be the same as the buffer occupancy level $y(t)$. Then,

$$\dot{x} = u - w$$

is the state update equation of the system. Clearly, the output of the system is identical to the buffer occupancy level $y(t)$.

■

The state space model can be easily extended in two ways. First, it can be used to represent systems with multiple inputs and outputs. In this case, the input u is replaced by the vector $\boldsymbol{u}$, the output y is replaced by the vector $\boldsymbol{y}$, and the disturbance w by the disturbance vector $\boldsymbol{w}$. The details are straightforward and are left as an exercise to the reader.

Second, the state space model can also be used to represent non-linear time-invariant systems as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \end{aligned} \quad \text{(EQ 3)}$$

where the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the state and output vectors of a multiple-input multiple-output system with input vector $\boldsymbol{u}$ and disturbance vector $\boldsymbol{w}$, and the vector functions $\boldsymbol{f}$ and $\boldsymbol{g}$ represent arbitrary functions. It is often possible to **linearize** such a system around an operating point $\boldsymbol{x_0}, \boldsymbol{u_0}, \boldsymbol{w_0}$ by approximating the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ using the multivariable form of the **Taylor expansion**:

$$\dot{\boldsymbol{x}} \approx \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0, \boldsymbol{w}_0) + (\boldsymbol{x} - \boldsymbol{x}_0)^T \nabla x \boldsymbol{f} + (\boldsymbol{u} - \boldsymbol{u}_0)^T \nabla u \boldsymbol{f} + (\boldsymbol{w} - \boldsymbol{w}_0)^T \nabla \boldsymbol{w} \boldsymbol{f} \quad \text{(EQ 4)}$$

where $\nabla x\boldsymbol{f}$, $\nabla u\boldsymbol{f}$, and $\nabla w\boldsymbol{f}$ are the gradients of $\boldsymbol{f}$ with respect to the $\boldsymbol{x}$, $\boldsymbol{u}$, and $\boldsymbol{w}$ vectors respectively. $\boldsymbol{g}$ can be similarly linearized. Note that this approximate linearization is only valid in the immediate neighbourhood of the equilibrium operating point $\boldsymbol{x_0}$.

Impulse response model

Many practical systems of interest have a single scalar input u. In this case, it is possible to represent the system by its **impulse response**. Recall from Section 5.5.3 on page 137 that if we know the response $g(t)$ of an LTI system to a Dirac delta input, then we can compute its response to *any* input $u(t)$ by convolving the input signal $u(t)$ with the impulse response $g(t)$. When using this model, we assume that the system is in equilibrium at time t=0, i.e. $\boldsymbol{x}(0) = \boldsymbol{0}$. Also, if we are modelling the entire system (not just the plant), then the input signal should incorporate the disturbance. The output $y(t)$ is given by:

$$y(t) = \int_0^t u(\tau)g(t-\tau)d\tau = u(t) \otimes g(t) \quad \text{(EQ 5)}$$

The impulse response model is rarely used in practice: its main purpose is as a step in the derivation of the *transfer function model*, that we describe next.

Transfer function model

Consider a system at equilibrium at time 0 that is described by the impulse response function $g(t)$, so that the output $y(t)$ is given by Equation 5. By taking the Laplace transform of both sides:

$$Y(s) = U(s)G(s) \quad \text{(EQ 6)}$$

In other words, we obtain the Laplace transform of the output by multiplying the Laplace transform of the input (which, if necessary, incorporates a disturbance input) with the Laplace transform of the impulse response rather than computing a convolution, as is required with the impulse response function. This simple form of the output makes it convenient to represent a single-input single-output causal system by the Laplace transform of its impulse response, which we call its **transfer function**.

The transfer function of many common systems takes the form $N(s)/D(s)$, where $N(s)$ and $D(s)$ are polynomials in s. The highest power of s in $D(s)$ is called the **order** of the transfer function. Consider a denominator polynomial $D(s)$ of order m. The m roots of $D(s)$, that is, the m values of s for which $D(s)$ is zero, denoted $\alpha_1, \alpha_2, \ldots, \alpha_m$, are the values of s for which

$Y(s)$ is undefined, and these are called the **poles** of the system. It turns out that these poles have a deep significance. This is because, using the technique of partial fraction expansion discussed in Section 8.12 on page 255, $Y(s)$ can be written as a sum of terms of the form $\frac{A}{(s-\alpha)^r}$. Referring to Row 6 of Table 4 on page 152 and recalling that the Laplace transform is linear, we see that the corresponding term in $y(t)$, obtained by taking the inverse Laplace transform, is $A\frac{t^r}{r!}e^{\alpha t}$. Therefore, if the pole is negative, the corresponding term in the output decays with time, and, on the other hand, if it positive, the output grows with time. We like a system to be **stable**, which informally means that for a bounded input, the output should also be bounded. This is only possible if all system poles lie in the left half of the complex s plane. (We discuss this notion in more detail in Section 8.9 on page 245.) Thus, the nature of $D(s)$ allows us to quickly determine the overall behaviour of the system, reinforcing the usefulness of the transfer function system model.

EXAMPLE 4: TRANSFER FUNCTION MODEL

We now derive the transfer function model of the web server system. Recall that the state space representation of the system is:

$$\dot{x} = u - w$$

This is a single-input single-output system the system is initially at equilibrium so that $y(0) = 0$. Therefore, we can use a transfer function to model it. The output y is the same as the state variable x. To incorporate the disturbance into the input signal, we write:

$$\begin{aligned} v &= u - w \\ \dot{x} &= v \end{aligned}$$

We obtain the transfer function model by taking the Laplace transform of both sides:

$$\begin{aligned} sY(s) - y(0) &= V(s) \\ Y(s) &= V(s)\left(\frac{1}{s}\right) + \frac{y(0)}{s} \end{aligned}$$

so that

$$G(s) = \frac{1}{s}$$

is the required transfer function model of the system.

■

'Modern' vs. 'Classical' control

The classical approach to control uses transfer function models. Although the results are intuitive, this approach does not permit the modelling of multiple-input multiple-output and non-linear systems. The 'modern' approach, instead, uses the state variable model, which easily generalizes to these domains and can also draw upon linear algebra for compact notation and a deep theoretical basis. In keeping with our goal of presenting intuition rather than formalisms, we will use the classical approach, with a brief treatment of the state variable approach in Section 8.9 on page 245.

8.4 A First-order System

One of the simplest systems that can be characterized by a transfer function is a **first-order** system with a transfer function:

$$G(s) = \frac{K}{1 + \tau s} \tag{EQ 7}$$

where K is called the **gain parameter** and τ is called the **system time constant**. The reason for this nomenclature is best understood by consider the response of this system to a step input $u(t)$, that is, an input that is one unit for all $t > 0$. Recall that the Laplace transform of a step input is $1/s$. Therefore, the Laplace transform of the output, i.e., $Y(s)$ is given by:

$$Y(s) = \frac{K}{s(1 + \tau s)}$$

It is easy to see that the roots of the denominator are s=0 and $s = -\frac{1}{\tau}$. We use this fact to compute the partial fraction expansion as

$$Y(s) = K\left[\frac{1}{s} - \frac{\tau}{1 + \tau s}\right]$$

Using Table 4 on page 152, we compute

$$y(t) = K\left(1 - e^{\frac{-t}{\tau}}\right) \tag{EQ 8}$$

which is shown in Figure 3 for K=2.5 and $\tau = 2$.

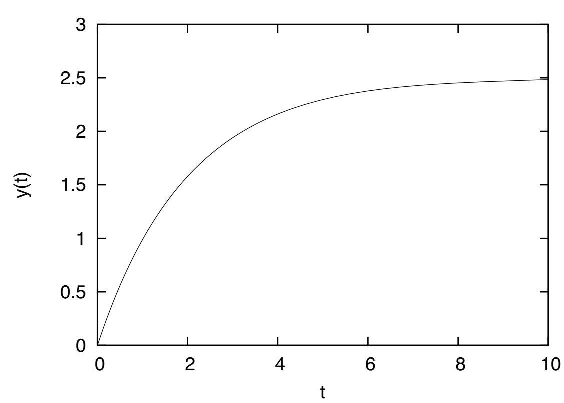


FIGURE 3. Step response of a first-order system with gain 2.5 and time constant 2

A close examination of Equation 8 and Figure 3 reveals that:

- In response to a step input, the output rises rapidly, eventually asymptoting to a value that is K times the input value of 1. In other words, in steady state, the system magnifies the input by a factor of K, thus justifying K as the system gain parameter.
- The system reaches (1-1/e) or roughly 63% of its final value when $t = \tau = 2$ and 99.33% of its final value when $t = 5\tau = 10$. For all practical purposes, the system reaches its asymptotic value at time 5τ, justifying τ as the system time constant.

- A step input is like a ‘sharp kick’ to the system. We can write the system’s **step response** to this input as $y(t) = K(1-e^{t/\tau}) = K - Ke^{t/\tau}$. The first term is the **steady state response** and the second term is the **transient response**. In general, given any sharp change in input, any system will exhibit both forms of response. We are interested in quantifying both. Given the particularly simple form of the Laplace transform of a step function, we often characterize a system by its step response.
- Step inputs are easy to generate. Therefore, if we can model a system as a first-order system, it is possible to numerically obtain the value of *K* and τ by giving it a step input and plotting its output.

8.5 A Second-order System

Intuitively, a second-order system arises when the system responds to the input by accumulating and then releasing energy. A classic example is a violin string excited by a bow. As the bow moves over the string, the string distorts, accumulating energy. At some point, the accumulated energy is sufficient to overcome the motion of the bow and the string returns, then overshoots its neutral position. The process repeats, resulting in a oscillation, powered by the energy of the bow.

Compared to a first-order system, a second-order is more complex, but still relatively easy to analyze. Its transfer function is given by

$$G(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\varsigma s}{\omega_n} + 1} \qquad \textbf{(EQ 9)}$$

A second-order system is characterized by three factors: its **gain**, *K*, its **natural frequency** ω_n (in radians/second), and its **damping ratio** ς. As before, to understand the physical meanings of these terms, we consider its unit step response. For a step input, we have

$$Y(s) = \frac{K}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\varsigma s}{\omega_n} + 1\right)}$$

We study three cases corresponding to different values of ς.

8.5.1 Case 1: $\varsigma = 0$ - an *undamped* system

When $\varsigma = 0$,

$$Y(s) = \frac{K}{s\left(\frac{s^2}{\omega_n^2} + 1\right)} = \frac{K\omega_n^2}{s(s^2 + \omega_n^2)} = K\left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right)$$

Using Table 4 on page 152, we compute

$$y(t) = K(1 - \cos\omega_n t) \qquad \textbf{(EQ 10)}$$

which is shown in Figure 4 for $K = 2.5, \omega_n = 2$.

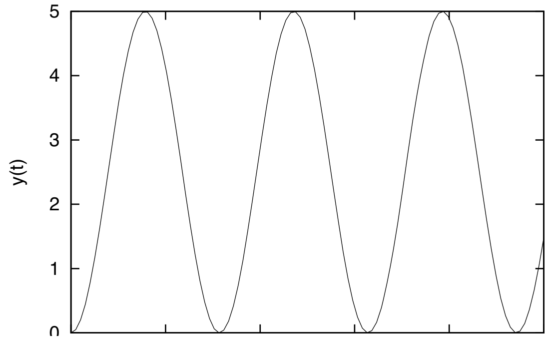


FIGURE 4. **Unit step response of an undamped second-order system with** $K = 2.5, \omega_n = 2$

The output of an undamped second-order system is oscillatory, with a frequency of ω_n. The gain parameter determines the amplitude of the oscillations. These justify naming K and ω_n as the gain and natural frequency respectively.

8.5.2 Case 2: $0 < \varsigma < 1$ - an *underdamped* system

When $0 < \varsigma < 1$, the Laplace transform of the step response is given by:

$$Y(s) = \frac{K}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\varsigma s}{\omega_n} + 1\right)} = \frac{K\omega_n^2}{s(s^2 + 2\varsigma s + \omega_n^2)} = K\left(\frac{1}{s} - \frac{(s + \varsigma\omega_n)}{(s + \varsigma\omega_n)^2 + \omega_d^2} - \frac{\varsigma\omega_n}{(s + \varsigma\omega_n)^2 + \omega_d^2}\right)$$

$$\text{where } \omega_d^2 = \omega_n^2(1 - \varsigma^2)$$

Using Table 4 on page 152, we compute

$$y(t) = K\left(1 - e^{-\varsigma\omega_n t}\left(\cos\omega_d t + \frac{\varsigma}{\sqrt{1 - \varsigma^2}}\sin\omega_d t\right)\right)$$

$$y(t) = K\left(1 - \frac{e^{-\varsigma\omega_n t}}{\sqrt{1 - \varsigma^2}}(\sqrt{1 - \varsigma^2}\cos\omega_d t + \varsigma\sin\omega_d t)\right)$$

Let $\zeta = \cos\theta$ so that $\theta = \cos^{-1}\zeta$. Then, $\sqrt{1 - \varsigma^2} = \sqrt{1 - (\cos\theta)^2} = \sin\theta$, so that

$$y(t) = K\left(1 - \frac{e^{-\varsigma\omega_n t}}{\sqrt{1 - \varsigma^2}}(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t)\right)$$

From the identity $\sin(A + B) = \sin A\cos B + \cos A\sin B$, we get

$$y(t) = K\left(1 - \frac{e^{-\varsigma\omega_n t}}{\sqrt{1 - \varsigma^2}}\sin(\omega_d t + \cos^{-1}\zeta)\right); t \geq 0 \qquad \text{(EQ 11)}$$

From the form of Equation , we see that the sinusoidal oscillation is modulated by a decaying exponential of the form $e^{-\varsigma\omega_n t}$. For a fixed value of ω_n, as ς increases, the exponent dies down more rapidly, 'damping' the system. This justifies calling ς the damping ratio. We also see that the phase of the sinusoid is shifted (compared to the undamped system) by a phase angle $arc\tan\frac{\sqrt{1-\varsigma^2}}{\varsigma}$, a function of ς.

Figure 5 shows $y(t)$ for different values of ς, while keeping the values of K and ω_n the same as in the case of the undamped system, i.e., $K = 2.5, \omega_n = 2$. As ς approaches 1, the transient response of system becomes less oscillatory. For example, when $\varsigma = 0.2$, the system continues to oscillate even after five time periods (i.e., t=10). On the other hand, when $\varsigma > 0.8$, the output steadily approaches the asymptotic value of 2.5.

Intuitively, the damping ratio determines the responsiveness of the system. When this ratio is small, the system is 'jumpy' responding immediately to a stimulus, but takes a long time to reach steady state after receiving a shock. As the ratio increases, the system is more 'sluggish,' but reaches its asymptote smoothly.

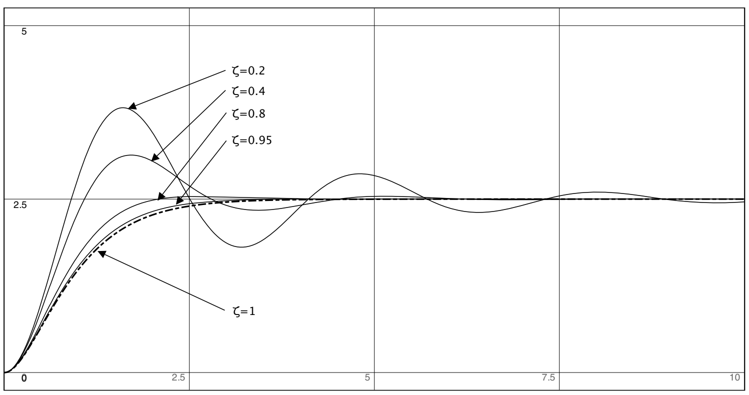


FIGURE 5. Unit step response of an underdamped second-order system with $K = 2.5, \omega_n = 2; \varsigma = 0.2, 0.4, 0.8, 0.95, 1.0$

8.5.3 Critically damped system ($\varsigma = 1$)

When $\varsigma = 1$, we say that the system is **critically damped**. When subjected to a unit step input, this system is neither too responsive nor too sluggish, rising smoothly to its asymptotic value. Of course, choosing parameters to achieve critical damping for any real system can be a challenge!

To mathematically study such a system, we use the partial fraction expansion of $Y(s)$ as before to find

$$Y(s) = \frac{K\omega^2}{s(s^2 + 2\omega s + \omega^2)} = K\left[\frac{1}{s} - \frac{\omega}{(s+\omega)^2} - \frac{1}{s+\omega}\right]$$

$$y(t) = K[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}] \qquad \textbf{(EQ 12)}$$

Note that the output $y(t)$ has no sinusoidal components. The corresponding function is shown in Figure 5. Comparing the output of a critically damped system with that of an undamped and underdamped system, it should be clear why a critically damped system is ideal.

8.5.4 Overdamped system ($\varsigma > 1$)

When the damping ratio exceeds 1, the output of a second order system becomes less and less responsive to a step input. Such a system is **overdamped**. We can mathematically study the system as follows. We have

$$Y(s) = \frac{K\omega_n^2}{s(s^2 + 2\varsigma\omega_n s + \omega_n^2)}$$

The roots of the second term in the denominator, from the quadratic formula, are $\frac{-2\varsigma\omega_n \pm \sqrt{4\omega_n^2\varsigma^2 - 4\omega_n^2}}{2}$, so that

$$Y(s) = \frac{K\omega_n^2}{s(s + \varsigma\omega_n + \omega_n\sqrt{\varsigma^2 - 1})(s + \varsigma\omega_n - \omega_n\sqrt{\varsigma^2 - 1})}$$

For convenience, let $\gamma = \sqrt{\varsigma^2 - 1}$. From the inverse Laplace transform of the partial fraction expansion, we find that

$$y(t) = K\left[1 + \frac{e^{-(\varsigma + \gamma)\omega_n t}}{2\gamma(\varsigma + \gamma)} - \frac{e^{-(\varsigma - \gamma)\omega_n t}}{2\gamma(\varsigma - \gamma)}\right] \quad \text{(EQ 13)}$$

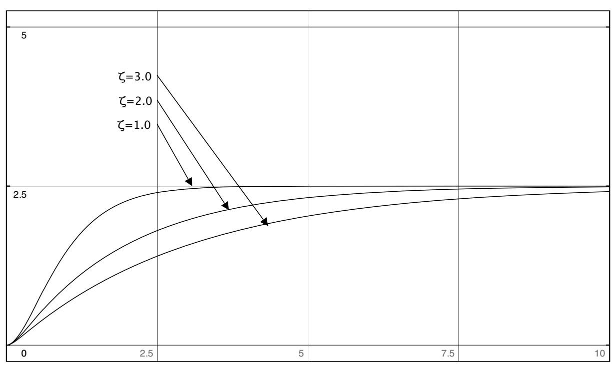


FIGURE 6. Unit step response for an overdamped system with $K = 2.5, \omega_n = 2; \varsigma = 1.0, 2.0, 3.0$

The step response has no sinusoidal components being the difference between two exponential curves. The function is plotted in Figure 6, with the same values of K and ω_n as before. We see that as the damping ratio increases, the system takes progressively longer to reach its asymptote.

To summarize, the behaviour of a second order system is controlled by its gain, damping ratio, and natural frequency parameters. The gain controls the steady state response to a unit step input. When underdamped, the natural frequency controls the step response oscillation frequency. Finally, the damping ratio controls the responsiveness of the step response. In designing a control system, it is usually possible to choose arbitrary values for these three parameters. Our goal, therefore, is to tune the system to achieve the desired gain and critical damping.

In practice, no system is truly a second-order system: this is just a convenient approximation. However, the system gives us valuable insights into the behaviour of many real systems, as the next example demonstrates.

EXAMPLE 5: ROUTE FLAPPING AS A SECOND-ORDER SYSTEM

In a computer network, the dynamic selection of least-delay routes between two nodes can be modelled as a second-order system. Consider two nodes A and B that are connected by two edge-disjoint paths. Suppose all the traffic from A to B goes over the first path. This load increases the delay on this path, which we can model as the accumulation of energy. At some point, the energy on the path is so large that the routing algorithm running at node A decides to route traffic destined to B on the second path. This reduces the energy on the first path and increases it on the second. As before, this results in an over-accumulation of energy on the second path, resulting in an oscillatory system exhibiting **route flapping**.

To make this more precise, consider a network where both paths from A and B have a capacity of 1unit/second. As long as the load from A to B is smaller than 1 unit/second, both paths are uncongested and the system does not exhibit oscillations. However, suppose that A injects load at the rate of 2 units/second into the first path. This will result in the buffer on the path building up at the rate of 1 unit/second. Suppose that the routing protocol running at A switches paths when the path delay exceeds 5 seconds. This will happen at t=5 seconds. At this point, the load switches to the second path and the queue there will build up at the rate of 1 unit/second, until at time t=10 seconds traffic will revert to the first path. Clearly, the oscillations have a period of 10 seconds.

■

EXAMPLE 6: DAMPING ROUTE FLAPPING

We saw the oscillatory behaviour of a naive threshold-based routing protocol in Example 5. We can use the insights from second-order systems to reduce oscillations using damping. The essence of damping is to reduce system responsiveness. We use this insight as follows: suppose that the routing algorithm, instead of sending *all* data on the shortest path, distributed load inversely proportional to the measured path delay. Then, the load from A to B would be spread on both paths, increasing overall capacity. Moreover, if there were a sudden increase in delay along one path, load would be proportionally redistributed to another path. By reacting less aggressively to an increase in load, this damped routing algorithm would reduce the oscillatory behaviour of the network. The OSPF routing protocol in the Internet allows multipath routing for equal-cost paths, so this form of damping is feasible even in practice.

■

8.6 Basics of Feedback Control

We are now in a position, finally, to investigate the basics of feedback control. We use the transfer function model to study a simple feedback system. We also investigate the goals of a control system and learn how these goals can be achieved using feedback control.

Recall the abstract model of a continuous, linear, time-invariant control system in Figure 1. Assume that all inputs and outputs are scalar. Then, we can use transfer functions to model the behaviour of each of the control blocks. That is, if the input to a control block G is the signal $u(t)$ and its output is the signal $y(t)$, then we can model the control block using the transfer function $G(s)$, so that the Laplace transform of the output, $Y(s)$ is merely the product of the transfer function and the Laplace transfer of the input $U(s)$:

$$Y(s) = G(s)U(s)$$

This allows us to redraw Figure 1 replacing each control block by its corresponding transfer function and each signal by its Laplace transform, as shown in Figure 7.

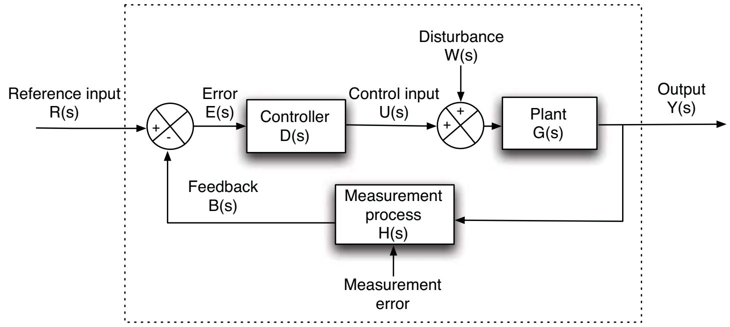


FIGURE 7. **Transfer-function model of a single-input single-output linear time-invariant feedback control system**

The use of the transfer function model allows us to state the system equations as follows (we omit the argument s for clarity):

$$Y = GU + GW \quad \text{(EQ 14)}$$

$$E = R - B = R - HY \quad \text{(EQ 15)}$$

$$U = DE = D(R - HY) \quad \text{(EQ 16)}$$

From Equation 14, Equation 15, and Equation 16, we can write

$$Y = GDR - GDHY + GW$$

$$Y(1 + GDH) = GDR + GW$$

$$Y = \left(\frac{GD}{1 + GDH}\right)R + \left(\frac{G}{1 + GDH}\right)W \quad \text{(EQ 17)}$$

Equation 17 is the fundamental equation of a feedback control system. It shows that the output is a linear combination of two terms, one arising from the Laplace transform of the reference signal $R(s)$ and the other from the Laplace transform of the disturbance $W(s)$. Note that the only variable that can be modified by a system designer is D, which represents the controller. A common situation is where the measurement process simply feeds the output back to the input, so that $H(s) = 1$: such a system is called a **unity feedback system**.

We now use this equation to investigate the goals of a control system and how they are achieved by a feedback control system. Essentially, we will compare the controlled system whose output is given by Equation 17 with a system with no feedback (i.e. $H = 0$), whose output is given by $Y = G(DR+W)$.

8.6.1 System goal

The primary goal of any controller is to ensure that $y = r$ despite w. We assume that we can design the controller, and therefore choose D, more or less as we wish (within some limits, as discussed below) to meet this goal.

It is illuminating to consider how this system goal is achieved in the context of three systems: an uncontrolled system, a controlled system with no feedback and a system with feedback control. The output of an uncontrolled system is affected by disturbances and so the control goal is achieved only if $w = 0$. In contrast, a controller that lacks feedback can compensate for a disturbance by predicting it. If it can predict disturbances accurately, then it can achieve the system goal. For instance, suppose that an oven temperature controller knew that every time the oven door was opened, a certain amount of heat is lost, and so the heating element needs to be turned on for a certain duration. This controller could maintain oven temperature despite a disturbance created by a door opening by sensing the state of the oven door. In contrast, a feedback control system is able to learn the effect of a disturbance on the output (for example, the effect of a door opening on the oven temperature) and can use the system transfer function to determine the appropriate input that removes the effect of the disturbance.

In many situations, the command *r* is held constant for a reasonably long duration of time. In this case, the goal of a controller is first, to ensure that the output quickly reaches *r* and second, that the output stays at *r* despite disturbances. The time taken for the output to get 'close' to *r* from its initial state is called the **settling time**. The discrepancy between the actual output and the reference input after the settling time is past is called the **steady state error**.

8.6.2 Constraints

In achieving the system goal, a controller must take the following constraints into account. The constraints are *stability, disturbance rejection, transient response, linearity, measurement error,* and *robustness.* We consider each in turn.

Stability

Feedback control systems can become unstable if they are improperly designed. Stability essentially means that the output returns to the desired value chosen by a command input despite small variations in the control input or the disturbance. Two widely used stability criteria are **bounded-input bounded-output (BIBO) stability** and **zero-input** stability.

- A system is BIBO stable if, given that the system is initially in equilibrium, any input with bounded magnitude results in an output that is also bounded (though the system may potentially oscillate indefinitely within some bound).
- A system is zero-input stable if, given that the input is zero, all initial states result in the system eventually reaching the zero equilibrium state.

EXAMPLE 7: STABILITY

The web server system of Example 2 is BIBO-stable if, starting with a zero initial queue size, the request queue size does not exceed some bound for all request arrival streams. The system is zero-input stable if, given that the initial queue length is finite, as long as no more requests are accepted into the queue, the queue size eventually drops to zero.

■

Feedback control systems typically achieve stability by ensuring that the control does not over-react to a disturbance. We study this in more detail in Section 8.9 on page 245.

Disturbance rejection

A control system should reject any changes in the output due to a disturbance. From Equation 17, we see that the noise input W is attenuated by a factor of $\frac{G}{1+GDH}$. In contrast, an uncontrolled system or a system without feedback control is subjected to the entire disturbance. This shows the benefit of feedback in rejecting disturbance.

Transient response

When a system in its steady state is subjected to a sudden disturbance, it takes a while for the system to return to equilibrium. This is like a jolt to the handle bar of a bicycle: it takes some time for the bicycle to return to its steady path. The behaviour of the system during this time is called its **transient response**. If a control system is not properly designed, a sharp disturbance can result in the system becoming unstable or collapsing.

A feedback control system gives designers a control 'knob' to shape a system's dynamic response. To see this, we write Equation 17 as

$$Y = \frac{GDR + GW}{1 + GDH}$$

Note that the behaviour of $y(t)$ is given by the inverse Laplace transform of this expression, which, in turn, depends on the roots of the denominator. For instance, if all the roots of the denominator have a strictly negative real component then $y(t)$ is the sum of decaying exponentials so that the transient response eventually dies down. A designer, therefore, can choose *D* so that the denominator polynomial has this behaviour. This intuition lies at the heart of **pole placement**.

Linearity

A control system designed assuming that the system is LTI will not function correctly when the system is driven into a range where it is non-linear. Therefore, we must be careful to ensure that the system always stays in the linear range, and, if a disturbance pushes it into a non-linear regime, then the control input brings the system back into a linear regime. In a feedback control system, the system is likely to continue to be linear when the feedback error is small. A designer may need to take special measures to deal with the situation when the feedback error is large, as the next example demonstrates. For example, in a computer network, when the network is congested, packet losses from overflowing buffers make the system non-linear. One reasonable control rule is for sources to drastically reduce their data transmission rate in the presence of packet loss to restore normal operation (in practice, a TCP source can viewed as intentionally placing the system into near-congestion to maximize network utilization).

EXAMPLE 8: LINEARITY

Consider the web server of Example 2. This system is linear as long as the request queue is neither overflowing nor underflowing. If the queue is, for example, overflowing, then the error term will be large. In this case, a designer could add a special control rule to set the input rate u to 0, so that the queue would drain as fast as possible, and the system re-enters the linear regime.

■

Measurement error

A feedback system measures the output using a sensing or measurement device. Importantly, the feedback system *cannot* compensate for errors in this sensor. For instance, if a thermostat sensor is off by one degree, then the controlled system has a steady state error of one degree. Therefore, it is critical that the measurement process by as error-free as possible.

A second form of measurement error arises due to delays in the measurement process. Consider a thermostat that takes five minutes to measure the room temperature. In such a system, once the setpoint was reached, the furnace would be left on five minutes too long, raising the room temperature beyond the setpoint. At this point, the furnace would be turned off and the room temperature would return to the setpoint, then drop below it for at least five minutes before corrective action could be taken. It is clear that the delayed measurement results in a persistent oscillation in the output signal (i.e., the room temperature). In general, feedback delays can cause oscillations or even instability. The general approach to deal with such delays is to be cautious in responding to a measurement, recognizing that it may reflect stale input. This can be achieved by damping the control input. We will consider dealing with feedback delays more fully in Section 8.8 on page 241.

Robustness

A controller is designed subject to many assumptions. These include assumptions that the system is linear, that the system model is accurate, that the measurement is correct, and that the measurement is not subjected to too much delay. Moreover, we must also assume that the system behaviour does not change over time, due to wear and tear or software upgrades. These are strong assumptions. Therefore, it is necessary to design a controller that does not exhibit poor behaviour when one or more of these assumptions are violated. The theory of **robust control** deals with the design of control systems that work well despite bounded variations in the system parameters and is described in more advanced textbooks on control theory.

8.7 PID Control

Proportional-integral-derivative (PID) control is the simplest classical approach to controller design. It refers to controllers whose control input is either proportional to the error, to the integral of the error, or to the derivative of the error. PID controllers may use two or even three modes simultaneously. We study each type of controller in turn.

8.7.1 Proportional mode control

With proportional control, the control signal is a scalar multiple of the feedback error, so that $u = K_p e$ or, in the transform domain, $U=K_pE$, where K_p is called the **loop gain**[2]. The greater the loop gain, the larger the control action corresponding to a disturbance. Therefore, a large loop gain makes the controller more responsive.

Note that the control input is proportional to the error, so that, if the system in steady-state needs a non-zero control input, then there needs to be a non-zero steady state error, also called the **droop**. To keep the steady-state error small, we need to make the loop gain large. However, this reduces damping and can make the system unstable (we will return to this topic when we study system stability in Section 8.9 on page 245).

EXAMPLE 9: PROPORTIONAL MODE CONTROL

Consider the design of a proportional controller for the web server in Example 2. Recall that its transfer function $G(s)$ is given by $\frac{1}{s}$. Suppose that the command r is the desired setpoint of the request buffer. Then, the error is given by $r - y$. For proportional control, the control input is proportional to this error[3]. Specifically, the controller chooses an input u such that, if there is no further disturbance, the system output y returns to the desired setpoint r after a time τ. That is,

$$u = \frac{r-y}{\tau}$$

so that the loop gain is $\frac{1}{\tau}$. Recall that $Y = \frac{1}{s}(U - W) = \frac{1}{s}\left(\left(\frac{R}{\tau} - \frac{Y}{\tau}\right) - W\right)$, which can be rearranged to get

$$Y = \frac{R}{1 + s\tau} - \frac{W\tau}{s(1 + s\tau)}$$

To study the step response of this controller, set $W = 0$ and $R = 1/s$. Then, $Y = \frac{1}{s(1 + s\tau)}$. Using partial fractions to expand this, $Y = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$, and taking the inverse Laplace transform, we find $y(t) = 1 - e^{t/\tau}$. That is, when there is no disturbance, if the input command is a step signal, then the output is a step signal (i.e., the desired value because $y = r = 1$) along with a transient response that decays exponentially over time. The smaller the value of τ the larger the loop gain and the faster the decay of the transient response.

We investigate the steady-state error in the step response (assuming zero disturbance) using the final value theorem, which states that $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$. Thus,

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE = \lim_{s \to 0} s\left(\frac{1}{s} - Y\right) = \lim_{s \to 0} s\left(\frac{1}{s} - \frac{1}{s} + \frac{1}{s + \frac{1}{\tau}}\right) = 0$$

This shows that, independent of the loop gain, the steady-state error is zero.

2. This is not the same as the system gain. The loop gain is the ratio of the command to the error, whereas the system gain is the ratio of the output to the input, which, in the case of a perfectly controlled system, should be 1.
3. If the error is negative (that is, the buffer occupancy is greater than the setpoint), then this requires the controller to send ‘negative’ packets into the buffer. For simplicity, we will interpret this to mean that the controller sends special ‘cancel’ requests that have the effect of removing pending requests from the web server’s request buffer.

■

8.7.2 Integral mode control

The motivation for integral control is the intuition that the magnitude of the control input should depend not only on the current value of the error, but the total error built up in the past: the greater this overall accumulated error, the greater the correction. Mathematically, with integral control, $u = K_i \int e$ or $U = K_i \frac{E}{s}$ where K_i is the loop gain. Integral control removes the need for a steady-state error term, but usually makes the control less stable, as the next example demonstrates.

EXAMPLE 10: INTEGRAL MODE CONTROL

Consider the design of a integral controller for the web server in Example 2 with transfer function $G(s) = \frac{1}{s}$. For integral control, the control input *u* is proportional to the integral of the error. $e = (r - y)$ so $u = K_i \int (r - y)$ and $U = \frac{K_i(R - Y)}{s}$ where the loop gain is K_i. Now, $Y = \frac{1}{s}(U - W) = \frac{1}{s}\left(\left(\frac{K_i(R - Y)}{s}\right) - W\right)$, which can be rearranged to get

$$Y = \left(\frac{K_i}{s^2 + K_i}\right)R - \left(\frac{s}{s^2 + K_i}\right)W$$

To study the step response of this controller, set *W = 0* and *R = 1/s*. Then, $Y = \frac{K_i}{s(s^2 + K_i)} = \frac{1}{s} - \frac{s}{s^2 + K_i}$. Taking the inverse Laplace transform, we find $y(t) = (1 - \cos\sqrt{K_i}t)$. That is, when there is no disturbance, if the input command is a step signal, then the output is the step signal along with a persistent sinusoidal oscillation with magnitude 1 and period $\frac{1}{\sqrt{K_i}}$. The transient response neither grows nor decays with time: we call a system with such a response **marginally stable**. This demonstrates our claim that the introduction of integral-mode control diminishes system stability. Note that the larger the loop gain, the smaller the magnitude of the response and the slower the oscillation frequency.

■

8.7.3 Derivative mode control

Derivative mode control recognizes the fact that control actions (for the same degree of error) should differ depending on whether the error trend is increasing or decreasing. If the error trend (its derivative) is increasing, then the controller needs to take stronger corrective action, but when it is decreasing, it must back off the control action to prevent over-reaction. Thus, derivative mode control tends to dampen the actions of a controller.

Derivative mode control cannot be used in isolation because a constant steady-state error, no matter how large, has a zero derivative. To avoid steady-state error, derivative mode control is combined with either proportional mode or both integral and proportional mode control.

Note also that derivative model control is very sensitive to high-frequency noise in the measurement process: spikes in measurement noise are interpreted as a sharp increase in system error. In practice, to avoid this problem measurements must be filtered using a low-pass filter before using them as the basis of derivative mode control.

Mathematically, with derivative mode control, $u = K_d\left(\frac{de}{dt}\right)$ or $U = K_d Es$ where K_d is the loop gain. We demonstrate derivative model control through an example.

EXAMPLE 11: DERIVATIVE MODE CONTROL

Consider a derivative controller for the web server in Example 2 with transfer function $G(s) = \frac{1}{s}$. For derivative control, the control input is proportional to the derivative of the error so that $U = K_d(R - Y)s$ where the loop gain is K_d. Now, $Y = \frac{1}{s}(U - W) = \frac{1}{s}((K_d(R - Y)s) - W)$, which can be rearranged to get

$$Y = \left(\frac{K_d}{1 + K_d}\right)R - \left(\frac{1}{s(1 + K_d)}\right)W$$

To study the step response of this controller, we set $W = 0$ and $R = 1/s$. Then, $Y = \frac{K_d}{s(1 + K_d)}$. Taking the inverse Laplace transform, we find $y(t) = \frac{K_d}{1 + K_d}$. That is, when there is no disturbance, if the input command is a step signal, then the output is an attenuated step signal with no transient response. Note that the larger the loop gain, the smaller the attenuation in the input signal.

We can investigate the steady-state error in the step response (assuming zero disturbance) using the final value theorem. We have $Y = \frac{K_d}{s(1 + K_d)}$, so from the final value theorem, the asymptotic value of $y(t)$ is $\frac{K_d}{(1 + K_d)} \neq 1$, which demonstrates our earlier claim that derivative mode control results in a non-zero steady-state error. However, note that the steady state error is a decreasing function of the loop gain, so the steady state error can be reduced by increasing the loop gain.

■

8.7.4 Combining modes

Each of the three basic control modes--proportional, integral, and derivative--has its advantages and disadvantages. Therefore, it is common to combine modes in a single so-called **PID controller**. Such a controller has three control parameters that correspond to loop gains in each mode. That is, for an error E, we have

$$U = \left(K_p + \frac{K_i}{s} + K_d s\right)E$$
$$D = \left(\frac{K_d s^2 + K_p s + K_i}{s}\right) \qquad \text{(EQ 18)}$$

The relative contributions of the three modes can be adjusted using the gain factors. Intuitively, the use of proportional mode corrects for error, the use of integral mode removes steady-state error, and the use of derivative mode corrects for instability that may be produced by the integral mode.

8.8 Advanced Control Concepts

Having studied the basic PID control, we now examine, at an intuitive level, some more advanced control concepts that are applicable to control systems in computer networks.

8.8.1 Cascade control

When trying to accomplish a complex control objective for a plant or process that has multiple time-scales of behaviour, it is often useful to layer control. For example, with two layers of control, a lower-layer controller, operating at a faster time-scale, controls a portion of the system to achieve a desired setpoint, and the higher-layer controller, operating at a slower time-scale, dynamically chooses the setpoint to achieve the top-level control objective. This is called **cascade control**.

EXAMPLE 12: CASCADE CONTROL

Consider the use of cascade control to optimise the communication between a web browser client and a web server. A web browser can have multiple connections open to multiple web servers, where each connection queries for and fetches a set of data objects to be rendered on the screen[4]. Ideally, the connections should make progress at rates proportional to the amount of data being transferred over the connection, so that all the elements of the web page can be rendered at the same time.

Recall that query request rate control operates at the tens to hundreds of milliseconds timescale and adjusts the sending rate from a source to ensure that the buffer occupancy level at the web server is close to a desired setpoint. Recall from Example 2 that the greater this setpoint, the greater the achievable end-to-end throughput. This suggests the use of a higher-layer controller that dynamically modifies the buffer occupancy level setpoint to achieve the overall control objective. Specifically, the higher-layer controller has, as its control goal, the desire to have all connections make equal fractional progress on their queries. The error term for this controller is the difference between the actual fractional progress made by a connection and the mean fractional progress made by all the clients. If this error term is negative, the connection is a lagging connection, and its setpoint is increased using proportional, integral, or differential mode control. On the other hand, if this error term is positive, the connection is a leading connection, and its setpoint is similarly decreased.

To make this example concrete, consider a browser that has three connections to three different web servers, where the first connection fetches 3 query responses, the second fetches 5 query responses, and the third 10 responses. Therefore, they should make progress at the relative rates of 3:5:10. Suppose that at some point in time the higher layer controller finds that the first connection has retrieved 2 responses, the second has retrieved 3 responses and the third has retrieved 5 responses. The fractional progress made, therefore, is 2/3:3/5:5/10 = 0.66 : 0.6 : 0.5. The mean progress is 0.59. Therefore, the higher-layer controller decreases the setpoint of the first connection, leaves the second connection’s setpoint nearly unchanged, and increases the setpoint of the third connection.

■

8.8.2 Control delay

Our discussion so far has assumed that the measurement of the plant output is instantaneously available to the controller. This is, of course, almost never the case. Delays in measurement are particularly challenging for controllers developed for computer networks, where measurement delays can be significant. We first describe the mathematical modelling of measurement delay, then describe the effects of delay on control.

Measurement delays are easy to represent mathematically. Recall from Table 3 on page 152 that a delay of τ seconds corresponds to multiplying the Laplace transform of a signal by $e^{-s\tau}$. Therefore, with feedback delay, we find that the error E is given by

$$E = R - e^{-s\tau}Y \tag{EQ 19}$$

We can then use this input as the basis for, say, proportional control, where $U = K_pE = K_p(R - e^{-s\tau}Y)$. The subsequent analysis of this system proceeds as before, except that going from the Laplace domain to the time domain is complicated because we cannot use partial fraction expansion, which only apply to ratios of polynomials. Instead, we can use the **Pade approximations** of the exponential function:

4. As a simplification, we assume that all queries are equal-sized and take equal amounts of time to process, so that we only need to control the number of queries, independent of their size. Generalizing from this simplified case to the true situation is left as an exercise to the reader.

$$e^{-s\tau} \approx \frac{1 + \frac{(-s\tau)}{2}}{1 - \frac{(-s\tau)}{2}}$$

$$\approx \frac{1 + \frac{(-s\tau)}{2} + \frac{(-s\tau)^2}{12}}{1 - \frac{(-s\tau)}{2} + \frac{(-s\tau)^2}{12}}$$

which replace the exponential with increasingly accurate polynomial ratios. More elaborate Pade approximations can be found in the literature.

The general treatment of control systems with delays is complex and beyond the scope of this book. However, two observations are relevant. First, a practical approach to deal with delays is to predict the current system state from past measurements, using an appropriate predictor. Once this is done, then the problem reduces to the standard zero-delay control system. Many sophisticated techniques for time-series prediction can be used for this purpose. Second, the general effect of delays is that the controller does not know the response to its prior control action. To prevent osciallations (and instability), it may choose to be cautious in its actions, delaying its next control action until it has learnt the respone to its prior action. This leads to a fundamental tradeoff between responsiveness and stability. In most practical systems, stability is more important and comes at the cost of decreased responsiveness. For example, in TCP, to dampen the control input, control actions are taken no faster than once per round-trip-time, and the response to a loss event is artificially delayed by up to half a second. This reduces oscillations, though it reduces responsiveness.

EXAMPLE 13: EFFECT OF CONTROL DELAY

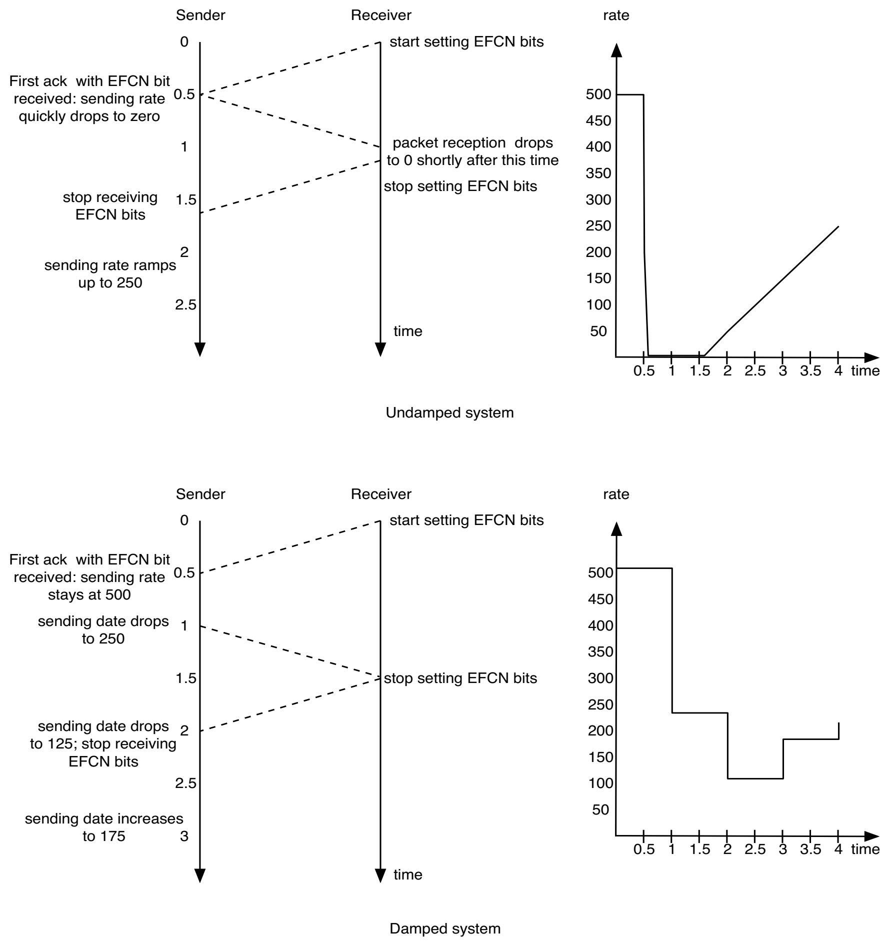


FIGURE 8. **Figure for Example 13**

Consider a transport-layer flow control algorithm that takes control actions in one of two ways: if it receives an acknowledgement with the Explicit Forward Congestion Notification (EFCN) bit set to zero, indicating that the corresponding data packet did not experience congestion, then it increases its sending rate additively; otherwise, it decreases its sending rate multiplicatively. This is a controller with an undamped response. In contrast, consider another flow control algorithm that collects the EFCN bits set on packets sent during each window of duration one round trip time (measured from the start of the connection), then takes control actions once per round-trip-time depending on whether the majority of the bits were set. This controller dampens its response.

Let us do a thought experiment to study the behaviour of the two systems. To exaggerate the difference in their behaviour, we will assume that the source has an infinite number of packets to send and that its fastest sending rate is 1000 packets per second. Assume also that the delay from the time that an EFCN bit is set to the time that it is received by a sender is 0.5 seconds, and that the end-to-end delay is also 0.5 seconds (i.e., the single bottleneck link is next to the receiver), so that the round-trip time (RTT) delay is 1s. Finally, assume that the capacity of the bottleneck link is 500 packets per second, the additive increase factor is 50 packets/RTT, and that the multiplicative decrease factor is 0.5.

We conveniently define the system to be in equilibrium when the source is sending packets at 500 packets/s and the bottleneck link is serving packets at 500 packets/s. Consider the time-evolution of control for the damped and undamped controllers when the system is initially at equilibrium and when at time 0 the bottleneck service rate decreases abruptly to 250 packets/s due to the arrival of another flow.

With both the undamped and the damped controllers, packets with EFCN bits arrive to the sender at time 0.5. With the undamped system, on the arrival of the first acknowledgement packet with an EFCN bit set, the source immediately reduces its sending rate to 250 packets/s, further reducing its rate with every additional acknowledgement. Note that in equilibrium, the source has 500 packets outstanding, all with bits set. So the source reduces its rate to nearly 0 by time 1.5, when the last of the EFCN bits is received. Then, no more bits are set, so the source increases its rate by 50 packets/RTT, reaching its new equilibrium value of 250 at time 6.5. The subsequent time evolution is straightforward. Note that the effect of this immediate response is to quickly drain the bottleneck queue of any packets that are buffered there due to the drop in the link service rate.

With the damped system, the source continues to send at the rate of 500 packets/s until time 1 because it takes control actions only at time 1, 2, 3, and so on. At time 1, it has received 500 acknowledgments, of which 250 have bits set and 250 do not. Assume that, in this borderline case, the source reduces its rate to 250 packets/s at time 1. In the time interval from 1 to 2 seconds, it sends 250 packets, but receives 500 packets with EFCN bits set (left over from the first round trip time). So, it reduces its rate once more to 125 packets/s. No more bits are set subsequently, and the source ramps up its rate to 250 over the next three seconds. Note that, due to the delayed response, the bottleneck queue is drained much more slowly than with the undamped system.

Damping decreases responsiveness so the effect of a disturbance persists longer, but the magnitude of the impulse response diminishes and the system returns to equilibrium sooner. This is characteristic of any control system with control delays.

■

8.9 Stability

Consider a system that is not subject to a disturbance for some length of time. We expect such a system to eventually settle down so that its output y matches the command input r. A system whose output persists at its setpoint is said to be in **equilibrium** or **relaxed**. By convention, we denote this state (and the corresponding command and output) by a **zero** vector. In many cases, this ideal equilibrium state may never be achieved. Yet, it is important because it is the state that a system *strives* to achieve.

Suppose that a system in equilibrium is subjected to a time-limited disturbance. Ideally, if the disturbance is small then the change in the output should also be small. Moreover, the system should rapidly return to equilibrium when the disturbance ceases. We call such an ideal system a **stable** system.

To further grasp the concept of stability, imagine a cone that is lying on its side and another that is balanced on its point. A small disturbance to a cone balanced on its point results in a large change (it falls down) and, when the disturbance is removed, the cone does not return to its previous balance. In contrast, a small disturbance to a cone lying on its side results in a small change in position (although it may not return to the original equilibrium state when the disturbance is removed).

Besides these two responses to a small disturbance, a third possible response is for the system to oscillate, where the magnitude of the oscillations decays over time. This is akin to a rocking chair or guitar string subjected to a small impulse disturbance. Depending on the desired output behaviour of the system, such a response could be characterized as either a stable or an unstable response.

When designing a controller, it is very desirable that the controlled system be stable. In this section, we will study the conditions under which a controlled system can be guaranteed to be stable. Before doing so, it is worth distinguishing between two types of stable responses:

- A system is said to be **bounded-input bounded-output** or **BIBO** stable if the output of the system is bounded in magnitude by some value M as long as the disturbance is bounded in magnitude by m
- A system is said to be **zero-input** stable if, for all initial states of the system (including non-equilibrium states), when both the command and disturbance inputs are zero, the system eventually reaches the equilibrium state

Both BIBO and zero-input stability are relatively easy to characterize for LTI systems. For time-varying or non-linear systems, a system may be stable for classes of disturbances but not others, or may be stable for small disturbances but not large ones. Such systems may also have multiple equilibrium states and may move from one equilibrium state to another when subject to small disturbance (much like a round pencil lying on its side when subjected to a small torque). Such multiple equilibria arise naturally in computer networks.

EXAMPLE 14: META-STABILITY IN THE TELEPHONE NETWORK

The telephone network core is organized as a clique where traffic from any node to any other node follows a one-hop path. This routing is simple but inefficient, in that the one-hop path does not make use of capacity available on two-hop paths. Consider a clique where all links have unit capacity so that the one-hop path from any node A to any other node B has capacity 1. If the clique has *n* nodes, there are an additional *n*-2 two-hop paths between them also with unit capacity. So node A could send traffic to node B at a rate as high as *n*-1. However, using two-hop paths comes at the cost using twice the network capacity per traffic flow (i.e., two links instead of one). This has the following effect: if the network has a high traffic load so that most one-hop paths are nearly at capacity, the addition of a small additional load on a one-hop path could cause diversion of the excess load to one of the alternative two-hop paths. If these two-hop paths are also close to capacity, additional traffic on either of these hops would cause the use of additional two-hop paths. At high loads, this process continues until most traffic traverses two-hop paths, reducing network efficiency to 50%. This inefficient state turns out to be persistent: even if traffic volume decreases, most traffic would still traverse two hop paths. Thus, the system has more than one equilibrium[5].

Despite the presence of multiple equilibria, note that the system is both BIBO and zero-input stable. To see this, we define the input to the system to be its traffic load and the output as the carried load. The system is in its relaxed state when traffic load is zero, corresponding to a zero carried load. If a small disturbance adds traffic to the system, it is carried on a one-hop path and the carried load equals the traffic load. Therefore, as long as the input is bounded, so is the output, showing BIBO stability.

To prove zero-input stability, consider a network where the traffic load is non-zero at time zero but there is no additional load after time zero. In this case, once the traffic is delivered to its destination, the network always eventually returns to the zero state, as desired.

The instability, or rather, change in state from one equilibrium state to another, is caused by a disturbance when the system load is *high*. We see, therefore, that BIBO and zero-input stability criteria are rather weak. Perhaps a better way to view them is that a system that fails even these two weak stability criteria is not particularly robust.

■

EXAMPLE 15: COMPUTER NETWORK STABILITY

A proof along the lines of Example 14 shows that nearly all computer networks are trivially both BIBO and zero-input stable if we accept the definition of the output of the system as the overall carried load and the zero state as the zero-load state. We could also define the “output” of the network as the sum of all the router buffer occupancies, so that a congested network would have a large output amplitude. But all buffers, in practice, are bounded in size. So, no matter what the input load, they cannot exceed a pre-defined constant. From this perspective, again, computer networks are trivially stable. (The most common manifestation of marginal stability is probably the presence of a persistent oscillation, as we saw in Example 5.)

■

8.9.1 BIBO Stability Analysis of a Linear Time-invariant System

This section presents the conditions under which a continuous single-input single-output LTI system is BIBO stable. If the system has input $r(t)$, output $y(t)$, and impulse response $g(t)$, then, from Equation 28 on page 137:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)r(t-\tau)d\tau = \int_{0}^{\infty} g(\tau)r(t-\tau)d\tau$$

5. This unstable equilibrium can be avoided by reserving a fraction of each link’s capacity for one-hop paths.

where the second step follows from the fact that for a causal system the impulse response is zero when $t < 0$. The magnitude of the output is bounded by

$$\left|\int_0^{\infty} g(\tau)r(t-\tau)d\tau\right| \le \int_0^{\infty} |g(\tau)||r(t-\tau)|d\tau \le r_{max}\int_0^{\infty} |g(\tau)|d\tau$$

where r_{max} is the maximum value of the input. The output is bounded as long as the quantity $\int_0^{\infty} |g(\tau)|d\tau$, which is the integral of the absolute value of the impulse response $g(t)$, is bounded. To compute this quantity, we first consider the Laplace transform of the impulse response, i.e., the transfer function $G(s)$. It can be shown that the transfer function of any LTI system that can be modelled as a set of differential equations can be written as the ratio of two polynomials in *s:*

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1+\ldots+a_n}} \qquad \textbf{(EQ 20)}$$

The impulse response $g(t)$ of any real system is real, that is, it has no complex coefficients. Therefore, by the definition of the Laplace transform, its Laplace transform $G(s)$ also has real coefficients and the roots of its denominator polynomial $s^n + a_1 s^{n-1+\ldots+a_n}$ must be either real and distinct, real and repeated, or conjugate complex (and it cannot have non-conjugate complex roots). Using partial fraction expansion (see Section 8.12 on page 255), such a transfer function can be written in the form:

$$G(s) = \begin{cases} \dfrac{a_1}{(s-\alpha_1)} + \dfrac{a_2}{(s-\alpha_2)} + \ldots + \text{other distinct real roots} \\ \dfrac{a_k + jb_k}{(s-(\alpha_k + j\beta_k))} + \dfrac{a_k - jb_k}{(s-(\alpha_k - j\beta_k))} + \ldots + \text{other complex conjugate roots} \\ \dfrac{a_{m(r)}}{(s-\alpha_m)^r} + \dfrac{a_{m(r-1)}}{(s-\alpha_m)^{r-1}} + \ldots + \dfrac{a_{m(1)}}{(s-\alpha_m)} + \text{other repeated roots} \end{cases} \qquad \textbf{(EQ 21)}$$

What does this look like in the time domain? The Laplace transform is linear, so we can take the inverse transform term by term.

- For a term of the form $\frac{a_i}{(s-\alpha_i)}$, the corresponding time-domain term is $a_i e^{\alpha_i t}$. This decays exponentially to zero whenever $\alpha_i < 0$, i.e., the root lies on the X axis in the left half of the complex *s* plane. In this case, the integral of the absolute value of the impulse response is bounded and the system is stable. If $\alpha_i = 0$, then the output is constant, the integral is unbounded, and the system in unstable.
- For complex conjugate terms of the form $\frac{a_k + jb_k}{(s-(\alpha_k + j\beta_k))} + \frac{a_k - jb_k}{(s-(\alpha_k - j\beta_k))}$, the corresponding time-domain term is $2e^{\alpha_k t}(a_k \cos\beta t - b_k \sin\beta t)$. Note that the expression in the parenthesis is the difference between two sinusoids and therefore is bounded in magnitude. The output corresponding to this pair of terms decays to zero, corresponding to a bounded integral, if and only if α_k, the real part of the complex conjugate root, is negative, so that the pair of roots lie in the left half of the complex *s* plane. If the real part is zero (i.e., the root lies on the *j* axis), then the system oscillates and this oscillation does not decay over time. This system is marginally stable.
- A similar but somewhat more complicated analysis shows that if the system has repeated real or complex roots, then the corresponding time-domain output decays to zero, and the integral of the impulse response is bounded, if and only if the

real part of the root lies strictly in the left half of the complex *s* plane: repeated roots on the *j* axis lead to marginally stable systems and repeated roots in the right half of the complex *s* plane lead to unstable systems.

Summarizing, we see that an LTI system is BIBO stable if and only if *all* the roots of the denominator polynomial of its transfer function lie in the left half of the complex *s* plane. If even one root lies on the right half of the plane, then the system is unstable. For the special case of conjugate roots on the j axis: (a) if the pairs of conjugate roots on this axis are distinct, then the system is oscillatory but bounded (b) if the pairs of conjugate roots on this axis are repeated, then the system is unstable. This simple characterization allows us to quickly determine whether a single-input single-output LTI system is BIBO stable simply by finding the roots of its denominator polynomial: these are also called the **poles** of the system. This can be determined using numerical algorithms such as the Jenkins-Traub algorithm.

EXAMPLE 16: BIBO STABILITY OF AN LTI SYSTEM

Determine the stability of an LTI system with denominator polynomial $3s^4 + 5s^3 + 7s^2 + 12s + 1$.

Solution: Using a numerical solver, we find that the roots of this polynomial are -0.08753, -1.6486, $0.03475 \pm j1.51936$. Since the conjugate roots lie in the right half plane, the system is unstable.

■

EXAMPLE 17: BIBO STABILITY OF AN LTI SYSTEM

Determine the stability of an LTI system with denominator polynomial $s^4 + 50s^2 + 625$.

Solution: We factor the polynomial as $(s^2 + 25)^2 = ((s + j5)(s - j5))^2$, so that there are two repeated roots at $\pm j5$. The system has repeated roots on the *j* axis and is therefore unstable.

■

EXAMPLE 18: STABILITY OF A WEB SERVER CONTROLLER

Determine the stability of the proportional, integral, and derivative mode controllers of Examples 9-11.

Solution:

For the proportional mode controller, we have $Y = \frac{R}{1 + s\tau} - \frac{W}{s(1 + s\tau)}$. Ignoring the disturbance, the transfer function of the controlled system is given by $G = \frac{1}{1 + s\tau}$. The root is at $s = -\frac{1}{\tau}$. Since $\tau > 0$ (it is a time constant), the system's single pole (which corresponds to this root) is always in the left hand of the complex *s* plane and is therefore always BIBO stable.

For the integral mode controller, we have $Y = \left(\frac{K_i}{s^2 + K_i}\right)R$ so that $G = \left(\frac{K_i}{s^2 + K_i}\right)$ and the roots are at $\pm jK_i$. The conjugate poles on the *j* axis indicate that the system is oscillatory and therefore bounded but BIBO marginally stable.

For the derivative mode controller, we have $Y = \left(\frac{K_d}{1 + K_d}\right)R$, so $G = \left(\frac{K_d}{1 + K_d}\right)$ which is a constant that is never 0 (i.e., the system has no pole). The effect of the controller is to act as a constant scalar multiplier to the Laplace transform of the control input. If the input is bounded, so will the output, which implies that the system is BIBO stable.

■

There is one corner case that requires special care. Suppose $G(s)$ has a single set of complex conjugate roots on the j axis, which corresponds to an marginally stable bounded oscillatory system. Suppose it excited by an input $R(s)$ that also has complex conjugate roots at the same locations on the j axis. Then, $Y(s) = G(s)R(s)$ has conjugate roots of multiplicity 2 on the j axis, making it both unbounded and unstable. The physical interpretation of this phenomenon is that an input of a sinusoid at a frequency that exactly matches one of the natural frequencies of the system leads to instability due to **resonance**.

8.9.2 Zero-input Stability Analysis of a SISO Linear Time-invariant System

We now turn our attention to zero-input stability of a single-input-single-output system. In this context, 'zero-input' means that the command given to the system (i.e., r) is 0 and does not change over time. Recall that a system is zero-input stable if, in the absence of any input or disturbance, no matter what the initial state at time zero, the system always eventually returns to its equilibrium state. It is convenient to study zero-input stability using the state-space representation of a system (Equation 2 on page 227):

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{F}\boldsymbol{w} \\ y &= \boldsymbol{c}\boldsymbol{x} + du + \boldsymbol{e}\boldsymbol{w} \end{aligned}$$

We have $\boldsymbol{w} = u = 0$, so the state evolution is given by:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

Taking the Laplace transform of both sides,

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s)$$

$$\begin{aligned} \boldsymbol{X}(s) &= (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) \\ &= \frac{[C(s\boldsymbol{I} - \boldsymbol{A})]^T}{|s\boldsymbol{I} - \boldsymbol{A}|}\boldsymbol{x}(0) \end{aligned} \quad \text{(EQ 22)}$$

where the second step expands the inverse in terms of the co-factor and the determinant (see Section 3.4.4 on page 85). The numerator is a matrix whose elements are polynomials in s and the denominator is polynomial in s. Therefore, $\boldsymbol{X}(s)$ is a vector whose elements are ratios of two polynomials in s. As before, we can expand each element as a partial fraction. For the system to be zero-input stable, we require that every element of the state vector eventually decay to zero. This will be true if and only if the roots of the equation $|s\boldsymbol{I} - \boldsymbol{A}| = 0$, which are the eigenvalues of $\boldsymbol{A}$, lie in the left half of the complex plane. Because of the importance of this equation, it is also called the **characteristic equation** of the system. Under mild conditions, it can be shown that the roots of the characteristic polynomial are, in fact, identical to the poles of the corresponding transfer function. Therefore, under these conditions, a system is either both BIBO and zero-input stable or neither and only one of the tests needs to be carried out.

8.9.3 Placing System Roots

The roots of the denominator of the system transfer function or of the characteristic equation are critical in determining its stability. If a system designer has one or more free parameters or 'control knobs' to control system behaviour, these must be chosen such that these roots for the controlled system lie in the left half of the complex plane. This is called '**pole placement**,' since each root corresponds to a system pole.

Before the advent of digital computers, finding the roots of the characteristic equation was nearly impossible other than for trivial systems. System engineers, therefore, came up with approximation techniques such as the root locus method, Bode plots, and Nyquist plots to roughly determine root locations as a function of the system parameters. Today, sophisticated numerical algorithms have simplified the process of finding the roots of any polynomial. Nevertheless, an appreciation of the influence of the system poles on its behaviour is critical in designing a robust and stable control system. We will not explore this complex topic in any detail. However, it should be clear that the further to the left of the j axis that we place a pole, the faster the decay of impulse response transients, which makes the system more responsive. However, this can also lead to instability due to over-correction. The goal of the system designer is to balance responsiveness with stability.

8.9.4 Lyapunov stability

Most real systems, such as computer networks, are neither linear nor time-invariant. Yet, we would like to understand whether they are stable and to design stable control algorithms for them. Here, we outline the important concept of Lyapunov stability that can be used to study such systems.

Recall from Section 8.3 on page 225 that we can represent a system using its state variables in the form of a state vector. The system's state space is the set of all values that can be assumed by its state vector. We declare the ideal system state to be its equilibrium or relaxed state when all the system state variables are at their nominal zero value, that is **x** = **0**. The goal of the controller, then, is to move the system from its current state to this ideal state: the path so taken over time is called the system's **trajectory**. The trajectory is much like the path taken by a marble as it rolls down a slope, except that the system's position at any time is described by a state vector rather than a scalar.

A system is said to be **stable in the sense of Lyapunov** if all system trajectories can be confined to a bounded part of the state space as long as the system was initially confined to some other part of the state space. In other words, if we are allowed to choose the initial conditions, a Lyapunov stable system never wanders too far from equilibrium.

More formally, consider a system described by:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}(t)) \\ \boldsymbol{f}(x(t)=\mathbf{0}) &= \mathbf{0} \end{aligned} \qquad \textbf{(EQ 23)}$$

The first equation says that the system state trajectory can be described in terms of a first-order differential equation. The second equation says that if the system is in equilibrium, then it continues to be in equilibrium (it is a fixed point of **f**). This of course ignores disturbances.

Such a system is said to be **stable in the sense of Lyapunov** at $\boldsymbol{x} = \mathbf{0}$ if, for every real number $\varepsilon > 0$ there exists another real number $\delta > 0$ such that if $||\boldsymbol{x}(0)|| < \delta$ then for all $t \geq 0$ $||\boldsymbol{x}(t)|| < \varepsilon$, where $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$. So, as long as we can choose the initial conditions to be in a hypersphere centered at the origin and with radius δ, the system's trajectories will be confined to a hypersphere centered at the origin and with radius ε.

The system is **asymptotically stable** at $\boldsymbol{x} = \mathbf{0}$ if it is stable in the sense of Lyapunov, and, moreover, $\boldsymbol{x}(t) \to \mathbf{0}$ as $t \to \infty$. That is, we can choose initial conditions so that not only is further evolution bounded, but it also eventually converges to the equilibrium state.

Our overall goal is to prove that a non-linear or time-variant system is either stable in the sense of Lyapunov or is asymptotically stable. To do so, we define an auxiliary scalar function called **Lyapunov function** $V(\mathbf{x})$. The Lyapunov function can be nearly arbitrarily defined and corresponds to the amount of 'energy' in the system: the greater the energy, the larger the value of $V(\mathbf{x})$. We can think of a disturbance as adding energy to the system that is dissipated by the actions of the controller, so that the system eventually settles down to equilibrium. It can be shown that a system is stable in the sense of Lyapunov if it is possible to define a Lyapunov function for it such that the Lyapunov function decays over time (due to the actions of the controller) along all system trajectories.

More formally, the Lyapunov stability theorem states that a system described by Equation 23 is stable in the sense of Lyapunov at $\mathbf{x} = \mathbf{0}$ if it is possible to define a scalar function $V(\mathbf{x}(t))$ such that:

- $V(\boldsymbol{x}(t)=\mathbf{0}) = 0$
- $V(\boldsymbol{x}(t)) > 0;\ \boldsymbol{x}(t) \neq \mathbf{0}$
- $V(\boldsymbol{x}(t))$ is continuous and has continuous partial derivatives with respect to each component of **x**
- $\dfrac{dV(\boldsymbol{x})}{dt} \leq 0$ for all system trajectories

Moreover, if $\dot{V}(\boldsymbol{x}) < 0$ (strictly less than zero) along all system trajectories, then the system is asymptotically stable. This is also called **Lyapunov 2-stability** or stability using the second method of Lyapunov.

The use of a Lyapunov function is both powerful and somewhat open-ended: if such a function can be found, then even complex systems can be proved stable. On the other hand, finding such a function is a hit-or-miss process, other than for the simplest cases. In practice, we can use Lyapunov stability as a design tool by defining a plausible energy function and showing that control actions tend to always decrease system energy.

8.10 State-space Based Modelling and Control

Our discussion thus far has focused on the so-called *classical* approach to control theory. Although intuitive and powerful, this approach assumes a single-input single-output system and therefore cannot be easily applied to systems with multiple inputs and outputs. Moreover, it focuses only on the stability of the output. In some cases, although the system output may be bounded, internal state variables may oscillate or diverge (for instance, consider a network whose throughput is stable, but whose routing oscillates, causing highly variable delays). Finally, it does not allow for the design of controllers that use vector measurements of the internal system state: they can only be based on the observation of a single output variable. These limitations of classical control led to the development of 'modern' or state-space based control in the 1950s. This approach relies heavily on techniques from linear algebra, so the reader who wishes to master this section will need to be acquainted with the concepts in Chapter 3. We touch upon some elementary techniques in state-space based control in this section, deferring details to more advanced texts on this subject.

8.10.1 State-space based analysis

We begin with the canonical representation of a linear single-input single-output linear time-invariant system using the state-space model (Equation 2):

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{F}\boldsymbol{w} \\ y &= \boldsymbol{c}\boldsymbol{x} + du \end{aligned} \tag{EQ 24}$$

We first study the behaviour of the unforced system (i.e., with a zero input) in the absence of disturbances. In this case, the system reduces to

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{EQ 25}$$

This looks rather like a first-order homogeneous differential equation, so it is natural to expect that a solution would be of the form

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{k} \tag{EQ 26}$$

where the matrix exponential $e^{\boldsymbol{A}t}$ generalizes a standard exponential and can be viewed as a compact representation of the infinite series:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \frac{\boldsymbol{A}^3t^3}{3!} + \ldots \tag{EQ 27}$$

and $\boldsymbol{k}$ is a suitably chosen constant vector. Direct substitution shows that this indeed a solution of the unforced system. Moreover, by setting t to zero in Equation 26, it is easy to see that $\boldsymbol{k} = \boldsymbol{x}(0)$, so that the solution of the system is

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}(0) \tag{EQ 28}$$

We view the matrix $e^{\boldsymbol{A}t}$ as converting the initial state $\boldsymbol{x}(0)$ to the state at time t, i.e., $\boldsymbol{x}(t)$. Therefore, it is also called the **state transition matrix**.

Computing the matrix exponential can be difficult. There are two ways around it. The first is to transform the $\boldsymbol{A}$ matrix into a form whose powers are easy to evaluate using the similarity transformation discussed in Section 3.5.7 on page 96. This transformation diagonalizes $\boldsymbol{A}$ so that its powers, and the infinite sum, are easily computed (see Exercise 12). Note that the matrix can be diagonalized only if it has no repeated eigenvalues. Otherwise, the best we can do is to put it in Jordan Canonical Form, whose powers can also be computed, albeit with some effort. In this case, computing the infinite sum of Equation 27 is also somewhat more complex.

To avoid this computation, we can solve the system using the Laplace transform. Taking the Laplace transform of both sides of Equation 25, we get:

$$
\begin{aligned}
s\boldsymbol{X}(s) - \boldsymbol{x}(0) &= \boldsymbol{A}\boldsymbol{X}(s) \\
\boldsymbol{X}(s) &= (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) \\
\boldsymbol{x}(t) &= \mathcal{L}^{-1}((s\boldsymbol{I} - \boldsymbol{A})^{-1})\boldsymbol{x}(0)
\end{aligned}
\tag{EQ 29}
$$

which shows that the time evolution of the state of the unforced system is obtained by taking the inverse Laplace transform of the matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$.

A similar but somewhat more complex analysis allows us to compute state evolution of an SISO system when the input is non-zero as:

$$
\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)d\tau \tag{EQ 30}
$$

The first term of this equation describes how the system would evolve in the absence of control and the second term opens the possibility of setting the system state to any desired setpoint by appropriate choice of the control input u. To study this further, we first describe the conditions under which the system state can be arbitrarily controlled, then study how to choose the control input.

8.10.2 Observability and Controllability

The important principles of **observability** and **controllability** describe the conditions that allow a system to completely controlled: if a system is observable and controllable then it is possible to design a control input that moves the system from any initial state to any desired final state.

A system state $\mathbf{x}(0)$ is said to be controllable at time t if there exists a piecewise continuous input u defined over the time period $[0, t]$ such that, with this input, the system moves from state $\mathbf{x}(0)$ at time 0 to any desired state $\mathbf{x_d}(t)$ at time t. The system is said to be **completely controllable**, or simply, **controllable**, if every system initial state is controllable. It can be shown that this is true if and only if the matrix

$$
\left[\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \boldsymbol{A}^2\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{n-1}\boldsymbol{b}\right]
$$

has rank n. If $\boldsymbol{A}$ has distinct eigenvalues, then this test can be simplified as follows: let $\boldsymbol{P}$ be a non-singular matrix that diagonalizes $\boldsymbol{A}$. Then, the system is controllable if and only if $\boldsymbol{P}^{-1}\boldsymbol{b}$ has no rows that are all zero.

If the knowledge of the system state equations, the output y, and the input u over a finite time interval $[0, t]$ allows us to completely determine the initial state $\mathbf{x}(0)$, then that initial state is said to be observable. A system is **completely observable**, or simply, **observable**, if all initial states are observable. It can be shown that a system is observable if and only if the matrix

$$
\begin{bmatrix}
\boldsymbol{c} \\
\boldsymbol{c}\boldsymbol{A} \\
\boldsymbol{c}\boldsymbol{A}^2 \\
\ldots \\
\boldsymbol{c}\boldsymbol{A}^{n-1}
\end{bmatrix}
$$

has rank n. If $\boldsymbol{A}$ has distinct eigenvalues, then this test can be simplified as follows: let $\boldsymbol{P}$ be a non-singular matrix that diagonalizes $\boldsymbol{A}$. Then, the system is controllable if and only if the matrix $\boldsymbol{cP}$ has no zero columns.

8.10.3 Controller design

We now discuss how to design a controller for an observable and controllable SISO system. Without loss of generality, we will assume that the desired final state is the system equilibrium state, i.e., $\mathbf{x} = 0$. We will also focus on a particularly simple form of control input u, of the form

$$u = -\boldsymbol{k}\boldsymbol{x} \qquad \text{(EQ 31)}$$

where $\boldsymbol{k}$ is a constant row vector. That is, the scalar control input u is a linear combination of the values assumed by the state elements. First, assume that the disturbance is zero. Then, from Equation 24, the state equation is $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$. Substituting Equation 31, we get the control equation for the closed loop control system to be:

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k})\boldsymbol{x}$$

By analogy with Equation 29, we find

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}((s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}))^{-1})\boldsymbol{x}(0)$$

$$= \mathcal{L}^{-1}\left(\frac{[C(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}))]^T}{|s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k})|}\right)\boldsymbol{x}(0)$$

where, in the second step, as in Equation 22, we have expanded the inverse in terms of its cofactor and determinant. Note that $\boldsymbol{A} - \boldsymbol{bk}$ is a constant matrix. Therefore, as before, it can be shown that the term in the parentheses is a ratio of two polynomials in s. The poles of the system, and hence its stability, are determined by the roots of the characteristic equation of the controlled system $|s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k})| = 0$. Our task reduces to choosing $\boldsymbol{k}$ such that the roots of the characteristic equation are suitably placed in the left hand of the complex s plane. It can be shown that if a system is controllable, this can always be done (details can be found in any standard text on modern control).

The analysis thus far is for a single-input-single-output system. When dealing with multiple inputs and outputs, the matrix formulation allows us to easily generalize this analysis. Specifically, the vectors $\boldsymbol{b}$ and $\boldsymbol{k}$ are replaced by the $\boldsymbol{B}$ and $\boldsymbol{K}$ matrices and pole placement involves choosing the elements of $\boldsymbol{K}$ such that the poles of the characteristic equation $|s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})|$ are in the left half of the complex s plane.

In addition to achieving the desired output value $\boldsymbol{y}$ and placing the system poles in the left half plane, state-space based control can also be used simultaneously achieve two other objectives: minimizing the magnitude of the deviation of the system from the equilibrium state and minimizing the magnitude of the control input. The first objective bounds the deviation of the system's internal state, not just the output magnitude. The second objective prevents the system from becoming non-linear due to an overly large input. These objectives are typically expressed as a minimization of the sum

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ are diagonal matrices. A little thought shows that this represents a sum of the form $\sum q_{ii}x_i^2 + \sum r_{ii}u_i^2$, which is the square of the Euclidean distance from the $\boldsymbol{x}$ and $\boldsymbol{u}$ vector from $\boldsymbol{0}$, that is, their deviation from equilibrium, which can always be chosen to be at the origin. Such forms are called **quadratic forms** and the control systems that minimize these quadratic forms are called **linear quadratic controllers**. There is a rich literature on this topic that can be found in any advanced text on control theory.

In designing practical controllers, we often run into the problem that the state is unobservable. In this case, it is necessary to estimate the state from the output. In this case, the control problem proceeds in two steps. In the first step, we estimate the current system state from observations (the **estimation** problem). This itself can be treated as a control problem: our goal here is to minimize the error between the predicted output (from the estimated state) and the actual output (from the true state). In the second step, we control the system assuming that the estimation is accurate (the **control** problem). The **Separation Principle** states that the poles of a system that is designed in this fashion fall into two distinct sets: those that arise from the state estimation process and those that arise from the control process. Therefore, each can be independently determined.

In building a system with separate estimation and control, it is important to ensure that the estimation errors settle down before being fed into the control system. That is, the system time constant of the estimation component should be much smaller than the system constant of the control component. A good rule of thumb is to separate the estimation and control system time constants by one order of magnitude, that is, taking about ten measurements for each control decision. Although, with careful design, it is possible to violate this design law, such systems are usually not robust.

8.11 Digital control

Our discussion so far has focussed on continuous-time systems. Here, we briefly discuss systems whose inputs change only at discrete time intervals, and whose state, therefore, only changes at these time intervals. These are called **discrete-time** control systems or more commonly **digital** control systems.

Digital control systems arise in two contexts: from the sampling of inherently continuous systems and in systems that are inherently digital. The first context is typical in process control, for example in chemical plants, where the pressure or concentration of a reactor vessel is a continuous signal that is sampled by a measurement device. In such cases, the critical problem is to ensure that the sampling process is fast enough to capture the dynamics of the underlying system (the Nyquist criterion requires that the sampling rate be at least twice the fastest relevant dynamics).

Computers and computer networks, in contrast, are inherently digital systems because electronic latches prevent the system state from changing other than when a clock ‘ticks.’ It would appear, therefore, that such systems are best modelled as digital control systems. This reasoning, however, turns out to be naive for two reasons. First, in nearly all modern computers, the clock speeds are in the Gigahertz range, corresponding to a clock ticking faster than once per nanosecond. With such high-resolution clocks, system state can change every few nanoseconds, so the use of continuous-time control is a reasonable (and much more convenient) approximation. Second, in most computer systems, there is not a single system-wide clock, but multiple unsynchronized clocks, each with its own clock speed. Standard digital control systems do not adequately model systems with multiple clocks. For these two reasons, a continuous-time model may still be the best model even for inherently digital systems. Nevertheless, for completeness, we outline the basics of digital control.

The analysis of digital control systems proceeds along lines very similar to that of a continuous-time system. Using the state-space representation, we write the state evolution of a single-input single-output linear time system as:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) + \boldsymbol{F}\boldsymbol{w}(k) \\ y(k+1) &= \boldsymbol{c}\boldsymbol{x}(k) + du(k) + \boldsymbol{e}\boldsymbol{w}(k) \end{aligned} \qquad \textbf{(EQ 32)}$$

Note that, in general, the matrices that describe the system differ from the ones in a corresponding continuous system.

We analyze the evolution of the system, in the absence of a disturbance, by taking the Z transform of both sides of the first equation to get

$$\begin{aligned} zX(z) &= AX(z) + bU(z) \\ X(z) &= (zI - A)^{-1}bU(z) \\ x(k) &= Z^{-1}((zI - A)^{-1}bU(z)) \end{aligned}$$

Note the similarity between this analysis and that of a continuous-time control system. To first approximation, the same techniques of proportional, derivative, integral, and state-space control can be applied to digital control systems. One significant difference is that the stability criterion for BIBO and zero-input stability is that all poles need to be in the unit circle of the complex *z* plane, rather than on the left half of the complex *s* plane.

EXAMPLE 19: PROPORTIONAL MODE CONTROL OF A DIGITAL SYSTEM

To illustrate digital control, this example studies the design of a discrete-time proportional mode controller for the web server in Example 2. We first derive the transfer function in discrete time. Recall that the state space representation of the system in continuous time is:

$$\dot{x} = u - w$$

where x is the buffer occupancy level, normalized so that the zero level corresponds to the desired buffer setpoint r, u is the request rate at time step k, and w is the buffer service rate, modelled as a disturbance. In discrete time, we rewrite this as

$$x(k+1) - x(k) = u(k) - w(k)$$

where we are implicitly assuming that the system state and input values only change at discrete times. Note that the output y is the same as the state variable x so that

$$y(k+1) - y(k) = u(k) - w(k) \qquad \text{(EQ 33)}$$

Suppose that the command r is the desired setpoint of the request buffer. Then, the error is given by $r - y$. For proportional control, we will choose the control input to be proportional to this error. Specifically, suppose that the controller chooses an input such that, if there is no further disturbance, the system returns to the desired setpoint after T time steps. That is,

$$u = \frac{r-y}{T}$$

so that the loop gain is $\frac{1}{T}$. Substituting this in Equation 33, we get

$$y(k+1) - y(k) = \frac{r(k) - y(k)}{T} - w(k)$$
$$Ty(k+1) - (T-1)y(k) = r(k) - Tw(k)$$

Taking the Z transform of both sides, we get

$$Y(z)(Tz - (T-1)) = R(z) - TW(z)$$
$$Y = \frac{R}{Tz-(T-1)} - \frac{TW}{Tz-(T-1)}$$

To study the stability of this controller, we only have to check the location of the system poles, which are obtained by solving $Tz - (T-1) = 0$. This has a single solution of $z = \frac{T-1}{T}$. We see the $|z| < 1$ (the BIBO stability criterion for the Z transform) as long as $T \geq \frac{1}{2}$. Compare this with the criterion in Example 18.

■

8.12 Partial fraction expansion

Partial fraction expansion allows a complex function $f(s)$ that is the ratio of two polynomials in the complex variable s to be represented in an alternative form to which one can easily apply the inverse Laplace transform. To begin with, let

$$f(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1+\ldots+a_n}} = \frac{N(s)}{D(s)}$$

We first find the roots of $D(s)$ by solving the equation $D(s) = 0$ (this is usually computed numerically using algorithms such as the Jenkins-Traub algorithm). Let these roots be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, we can write

$$f(s) = \frac{N(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)}$$

The next steps depend on the nature of the roots. There are three cases.

8.12.1 Distinct roots

In this case, we write

$$f(s) = \frac{a_1}{(s-\alpha_1)} + \frac{a_2}{(s-\alpha_2)} + \ldots + \frac{a_n}{(s-\alpha_n)}$$

where $a_i \in C$. The coefficient a_i is given by

$$a_i = \lim_{s \to \alpha_i} (s-\alpha_i) f(s)$$

EXAMPLE 20: DISTINCT ROOTS

Find the partial fraction expansion of the polynomial fraction $\frac{1}{s^2+3s+2}$.

Solution: We write the fraction as $f(s) = \frac{1}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$. Then,

$a_1 = \lim_{s \to -1} (s+1)\left(\frac{1}{(s+1)(s+2)}\right) = \frac{1}{-1+2} = 1$ and $a_2 = \lim_{s \to -2} (s+2)\left(\frac{1}{(s+1)(s+2)}\right) = \frac{1}{-2+1} = -1$, so that $f(s) = \frac{1}{s+1} - \frac{1}{s+2}$.

■

8.12.2 Complex conjugate roots

Consider the case where $D(s)$ has one pair of complex conjugate roots. Note that they must be conjugate if the coefficients of $D(s)$ are real. Then we write

$$f(s) = \frac{a_1}{(s-(\alpha_1+j\beta_1))} + \frac{a_2}{(s-(\alpha_1-j\beta_1))} + \ldots + \frac{a_n}{(s-\alpha_n)}$$

It can be shown that a_1 and a_2 are the complex conjugates of each other. Moreover,

$$a_1 = \lim_{s \to (\alpha_1+j\beta_1)} (s-(\alpha_1+j\beta_1)) f(s).$$

EXAMPLE 21: COMPLEX CONJUGATE ROOTS

Find the partial fraction expansion of the polynomial fraction $\dfrac{1}{s^2 - 6s + 25}$.

Solution: We write the fraction as $f(s) = \dfrac{1}{(s-(3+j4))(s-(3-j4))} = \dfrac{a_1}{(s-(3+j4))} + \dfrac{a_2}{(s-(3-j4))}$. Then,

$a_1 = \lim_{s \to (3+j4)} (s-(3+j4))\left(\dfrac{1}{(s-(3+j4))(s-(3-j4))}\right) = \dfrac{1}{(3+j4-(3-j4))} = \dfrac{1}{j8}$ and $a_2 = \dfrac{1}{-j8}$, so that

$f(s) = \dfrac{1}{j8(s-(3+j4))} - \dfrac{1}{j8(s-(3-j4))}$.

■

8.12.3 Repeated roots

If $D(s)$ has repeated roots, the partial fraction expansion is somewhat more complex. Suppose that $D(s)$ has r repeated roots α_1 and that the other roots are distinct. Then,

$$f(s) = \frac{a_{1(r)}}{(s-\alpha_1)^r} + \frac{a_{1(r-1)}}{(s-\alpha_1)^{r-1}} + \ldots + \frac{a_{1(1)}}{(s-\alpha_1)} + \frac{a_2}{(s-\alpha_2)} + \ldots + \frac{a_n}{(s-\alpha_n)}$$

where the ith repeated root, $i = 0,1,\ldots,r-1$ is given by:

$$a_{1(r-i)} = \lim_{s \to \alpha_1}\left(\frac{1}{i!}\frac{d^i}{ds^i}((s-\alpha_1)^r f(s))\right)$$

EXAMPLE 22: REPEATED ROOTS

Find the partial fraction expansion of the polynomial fraction $\dfrac{1}{s^3 + 7s^2 + 16s + 12}$.

Solution: We write the fraction as $f(s) = \dfrac{1}{(s+2)^2(s+3)} = \dfrac{a_{1(2)}}{(s+2)^2} + \dfrac{a_{1(1)}}{s+2} + \dfrac{a_2}{s+3}$. Then,

$$a_{1(2-0)} = \lim_{s \to -2}\left(\frac{1}{0!}\frac{d^0}{ds^0}((s+2)^2 f(s))\right) = \lim_{s \to -2}\left(\frac{1}{s+3}\right) = 1$$

$$a_{1(2-1)} = \lim_{s \to -2}\left(\frac{1}{1!}\frac{d^1}{ds^1}((s+2)^2 f(s))\right) = \lim_{s \to -2}\left(\frac{d}{ds}\left(\frac{1}{(s+3)}\right)\right) = \lim_{s \to -2}\left(\frac{(s+3)\frac{d}{ds}(1) - 1\frac{d}{ds}(s+3)}{(s+3)^2}\right) = -1$$

$$a_2 = \lim_{s \to -3}(s+3)f(s) = \lim_{s \to -3}\left(\frac{1}{(s+2)^2}\right) = 1$$

$$f(s) = \frac{1}{(s+2)^2} - \frac{1}{s+2} + \frac{1}{s+3}.$$

■

8.13 Further reading

There are many excellent texts on control theory. A standard reference is by B.C. Kuo and F. Golnaraghi, Automatic Control Systems, John Wiley & Sons, 2002. Classical control is discussed in detail with numerous examples in M. Gopal, Control Systems: Principles and Design, McGraw Hill, 2008. The standard reference for digital control systems is K. Ogata, Discrete-Time Control Systems 2nd edition, Prentice Hall, 1995.

8.14 Exercises

1 A bandwidth management system

Consider the following problem: an ISP wants to ensure that, on a specific congested link, the fraction of P2P traffic is never more than r percent. If this fraction is exceeded, it sends TCP reset packets to P2P connection endpoints, terminating them, and thus reducing their load. For this system, identify the plant, the command, the control input, the disturbance, and the output.

2 Effort and flow

What are the effort and flow variables in the system of Exercise 1?

3 State space representation

Give a state space representation for the system of Exercise 1. Assuming that the control rule is to reduce the number of P2P connections by u connections over a time period T if the current P2P traffic fraction exceeds r percent, represent this in the state evolution equation. Note that the system is subject to a random disturbance at the instantaneous rate w.

4 Transfer function

What is the transfer function for the system of Exercise 1 in the regime when the congested link has more than r percent of P2P connections?

5 First order system

When given a step input, a first order system reaches 63% of its asymptotic value of 4.25 units at time 3s. What is its transfer function?

6 Second order system

Prove that for a critically damped second order system, $y(t) = K[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]$.

7 Proportional mode control

What is the relationship between the loop gain and the system pole in Example 9.

8 Integral mode control

What is the impulse response of the integral mode controller in Example 10?

9 Multiple mode control

What is the impulse response of the web server of Example 2 when simultaneously using derivative and integral control?

10 Stability

Use a numerical solver to determine the stability of an LTI system with denominator polynomial $3s^5 - 4s^3 + 7s^2 + 10s + 1$.

11 Matrix exponential

Prove that $e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \frac{\boldsymbol{A}^3t^3}{3!} + \ldots$ satisfies $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.

12 Matrix exponential

Show that for a diagonal matrix $e^{\boldsymbol{A}t} = \begin{bmatrix} e^{a_{11}t} & 0 & 0 & 0 \\ 0 & e^{a_{22}t} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & e^{a_{nn}t} \end{bmatrix}$. Use this to compute $e^{\boldsymbol{A}t}$ for $\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

13 Partial fraction expansion

Find the partial fraction expansion of $\frac{s}{(s+3)(s+5)}$.

14 Partial fraction expansion

Find the partial fraction expansion of the polynomial fraction $\frac{1}{s^2 + 4s + 29}$.

CHAPTER 9

Information Theory

9.1 Introduction

Communication takes many forms such as email, advertising on billboards, radio, and TV. Despite these different instantiations, every act of communication involves the same four abstract entities: a message **source** sending a **message** to one or more **recipients** over a communication **channel**. For example, when two people correspond by email, a message sender sends an email message to a recipient over the Internet. Similarly, an advertiser communicates to a mass audience by displaying a message to passers-by on a billboard. And, with a radio station, a broadcaster sends audio messages to listeners over a radio communication channel.

This suggests that it may be possible to study all forms of communication using the same underlying theoretical foundation. This foundation is provided by information theory, which allows us to precisely answer the following questions:

- How can we mathematically model communication?
- What is the minimal description of a message?
- How fast can we send messages over a noise-free channel?
- What is the effect of channel distortion on this rate?
- How can we send digital and analog messages over noisy analog channels?

The answer to the first question introduces the fundamental concept of **entropy**, which measures the degree of uncertainty in a system. The second question is answered by the design of optimal **source codes**. The answer to the third and fourth questions is provided by the well-known **Shannon capacity** of a channel. Finally, the fifth question leads to the study of **Gaussian channels**. We study these concepts in this chapter.

9.2 A Mathematical Model for Communication

Communication occurs when a message source sends a message to one or more recipients over a communication channel. For simplicity, let us first focus on a system with a single source, recipient, message, and channel. This system is in one of two states. In the first state, before the receipt of the message, the source knows something that the recipient does not. In the second state, after the receipt of the message and its processing by the recipient, the recipient knows something more than it did before. We attribute this increase in the state of knowledge of the recipient to its receipt of the message. This allows us to quantify the effect of the message, which we will call its **information content**, as its contribution to increasing the state of knowledge of the recipient.

Quantifying the state of knowledge of a recipient is a complex and perhaps impossible task (if it were easy, we wouldn't need so many examinations!). We must therefore restrict the scope of the problem to a situation where a recipient's knowledge state can be easily measured. Consider a situation where the source must choose between one of *N* equiprobable messages to send to the recipient, and *where the recipient knows this constraint*. Before the reception of the message, the recipient is uncertain about which of the *N* possible messages it will receive. The receipt of the message removes this uncertainty and therefore is its information content. We see that to quantify the message's information content we need to somehow quantify the initial degree of uncertainty or **entropy** on the part of the recipient.

To fix ideas, suppose that a source could choose between one of 16 equiprobable messages. How much uncertainty does the recipient have? For many good reasons, it turns out that the right way to measure uncertainty is by the base 2 logarithm[1] of the number of possible messages. Denoting the measure of uncertainty or entropy by *H*, we have

$$H = log_2 N = log_2 16 = 4 \qquad \textbf{(EQ 1)}$$

Note that this measure of uncertainty has no relationship to the semantic content of the messages: information theory regards all messages equally, independent of their semantic content.

A consequence of using a logarithmic measure for entropy is that if the number of possible messages doubles then entropy increases by a single unit. We interpret this as follows. When a source can choose to transmit one of 32 possible messages, we can divide these messages into two sets of 16. If the source were to tell the recipient which of these two sets contains its chosen message, the subsequent uncertainty on the part of the recipient reduces to the previous case. Therefore, doubling the number of possible messages adds only a single binary choice or one **bit** of additional uncertainty. This is one reason why a (base 2) logarithmic measure of uncertainty makes good sense. Thus, we can quantify the information content of a message chosen from one of *N* equiprobable messages as *logN* bits.

In discussion so far, we have assumed that all messages are equally likely. What if the receiver knows that the source almost always sends message number 13 and rarely any of the other messages? In this situation, if the recipient receives message 13, it does not learn very much, but if it receives an unlikely message, say message 2, it does. As the saying goes, "A dog biting a man is not news, but a man biting a dog is." We can account for this phenomenon by measuring the entropy of a message x_i whose probability of occurrence is $P(x_i)$ as $\log\frac{1}{P(x_i)} = -\log P(x_i)$. This assigns entropy in inverse proportion to message likelihood. Then, given a set of messages, we can compute their average entropy as

$$H = -\sum_i P(x_i)\log P(x_i) \qquad \textbf{(EQ 2)}$$

More formally, suppose that the random variable *X*, corresponding to a message chosen for transmission on a channel, assumes discrete values $x_i, x_2, \ldots, x_N$ according the probability mass function $P\{X = x_i\} = P(x_i)$. Then, the entropy of this random variable is given by:

$$H(X) = -\sum_i P(x_i)\log P(x_i) \qquad \textbf{(EQ 3)}$$

An alternative more compact notation is

$$H(X) = -\sum_X P(x)\log P(x) \qquad \textbf{(EQ 4)}$$

where the notation implicitly indicates that the range of the summation is over all values of $x \in X$.

We can summarize this discussion as follows. We view the act of communication as the removal of uncertainty on the part of a recipient as to which message *x* was chosen for transmission by a source on a communication channel. The message is

1. All logarithms in this chapter are base 2 unless otherwise specified.

modelled as a random variable X associated with a known probability mass function $P\{X = x_i\} = P(x_i)$. Then, the information content of the message, which corresponds to the uncertainty associated with X, is given by

$$H(X) = -\sum_i P(x_i) \log P(x_i).$$

EXAMPLE 1: ENTROPY OF A SIMPLE MESSAGE SOURCE

Suppose two generals are on the battlefield and want to coordinate an attack. This is done by sending a carrier pigeon from one general to another. The message, sent in the evening, says either “Attack at dawn” or “Don’t attack at dawn.” What is the entropy of this message?

Solution: There are two possible messages. Assuming they are equiprobable, the entropy = *log 2* = 1bit.

■

EXAMPLE 2: ENTROPY OF A SOURCE WITH EQUIPROBABLE MESSAGES

Prove that if messages are equiprobable, the definitions in Equation 1 and Equation 3 agree.

Solution: We have $P(x_i) = \frac{1}{N}$, so $H(X) = -\sum_i \frac{1}{N} \log \frac{1}{N} = -\frac{N}{N} \log \frac{1}{N} = -\log \frac{1}{N} = \log N$.

■

EXAMPLE 3: ENTROPY OF A RANDOM VARIABLE

What is the entropy of a random variable X whose probability mass function is given by $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$.

Solution: The entropy is $-(0.5 \log 0.5 + 0.25 \log 0.25 + 0.125 \log 0.125 + 0.125 \log 0.125) = 1.25$ bits.

■

EXAMPLE 4: ENTROPY OF A BINARY RANDOM VARIABLE

Suppose that a random variable X takes on the values {0,1} with probability {p, 1-p}. For what value of p is its entropy maximum? Minimum?

Solution: $H(X) = -\sum_i P(x_i) \log P(x_i) = p \log p + (1-p) \log (1-p)$. This function is plotted in Figure 1. It is clear that the maximum entropy is 1, when p = 0.5 and the minimum entropy is 0, when p = 0 or 1. In general, it can be shown that entropy is maximum when choices are equiprobable, and entropy is zero (there is no uncertainty) when the random variable is no longer random.

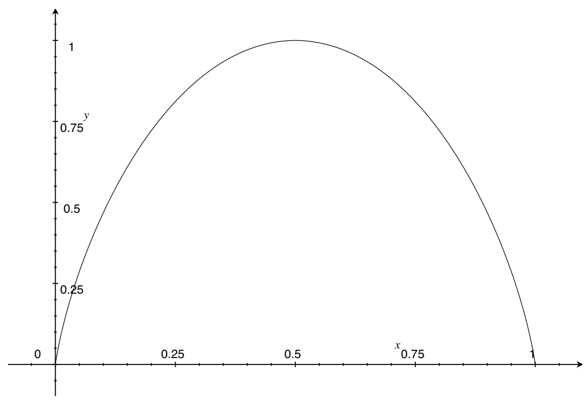


FIGURE 1. **Entropy of a binary random variable that takes values {0,1} with probability {*p*, 1-*p*}. The X axis shows *p* and the Y axis the corresponding entropy.**

■

An important property of entropy is that the entropies of independent random variables are additive. That is, if the entropy of random variable X is $H(X)$ and the entropy of random variable Y is $H(Y)$, and X and Y are **independent**, then the entropy of their joint distribution, $H(XY)$ is given by:

$$H(XY) = H(X) + H(Y) \qquad \textbf{(EQ 5)}$$

Intuitively, if a message source were to first send a message described by random variable X, then another message described by the random variable Y, then the pair of messages is described by their joint distribution XY. The overall entropy, however, adds up, as long as the two random variables are independent.

9.3 From Messages to Symbols

To compute the entropy of a source, we assume that the number of messages it can send is finite and that we can determine the probability with which it selects each message for transmission on a channel. This definition of a message source is rather unrealistic. After all, the set of messages transmitted by a radio or TV station or by email is practically infinite. Moreover, there is no obvious way to associate a probability with each message that could potentially be sent on these communication channels.

We address these issues as follows. First, note that, in practice, the infinite set of messages that could potentially be sent on any communication channel is always represented using **symbols** from a finite alphabet. Statistical analysis of large numbers of messages reveal symbol frequencies, which asymptotically correspond to symbol probabilities. This allows us to compute entropy at the symbol level, which we call **symbol entropy**.

EXAMPLE 5: SYMBOL ENTROPY OF ENGLISH TEXT

Although the set of books in the English language is infinite, they all are written with 26 characters with some additional punctuation marks. Statistical analyses of many types of English texts show that each character appears with a nearly constant character frequency independent of the text. We can interpret the measured symbol frequencies, therefore, as the corresponding symbol probabilities for 'typical' English text. This allows us to compute the symbol entropy of English text as approximately 1 bit/symbol.

■

EXAMPLE 6: SYMBOL ENTROPY OF DIGITAL IMAGES

Digital images are large matrices of picture elements or **pixels**. Each pixel in, for example, a red-green-blue or RGB digital image denotes the intensity of corresponding portion of the analog image in the red, green, and blue portions of the optical spectrum. Intensities in each part of the spectrum are commonly represented as one-byte or 8-bit quantities, so that each pixel corresponds to a 24-bit quantity. We can regard these as symbols in the language of images. This gives us an 'alphabet' of 2^{24} symbols to represent all possible images. In principle, we can compute the empirical probability of each symbols, to compute the symbol entropy of digital images.

■

Symbol entropy is not the same as message entropy. For example, studies have shown that the statistical frequency of the letter 'q' in a typical English text is 0.095% and that of 'u' is 2.758%. However, given that the prior letter is 'q,' the probability of sending a 'u' is nearly 100%. So, the entropy of the di-gram {"qu"} is smaller than the sum of the entropies of the individual symbols 'q', and 'u' (which proves that they are not independent). We see that increasing the message length by one decreases message source entropy. Intuitively, it should be clear that it is possible to asymptotically approach the true message probabilities, and therefore true message entropy, by considering the statistical frequencies of longer and longer symbol sequences in a representative message corpus. For example, if one were to analyse all English character and punctuations sequences say, 10,000 symbols long, it would be possible to compute the entropy associated with nearly every English sentence. Of course, the number of such messages would be incredibly large, so this would be a tremendously expensive computation. Nevertheless, at least in principle, it is possible to associate probabilities with nearly all English sentences.

To sum up, given a message source, by identifying the underlying alphabet of symbols and computing the relative frequencies of occurrence of longer and longer symbol sequences in a representative message corpus we can approximately determine its entropy.

9.4 Source Coding

We now consider the problem of optimally encoding messages from a message source. We assume that the source sends messages using symbols chosen from some known finite alphabet, and that, using the approach outlined earlier, we can compute the probability of selecting each message (or, at least, an asymptotic limit by considering increasingly longer symbol sequences). This permits us to compute the entropy of the source. Our task is to rewrite messages using a *code* such that coded messages are, on average, as short as possible. This allows us to transmit the fewest number of bits on a channel or store the messages in a file of the smallest possible size. We will demonstrate that using the optimal *Huffman* code the average code length is at most one bit more than the source entropy.

We model a source as a random variable *X* that takes on values corresponding to messages x_i where each message is composed of one or more symbols that are themselves sequences of elementary symbols chosen from an alphabet χ [2]. For example, for the case of a source of messages in English text, the alphabet χ of elementary symbols could consist of the 26 letters in the English alphabet, symbols could be English words, and messages would be sequences of words. Alternatively, each English letter could itself be treated as a separate message.

Coding consists of computing a coded message $c(x_i)$ for each possible message x_i. A coded message is composed from special symbols (called **codewords**) chosen from a **code** C. Without loss of generality, we will assume that the codewords are chosen from the alphabet $\{0, 1\}^+$, the set of binary strings of length 1 or more. For example, to code English text, we

2. Formally, a source is modelled as a stationary, ergodic Markov chain (see Section 6.2.7 on page 176) that may have state-dependent probabilities of message generation, but whose message-generation process is determined by its stationary distribution. If message x_j is generated with probability *P*(*i*,*j*) when the chain is in state *i*, and if the stationary probability that the chain is in state *i* is π_i, then the overall probability of generation of message x_j is given by $P(x_j) = \sum_i \pi_i P(i, j)$.

could assign one of 26 binary sequences to each English character. Alternatively, we could assign suitably chosen binary strings to all English words in a comprehensive dictionary.

In choosing a coding scheme, some rules are obvious. First, it does not make any sense to assign the same codeword to different message symbols: this prevents unambiguous decoding by the recipient of the code. Second, it is desirable that no codeword be the prefix of any other codeword: this allows a codeword to be decoded immediately on reception. A code that satisfies these two properties is said to be an **instantaneous code**.

EXAMPLE 7: INSTANTANEOUS CODES

Consider a message source that sends one of four messages ‘a’, ‘b’, ‘c’, or ‘d’. Two possible codes for this source are shown below:

Message symbol	**Code I**	**Code II**
‘a’	00	00
‘b’	001	01
‘c’	1	10
‘d’	11	11

With Code I, on receiving the coded message 0011, the receiver cannot decide after seeing the first two zeroes whether the source sent symbol ‘a’ or ‘b.’ Actually, the situation is a lot worse: this string cannot be unambiguously decoded, even though each message symbol has a unique code! In contrast, with Code II, no codeword is a prefix of any other codeword, so all received messages can not only be unambiguously decoded but also decoded immediately on receipt of the corresponding codeword. Therefore, Code II is instantaneous and Code I is not.

■

The previous example demonstrates that it is possible to come up with many possible coding schemes for the same message source. What coding scheme is the ‘best?’ Clearly, it is advantageous for a code to be instantaneous. But it is trivial to construct many equivalent instantaneous codes for the same message source (see Exercise 4). So, being instantaneous is not a sufficient criterion for optimality. Intuitively, we would like a code to not only be instantaneous but also the shortest possible. That is, given the probability with which messages are sent, we would like to minimize mean length of a coded message. This allows us to send the fewest bits on a potentially expensive communication channel.

An optimal coding scheme must take two factors into account. First, coding schemes typically become more efficient as message symbols are constructed from longer sequences of elementary symbols. Second, coding schemes become more efficient if shorter codes are assigned to more frequently used symbols. These factors are illustrated by the next two examples.

EXAMPLE 8: CODING DIGIT SEQUENCES

Consider coding messages that are sequences of decimal digits. If we assign a codeword to one digit at a time (i.e., a codeword corresponds to one elementary symbol), we need four bits to encode each of 10 possible symbols. This gives us 16 possible codewords, of which 6 are wasted. The mean coding efficiency is 4 bits/1 elementary symbol = 4 bits/elementary symbol. On the other hand, if we assign a codeword to two digits at a time (i.e., a codeword corresponds to two elementary symbols), there are 100 possible symbols, so we need at least 7 bits per codeword, leading to a coding efficiency of 7 bits/2 elementary symbols = 3.5 bits/elementary symbol. Arguing similarly, we find that when a symbol corresponds to a sequence of three elementary symbols, we need 10 bits per codeword, for a coding efficiency of 10 bits/3 elementary symbols = 3.33 bits/elementary symbol. This trend is not monotonic: with each message symbol corresponding to four elementary symbols, we need 14 bits per codeword, for a mean of 3.5 bits/elementary symbol. Nevertheless, the coding efficiency declines asymptotically.

■

EXAMPLE 9: CODING ENGLISH LETTERS

Consider coding messages that are sequences of English letters. If we consider coding one letter at a time, then we need 5 bits/elementary symbol, because there are 26 English letters (actually, 27, if we consider 'space' as a necessary elementary symbol). As before, we can increase coding efficiency by considering more than one letter at a time. For example, if we code five letters at a time, there are 27^5 = 14,348,907 combinations. These can be represented using 24 bits, for a coding efficiency 24/5 = 4.8 bits/elementary symbol. But an even greater increase in coding efficiency is gained by not coding for impossible sequences of characters. For example, we need not assign codewords to sequences such as 'qx' and 'zx.' Even better, we should assign short codewords to commonly occurring sequences and longer codewords to rarely occurring sequences. This would obviously minimize the mean length of a coded message.

■

A code has an intuitive geometric representation as a binary tree, with each codeword corresponding to a path from the root to a leaf or non-leaf node. For example, Figure 2 shows the binary trees corresponding to Code I and Code II in Example 7. In the tree for Code I, the path from the root to the node marked 'a' is labelled '00' which is the codeword representing 'a.' Note that the codeword for 'a' corresponds to a non-leaf node because it is on the path to the codeword for 'b.' This violates the prefix property, making it impossible to decode the string '00' until after seeing the next symbol at least (and, actually, not even after that!).

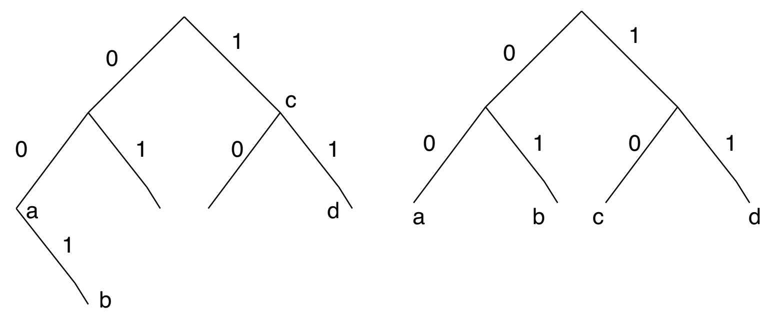


FIGURE 2. Binary trees corresponding to (left) Code I and (right) Code II in Example 7

Generalizing from this example, a little thought shows that for an instantaneous code (one that satisfies the prefix property), every codeword must correspond to a leaf in the binary tree representation. Consider a tree corresponding to an instantaneous code with codewords of length at most *L*. If we were to expand this to form a complete tree of height *L*, we would have 2^L leaves. Some of these leaves must correspond to actual codewords, others are descendants of codewords obtained by completing the tree, and still others are neither codewords nor descendants of codewords. A codeword *i* of length l_i corresponds to 2^{L-l_i} leaf nodes. For example, if *L=5*, and $l_3 = 2$, then the third codeword will correspond to 8 leaves. Each set of descendant leaf nodes is non-overlapping with any other set. Therefore

$$\sum_i 2^{L-l_i} \le 2^L$$
$$\sum_i 2^{-l_i} \le 1 \quad \text{(EQ 6)}$$

where the second equation is derived by dividing both sides of the first equation by the quantity 2^L. This is called the **Kraft inequality**. It constrains the choice of codeword lengths for instantaneous codes.

The Kraft inequality allows us to relate the minimum expected codeword length for an instantaneous code to the message entropy. To see this, note that the expected codeword length is given by

$$E = \sum_i P(i) l_i$$

To minimize this sum subject to the Kraft inequality, we can use the method of Lagrange multipliers to minimize

$$G = \sum_i P(i) l_i + \lambda \sum_i 2^{-l_i}$$

For the moment, we ignore the fact that codeword lengths are integral and treat them as reals. Then, we can find the minimal value of *G* by differentiating it with respect to l_i to find

$$\frac{\partial}{\partial l_i} G = P(i) - \lambda 2^{-l_i} \ln 2$$

Setting this to 0, we get

$$2^{-l_i} = \frac{P(i)}{\lambda \ln 2} \qquad \textbf{(EQ 7)}$$

When the constraint arising from the Kraft inequality is binding

$$\sum 2^{-l_i} = \sum \frac{P(i)}{\lambda \ln 2} = 1 \text{, so } \lambda = \frac{\sum P(i)}{\ln 2} = \frac{1}{\ln 2}.$$

Substituting in Equation 7, we get

$$P(i) = 2^{-l_i}$$

or

$$l_i = -\log P(i)$$

so that the expected codeword length

$$E = \sum_i P(i) l_i = -\sum_i P(i) \log P(i) = H(X) \qquad \textbf{(EQ 8)}$$

This is a beautiful and deep result! It shows that the expected length of a instantaneous codeword is lower-bounded by the entropy of the message source. Entropy can therefore be viewed as the intrinsic degree of randomness of the message source.

In deriving this result, we have ignored the integrality constraint (that is, codeword lengths must be integers). It can be shown that the result holds even if this constraint is observed. The consequence of having integer codeword lengths is essentially that real codes are slightly longer than optimal non-integral codes.

EXAMPLE 10: OPTIMAL CODES

Consider the two generals of Example 1. Suppose that the probability of sending the message “Attack at dawn” is 0.75 and the probability of the message “Don’t attack at dawn” is 0.25. Then, the entropy of the sending general is -(0.25*log 0.25 + 0.75 * log 0.75) = 0.81. This means that an optimal code could use as few as 0.81 bits. But, of course, no code can use fewer than one bit, so, due to integrality constraints, the minimum codeword length is 1 bit.

It is possible to use shorter codewords, even in this limited situation, if source messages are allowed to be concatenated to form longer messages. For example, if a general’s messages over a hundred days were to be aggregated together, then it would be possible to represent these hundred messages using fewer than 100 bits on average. However, no matter how clever we are, it is impossible to find an encoding that uses fewer bits on average than the message entropy.

■

The discussion above shows that entropy of a message source is a lower bound on expected codeword length. We now present the **Huffman code** that is guaranteed to have an average codeword length no longer than $H+1$, where H is the entropy of a set of messages or symbols. The intuition for this code is straightforward. We would like to have the longest codeword assigned to the least-likely message. We can arrange for this to happen as follows: we pick the two least-likely messages (or symbols) and make them leaf nodes in a binary tree with a common parent. Naturally, the probability associated with their parent is the sum of their probabilities. We can therefore replace the two messages in the set of messages with a virtual message (corresponding to the parent) with this probability. Note that the size of the message set has decreased by one. We now recursively apply this construction to the smaller set of messages, terminating when there are no more messages. It can be shown that this greedy recursive construction leads to a optimal code.

EXAMPLE 11: HUFFMAN CODE

Suppose that a message source generates messages with the following probabilities: ‘a’: 0.125, ‘b’: 0.125, ‘c’: 0.5, ‘d’: 0.10, ‘e’: 0.15. Construct the Huffman code for these messages. What is the mean code length? What is the message entropy?

Solution: From Figure 3, the optimal code: ‘a’: 001, ‘b’: 110, ‘c’: 1, ‘d’: 000, ‘e’: 011. The mean code length is (0.125*3 + 0.125*3 + 0.5*1 + 0.1*3 + 0.15 * 3) = 2. The entropy is given by -(0.125 log 0.125 + 0.125 log 0.125 + 0.5 log 0.5 + 0.1 log 0.1 + 0.15 log 0.15) = -(0.375 + 0.375 + 0.5 + 0.33 + 0.41) = 1.99. So, this code is not only optimal, but also very close to the entropy limit.

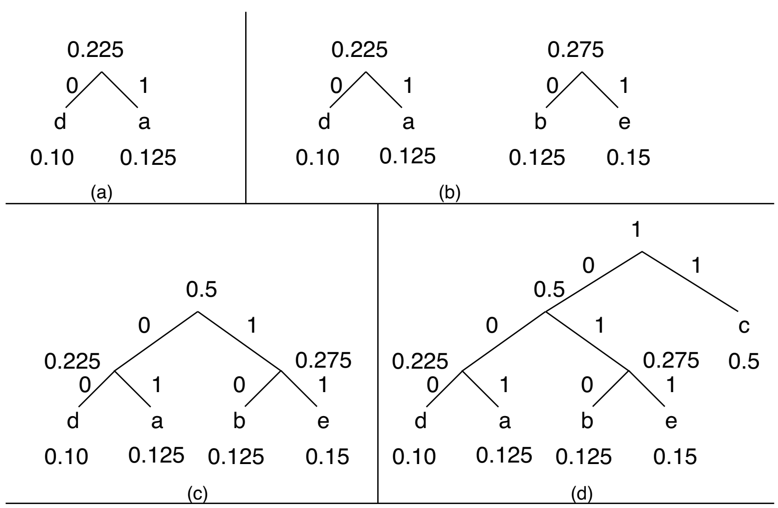


FIGURE 3. **Construction of the Huffman code for Example 10. (a) ‘d’ and ‘a’ have the lowest probabilities, so they form the first two leaves. (b) The intermediate node from the previous step has greater probability than ‘b’ and ‘e’, so ‘b’ and ‘e’ form the next two leaves. (c) The two intermediate nodes have the least probability, so they are joined in this step. (d) The final binary tree.**

■

Note that the Huffman code is not unique. In the previous example, for instance, we could have assigned 0 to ‘e’ and 1 to ‘d’in the first step to get a different but also optimal Huffman code. In general, inverting all the bits of a code, or swapping two codewords of the same length will preserve optimality.

Note also that to construct a Huffman code, we need to know the probability of occurrence of every message or symbol. Although these can be determined from a corpus of messages, the actual probability of occurrence of a symbol in any message instance will differ somewhat from that of the corpus. Therefore, a Huffman code, though optimal in a broad sense, may not be the optimal encoding for a specific message instance.

9.5 The Capacity of a Communication Channel

A communication channel repeatedly transfers symbols from a message source to one or more recipients. It is modelled as shown in Figure 4.

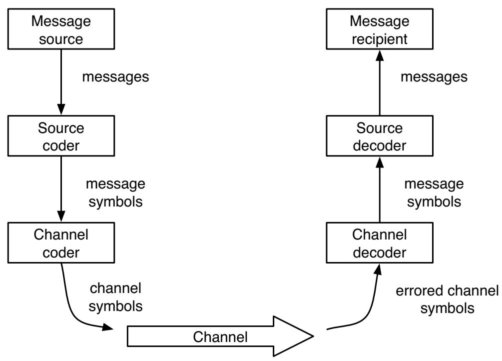


FIGURE 4. Model of a communication channel

Messages from a message source are encoded by a **source coder** that uses techniques such as those discussed in the previous section to eliminate redundancy. Thus, if the entropy of the message source is $H(X)$, then the average codeword length at the output of the source coder is at least $H(X)$. These codewords are created from an alphabet of **message symbols**.

We will demonstrate that it is inefficient to transfer message symbols directly on a channel. Instead, message symbols are translated into **channel symbols** by a **channel coder**. Channel symbols are chosen to counteract the effects of a noisy channel. Because they also are a code for source messages, the average codeword length of a channel codeword is also at least $H(X)$.

The channel transfers channel symbols from a message source to a message recipient. This process may be ideal, in the case of a **noiseless channel**, or may introduce errors, in the case of a **noisy channel**. The output of a noisy channel is a stream of errored channel symbols. These symbols are given to a **channel decoder** which attempts to determine the corresponding message symbols. If the channel code is well-chosen, in theory the error rate in determining the message symbols can be made vanishingly small. Otherwise, the decoded message symbols may contain a significant degree of errors. Finally, the **source decoder** translates from message symbols back to messages that are given to the message recipient.

Channels that transfer only binary channel symbols (i.e., ‘0’ or ‘1’) are called **binary channels**. If a binary channel can transfer C symbols per second, then its bitrate is C bits per second and is also called its **information capacity**. Channels can also transfer more complex symbols. For example, a Quadrature Phase Shift Keying (QPSK) channel can transfer one of four distinct symbols and a 16-Quadrature Amplitude Modulation (QAM16) channel can transfer one of 16 distinct symbols (see Section 5.2.3 on page 123 for more details). If a channel can transfer one of 2^k distinct symbols, we view as transferring k bits in parallel. If such a channel can transfer C symbols per second, its information capacity or bitrate is kC bits/second.

We will restrict our attention to **discrete memoryless** channels. Such channels transfer discrete symbols. Moreover, the probability of observing a particular output symbol when a particular symbol is input for transmission depends only on the input symbol itself and is independent of prior inputs or outputs.

We first characterize a message source, then study the capacity of an ideal noiseless channel. This sets the foundation to study the capacity of a noisy communication channel.

9.5.1 Modelling a Message Source

Recall that we model a message source as a random variable X that takes on values corresponding to messages x_i where each message is composed of one or more symbols. From Equation 8, each message can be represented by a binary code whose expected codeword length (in bits) is at least the message entropy $H(X)$. Therefore, a source that generates M messages per second must be coded at a rate of at least $MH(X)$ bits per second. This is called its **information rate**. We can equivalently view this source as generating entropy at the rate of $MH(X)$ bits per second. Therefore, the information rate is also called the **entropy rate** of the source. Intuitively, this is the rate at which a receiver becomes uncertain about the state of a message source.

Suppose that encoded messages generated by a source are independent of each other: this is a good approximation for codes over sufficiently long message lengths. Then, a source with entropy $H(X)$ that generates M messages per second can generate up to $2^{MH(X)}$ distinct messages per second. In general, if the information rate of a source is R, then it can generate up to 2^R distinct messages per second. Conversely, if a source can generate N distinct messages per second, its information rate is no more than $\log N$ bits per second (where equality is reached only if all messages are equiprobable). We summarize this as:

$$\text{information rate} \leq \log(\text{number of distinct messages/second}) \qquad \text{(EQ 9)}$$

EXAMPLE 12: SOURCE INFORMATION RATE

Consider a source that generates 10 independent messages/second. Let the per-message entropy be 3.5 bits. Then, the source can be viewed as generating entropy at the rate of 35 bits/second. Moreover, the source can generate up to 2^{35} distinct messages each second. This may sound like a lot, but a little thought reveals that the source is essentially generating 35 independent bits per second. The upper bound on the total number of distinct messages is, therefore, simply the total number of distinct binary strings of length 35 bits, which is 2^{35}.

■

A message source that selects message symbols independently and with identical distribution (*i.i.d.*) and that has an entropy $H(X)$ exhibits an interesting property: the messages that it generates fall into two distinct sets. One set, called the **typical** set has $2^{H(X)}$ elements and the other set, called the **atypical** set, has the rest. It can be shown that almost all messages generated by such a source fall into the typical set. It is therefore possible to code messages from the source with only $H(X)$ bits, yet find only a vanishingly small number of messages from the source to be uncodeable. This is also called the **Asymptotic Equipartitioning Property** and is a consequence of the law of large numbers, which states that the arithmetic average of an ensemble of i.i.d. random variables converges to their mean (see Section 1.7.4 on page 29).

EXAMPLE 13: TYPICAL MESSAGES

Consider a message source that generates ‘0’ symbols with probability 0.9 and ‘1’ symbols with probability 0.1. The probability that this source generates the sequence ‘111111111’ is 10^{-9}. Indeed, it is easy to see that if any message sequence has k or more ‘1’ symbols, then the probability of this message sequence is no greater than 10^{-k}. If we consider all sequences of length m symbols, where $m > k$, there are $\binom{m}{k} + \binom{m}{k+1} + \ldots + \binom{m}{m}$ sequences that have a probability no greater than 10^{-k}. For any probability level ε, we can choose an m such that $\binom{m}{k} + \binom{m}{k+1} + \ldots + \binom{m}{m}$ messages, of the 2^m possible messages, have a probability less than ε. These are the ‘atypical’ messages generated by the message source.

Note that the entropy of this source is 0.469 bits/symbol. So, if it generates independent messages with m symbols, the total entropy of this set of messages is 0.469m bits. The size of the typical set is therefore $2^{0.469m}$ elements. For example, if m = 100, the size of the typical set is $2^{(100)(0.469)} = 2^{46.9}$. This is a large number, but still much, much smaller than the total number of messages with 100 symbols, 2^{100} (their ratio is 1.04 10^{-16}).

■

9.5.2 The Capacity of a Noiseless Channel

The information capacity of a noiseless channel is trivially given by the rate at which it transfers symbols. Assume that there are 2^k distinct channel symbols. Then, a noiseless channel that can transfer C channel symbols per second has an information capacity of C symbols/second or kC bits/second.

One of the fundamental theorems of information theory, called the **channel capacity theorem**, states that a noiseless channel of capacity kC can carry messages from a source that generates entropy at the rate of $MH(X)$ bits per second if and only if

$$kC > MH(X) \quad \text{(EQ 10)}$$

EXAMPLE 14: CAPACITY OF A NOISELESS CHANNEL

Consider an ideal noiseless channel that can transmit one of four symbols, say ‘A’, ‘B’, ‘C’, or ‘D’ each second. We can view this as carrying the equivalent symbol sequences ‘00’, ‘01’, ‘10’, ‘11’ giving it an information carrying capacity of 2 bits per second. Recall that the message source in Example 10 has an entropy of 1.99 bits/message. Therefore, this channel can carry messages from this source if and only if the message generation rate is lower than 2/1.99 = 1.005 messages/second.

■

9.5.3 A Noisy Channel

All physical channels are **noisy**: they introduce errors so that the received channel symbol may differ from the one that is transmitted. Over a noisy binary channel, this implies that a ‘0’ transmitted symbol is sometimes received as a ‘1’ or *vice versa*. The presence of noise degrades a channel and reduces its capacity. The degree of capacity reduction depends on the characteristics of all three: the channel, the message source, and the channel coder.

EXAMPLE 15: A NOISY CHANNEL

Consider a noisy binary channel that corrupts the channel symbol ‘0’ with probability 0.9 and the channel symbol ‘1’ with probability 0.01 (see Figure 5). If message symbols are coded so that channel symbols consist predominantly of ‘0’ symbols, then many, perhaps all, of the transmitted messages will be corrupted. In contrast, if message symbols are coded so that the channel symbols consist predominantly of ‘1’ symbols, then few transmitted messages will be corrupted.

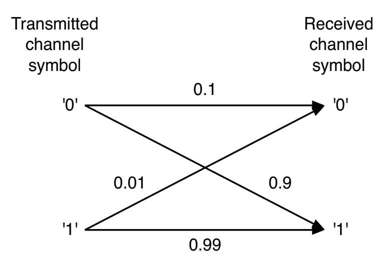


FIGURE 5. **Representation of a noisy channel**

■

Let us take a closer look at the effect of a noisy channel on channel symbols sent from a source. We study how a channel decoder, on receiving a channel symbol, can determine which channel symbol was actually transmitted[3].

3. For simplicity, from now on, when we write ‘symbol’ we will mean ‘channel symbol.’

Consider the channel shown in Figure 5. Suppose that the channel decoder receives symbol ‘1.’ With probability 0.9 it is due to the reception of a corrupted ‘0’ symbol and with probability 0.99 it is due to the reception of an uncorrupted ‘1’ symbol. Similarly, if the channel decoder receives symbol ‘0’, with probability 0.1 it is due to the reception of an uncorrupted ‘0’ and with probability 0.01 it is due to the reception of a corrupted ‘1.’ Thus, the probabilities with which the decoder receives a ‘0’ or a ‘1’ depends both on the probability with which the channel coder sends a ‘0’ or a ‘1’ *as well as* the way in which the channel corrupts a symbol. This is made explicit by considering the **joint probability distribution**[4] of the symbol sent and the symbol received.

Denote the transmitted symbol at the output of the channel coder by the random variable X and the received symbol at the input of the channel decoder by Y. Then, the discrete joint probability distribution $P(XY)$ assigns probabilities to each possible combination of X and Y. To make this concrete, consider a channel coder that sends a ‘0’ symbol with probability p and a ‘1’ symbol with probability 1-p over the channel in Figure 5 where symbol probabilities are independent and identically distributed (i.i.d). Then, the joint probability distribution of X and Y is given by Table 1.

***P*(*XY*)**	***X*=0**	***X*=1**
***Y*=0**	$0.1p$	$0.01(1-p)$
***Y*=1**	$0.9p$	$0.99(1-p)$

TABLE 1. The joint probability distribution of X and Y.

The **marginal** distributions of X and Y, $P(X)$ and $P(Y)$ are given by Table 2 and Table 3.

	***P*(*X*)**
***X*=0**	p
***X*=1**	$1-p$

TABLE 2. The marginal probability distribution of X.

	***P*(*Y*)**
***Y*=0**	$0.1p + 0.01(1-p)$
***Y*=1**	$0.9p + 0.99(1-p))$

TABLE 3. The marginal probability distribution of Y.

Moreover, the **conditional** distribution of X given that $Y = 0$ is given by Table 4 and the conditional distribution of X given that $Y = 1$ is given by Table 5.

	***P*(*X*|*Y*=0)**
***X*=0**	$0.1p/(0.1p + 0.01(1-p))$
***X*=1**	$0.01(1-p)/(0.1p + 0.01(1-p))$

4. Readers unfamiliar with joint distributions should review the material in Section 1.2.1 on page 5 at this point.

TABLE 4. The conditional distribution of X given that $Y = 0$.

	$P(X\|Y=1)$
$X=0$	$0.9p/(0.9p + 0.99(1-p))$
$X=1$	$0.99(1-p)/(0.9p + 0.99(1-p))$

TABLE 5. The conditional distribution of X given that $Y = 1$.

We similarly compute the conditional distribution of Y for the cases when $X = 0$ and $X = 1$ in Table 6.

	$P(Y\|X=0)$	$P(Y\|X=1)$
$Y=0$	0.1	0.01
$Y=1$	0.9	0.99

TABLE 6. The conditional distribution of Y given X.

These discrete probability distributions demonstrate two effects of a noisy channel. First, given a particular distribution of symbol probabilities $P(X)$ at the output of the channel coder, the noisy channel modifies the symbol probability distribution at the input of the channel decoder to $P(Y)$. This is evident when we compare Table 2 with Table 3. Second, a noisy channel makes it necessary for the channel decoder to work backwards to determine which symbol had been sent when a particular symbol is received. This is evident from Table 4 and Table 5.

Discrete probability distributions similar to those shown in Tables 1-6 can be computed for any channel coder, channel decoder, and channel. These distributions let us compute the corresponding entropies, respectively $H(XY)$, $H(X)$, $H(Y)$, $H(X|Y=0)$, $H(X|Y=1)$, $H(X|Y)$ and $H(Y|X)$. We interpret these entropies as follows:

- $H(X)$: The uncertainty at the channel decoder as to which symbol was sent by the channel coder.
- $H(Y)$: The uncertainty at the channel coder as to which symbol will be received at the channel decoder. This uncertainty depends on the characteristics of the message source, the message coder, the channel coder, and the channel.
- $H(XY)$: The uncertainty at an outside observer who can observe both the channel coder and the channel decoder of the occurrence of the symbol pair (transmitted symbol, received symbol).
- $H(X|Y=0)$: The uncertainty at the channel decoder as to the transmitted symbol conditional on the fact that a symbol ‘0’ was received. Formally, we define

$$H(X|Y=0) = P(Y=0)H(X|Y=0)$$

- $H(X|Y=1)$: The uncertainty at the channel decoder as to the transmitted symbol conditional on the fact that a symbol ‘1’ was received. Similarly,

$$H(X|Y=0) = P(Y=1)H(X|Y=1)$$

- $H(X|Y)$: The uncertainty at the channel decoder as to the transmitted symbol conditional on the fact that a symbol Y was received. From the additivity of entropy, we have

$$H(X|Y) = P(Y=0)H(X|Y=0) + P(Y=1)H(X|Y=1)$$

$$= \sum_Y P(y)H(X|y)$$

$$= -\sum_Y P(y)\sum_X P(x|y)\log P(x|y)$$

$$= -\sum_Y P(y)\sum_X \frac{P(xy)}{P(y)}\log P(x|y)$$

$$= -\sum_X\sum_Y P(xy)\log P(x|y)$$

- $H(Y|X)$: The uncertainty at the channel coder as to the received symbol conditional on the fact that a symbol X was transmitted. By symmetry, this is given by

$$H(Y|X) = \sum_X P(x)H(Y|x)$$

$$= -\sum_X\sum_Y P(xy)\log P(y|x)$$

These entropies allow us to define the information-theoretic meaning of communication over a noisy channel. Recall that this view is that the act of communication as the removal of uncertainty on the part of a recipient as to which symbol X was chosen for transmission by a source on a communication channel.

Let the **mutual information** $I(X;Y)$ corresponding to a particular source, channel coder, and channel and measured in units of bits/symbol be defined in as:

$$I(X;Y) = \sum_X\sum_Y P(xy)\log\frac{P(xy)}{P(x)P(y)} \quad \textbf{(EQ 11)}$$

It can be shown (see Exercise 14) that

$$I(X;Y) = H(X) - H(X|Y) \quad \textbf{(EQ 12)}$$

$$I(X;Y) = H(Y) - H(Y|X) \quad \textbf{(EQ 13)}$$

That is, the mutual information between a transmitter and receiver is the reduction in the uncertainty on the part of the receiver about the symbol sent by the transmitter due to the receipt of a particular symbol. Symmetrically, it is also the reduction in the uncertainty on the part of the transmitter about the symbol received at the receiver due to the transmission of a particular symbol. According to information theory, communication over a noisy channel can be precisely defined as the creation of mutual information between the transmitter and the receiver.

EXAMPLE 16: MUTUAL INFORMATION

Compute the mutual information for a channel coder that sends a ‘0’ symbol with probability 0.1 and a ‘1’ symbol with probability 0.9 over the channel of Figure 5.

Solution:

The probability of each symbol on the channel is given by:

	***P*(*X*)**
***X*=0**	0.1
***X*=1**	0.9

Therefore, $H(X)$ = -(0.1 log 0.1 + 0.9 log 0.9) = 0.469. To compute $H(X|Y)$, we first need to know the distribution of Y. From Table 3, we find this to be:

	***P*(*Y*)**
***Y*=0**	0.019
***Y*=1**	0.981

From Table 4, the conditional distribution of X given Y = 0 is

	***P*(*X*|*Y*=0)**
***X*=0**	0.526
***X*=1**	0.473

which has an entropy of -(0.526 log 0.526 + 0.473 log 0.473) = 0.624. It is interesting to note that when the received symbol is ‘0’, the receiver is more than 50% sure that the sender sent a ‘0’ even though the corruption rate for ‘0’ is 90%. This is because the corruption rate of ‘1’ is very low.

From Table 5, the conditional distribution of X given Y = 1 is

	***P*(*X*|*Y*=1)**
***X*=0**	0.092
***X*=1**	0.908

which has an entropy of -(0.092 log 0.092 + 0.908 log 0.908) = 0.443. We multiply these conditional entropies by the probability of Y being 0 or 1 respectively, to compute $H(X|Y)$ as 0.019* 0.624 + 0.981*0.443 = 0.446. Therefore, the mutual information is

$$I(X;Y) = 0.469 - 0.446 = 0.023 \text{ bits/symbol}$$

We interpret this to mean that the information content of each bit sent on the channel is only 0.023 bits. The remainder is wasted channel capacity due to the presence of channel noise. In contrast, with a binary noiseless channel, each transmitted symbol has an information content of 1 bit.

■

Note that the first term in Equation 12 depends only on the nature of the source but the second term depends jointly on the nature of the source, the nature of the channel coding scheme, and the nature of the communication channel. This opens the door to control channel symbol probabilities to compensate for the impairments introduced by the channel and maximize mutual information on the channel.

EXAMPLE 17: CODING FOR A NOISY CHANNEL

Suppose a channel can transmit one of four symbols, ‘A’, ‘B’, ‘C’, or ‘D’ and that the probability of corruption of the symbol ‘D’ is much greater than the probability of corruption of the other three symbols. Consider a source that can generate one of 16 message symbols, say the English letters a-p. A naive coding for this source would be to code each source symbol with a pair of randomly channel symbols. For example, we may code the source symbol ‘d’ as the channel codeword ‘AD and ‘f’ as ‘BB.’

The problem with this approach is that, due to the nature of the channel, channel codewords that contain a ‘D’ will be more likely to be corrupted than channel codewords that do not contain ‘D.’ If commonly occurring source symbols are allocated these error-prone channel codewords, $H(X|Y)$ will be greater, reducing the mutual information. It is more efficient to first determine the probability of generation of each source symbol a-p, and allocate channel codewords containing the channel symbol ‘D’ only to the least likely source symbols. Using this scheme, the least likely source symbol would be allocated the channel codeword ‘DD.’ This reduces the probability of channel codeword corruption, increasing the mutual information.

Alternatively, depending on the source symbol probabilities, it may be better to avoid using 'D' altogether when forming channel codewords. This would have the effect of making certain source symbols have a longer encoding on the channel (because at least some source symbols will correspond to a channel codeword that uses three rather than two channel symbols), but may still increase the mutual information on the channel.

■

Generalizing from this example we see that we can, in principle, given a probability distribution over source message sequences, choose an appropriate channel coding scheme to control the probability distribution over channel symbols. In other words, we can choose the channel coding scheme to control the distribution *P*(*X*) such that the mutual information *I*(*X;Y*) is maximized.

Shannon's fundamental theorem for a noisy channel proves that this maximum mutual information, given by $\max_{P(X)} I(X;Y)$ is also the greatest rate at which information can be carried over a noisy channel if the probability of decoding errors is to be made vanishingly small. It is not possible to send information faster than the maximum mutual information without introducing a significant number of uncorrectable errors. Moreover, there exists some coding scheme such that a source, when coded with this coding scheme, achieves a transmission rate arbitrarily close to this upper limit with an arbitrarily small error rate. Shannon's theorem does not, however, state how this coding scheme is to be found. However, both **turbo codes** and **Low DensityParity codes** come close to this ideal rate even in practice.

The proof of Shannon's theorem is elegant and instructive. It proceeds along the following lines. First, Shannon shows that if a source sends i.i.d. symbols drawn from a distribution *P*(*X*), then most sufficiently-long strings of symbols are *typical* and have an entropy close to the source's intrinsic entropy. Second, he computes the effect of sending a sequence of *n* i.i.d symbols over a noisy channel when using a *random* encoding. It can be shown that this maps each typical *n*-sequence of symbols to a set of $2^{nH(Y|X)}$ *non-overlapping* sequences of symbols all of which are equally likely. This allows the receiver to almost surely determine the symbol sequence intended by the transmitter, that is, the recipient's error in making this determination can be made as small as desired (though not zero). Third, Shannon shows that the maximum number of distinct messages arising from transmissions of typical strings is $2^{nH(Y)}$. This means that the transmitter can send at most $\frac{2^{nH(Y)}}{2^{nH(Y|X)}} = 2^{n(H(Y)-H(Y|X))}$ distinct messages that are almost surely decodable. Each message has *n* symbols, so the contribution of each symbol to the total set of messages is bounded by $2^{(H(Y)-H(Y|X))} = 2^{I(X;Y)}$. By analogy with the number of messages sent over a noiseless channel (see Section 9.5.1 on page 271), we conclude that the capacity of a noisy channel is $I(X;Y)$ bits/symbol. The channel capacity is achieved when this is maximized.

EXAMPLE 18: CAPACITY OF A NOISELESS CHANNEL

We revisit the capacity theorem for a noiseless channel here. For a noiseless channel, when *Y* is known, so is *X*. Therefore, the uncertainty in *X*, given *Y*, is zero, that is, *H*(*X*|*Y*) = 0. So, *I*(*X;Y*) = *H*(*X*) - *H*(*X*|*Y*) = *H*(*X*). Shannon's theorem tells us that the greatest rate at which information can be carried over this channel is given by $\max_{P(X)} I(X;Y) = \max_{P(X)} H(X)$. Obviously, such a channel can carry messages from any source with entropy *H*(*X*) smaller than this maximum. Thus, the capacity of a noiseless channel presented earlier is a special case of Shannon's theorem.

■

EXAMPLE 19: CAPACITY OF A SYMMETRIC BINARY CHANNEL

A symmetric binary channel is one that carries only two symbols, '0' and '1', and whose probability of symbol corruption, *e*, is the same for both symbols. Suppose that a source sends '0' symbols with probability *p* over this channel. Then, it is received as a '0' symbol with probability *p* and as a '1' with probability *pe*. Let us compute the mutual information of this channel. For this, we need to compute *H*(*X*) and *H*(*X*|*Y*). We have *H*(*X*) = -(*p* log *p* + (1-*p*) log (1-*p*)). Also, *P*(*Y*=0) = *P*(uncorrupted '0') + *P*(corrupted '1') = *p* + (1-*p*)*e* and *P*(*Y*=1) = *pe* + (1-*p*). Finally, when the channel decoder gets a 0 or a

1, it knows that it is an uncorrupted symbol with probability (1-e) and a corrupted symbol with probability e, so that the uncertainty in the transmitted symbol $H(X|Y) = -((1-e)\log(1-e) + e\log e)$. Putting these facts together,

$$H(X|Y) = P(Y=0)H(X|Y=0) + P(Y=1)H(X|Y=1)$$

$$= -(p + (1-p)e)((1-e)\log(1-e) + e\log e) -(pe + (1-p))((1-e)\log(1-e) + e\log e)$$

$$= -((1-e)\log(1-e) + e\log e)$$

Therefore,

$$I(X;Y) = H(X) - H(X|Y)$$

$$= -(p \log p + (1-p) \log (1-p)) +((1-e)\log(1-e) + e\log e)$$

The capacity of the channel is determined by maximizing this expression over p. But, from Example 4, we know that the greatest value is reached for $p = 0.5$, when $-(p \log p + (1-p) \log (1-p)) = 1$. Therefore, the capacity of the channel is

$$C = 1 + e\log e + (1-e)\log(1-e) \text{ bits/symbol.}$$

■

EXAMPLE 20: CAPACITY OF A 'NOISY TYPEWRITER'

Consider a typewriter that can type the lowercase letters a-z but whose output is noisy, so that when a character is depressed, the output is either the character itself or the next character in the alphabet modulo 26 with equal probability. What is the capacity of this 'channel?'

Solution:

We use the relationship $I(X;Y) = H(Y) - H(Y|X)$ to compute the mutual information. By symmetry, mutual information is maximized when the characters are chosen with equal probability (it could not be minimized: that happens when a single character is chosen with probability 1). In this case, $H(Y|X) = \log 2 = 1$, because on seeing a character in the output, there can be at most two possible inputs. The outputs are equiprobable because the inputs are equiprobable and the probability of corruption is 0.5. So, $H(Y) = \log 26$. Therefore, $I(X;Y) = \log 26 - 1 = \log 26 - \log 2 = \log (26/2) = \log 13$. This information capacity is reached if every alternate character in the input is used.

■

Computing the capacity of a channel is difficult because it requires a precise characterization of the error probabilities for every transmitted symbol as well as the inherent entropy of the message source. However, there is a special case where channel capacity can be determined using a different approach. This is the case of the *Gaussian channel* which we study next.

9.6 The Gaussian Channel

Our discussion so far has focussed on message sources that transmit digital information over a digital channel. Of course, in practice, both sources and channels are analog and a digital channel is an idealization of this reality. In this section, we discuss how to model an analog or continuous source and how we can compute the capacity of a noisy continuous channel.

9.6.1 Modelling a Continuous Message Source

Audio and video sources are continuous signal sources. We consider how such continuous signals can be sent over a digital channel. For simplicity, we first consider a specific audio source: human speech.

Human speech is generated as a continuous variation in air pressure produced by a combination of the lungs, diaphragm, vocal chords, throat, mouth, tongue, and lips. This signal is completely represented as a continuous function of pressure over time such as the one shown in Figure 6.

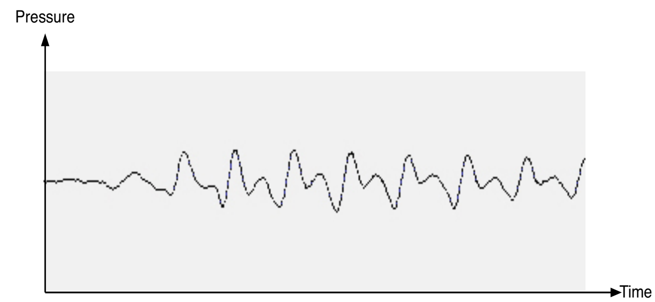


FIGURE 6. **A typical speech signal.**

To transmit this signal over a digital channel, we first need to **sample** the signal at evenly spaced points in time. This is equivalent to multiplying the signal with a set of evenly spaced Dirac delta functions or an **impulse train** (see Section 5.10.1 on page 153). Recall that, to prevent aliasing, the Nyquist criterion states that the sampling function should have a frequency that is at least twice that of the highest frequency component of a signal (see Section 5.10.3 on page 155). We will assume that the highest frequency component of the signal, that is, the greatest frequency at which the Fourier transform of the signal is non-zero, is W. Then, to prevent aliasing, the sampling function should have a frequency of at least $2W$, so that the continuous function should be represented by at least $2W$ samples per second.

To send these samples on a digital channel, we need to represent each sample as a binary number. This is done by choosing a set of **quantization levels** as shown on the Y axis of Figure 7. At each sample time, we encode the signal by the binary number corresponding to the closest quantization level. If the quantization levels are chosen with care, the error introduced due to quantization are negligible and, in most cases, below the threshold of detection by human receivers. For human speech, it turns out that the quantization levels are not evenly spaced, as one might expect, but closer spaced near the origin and wider spaced further away. This models the differential sensitivity of the human ear to sounds of different magnitudes.

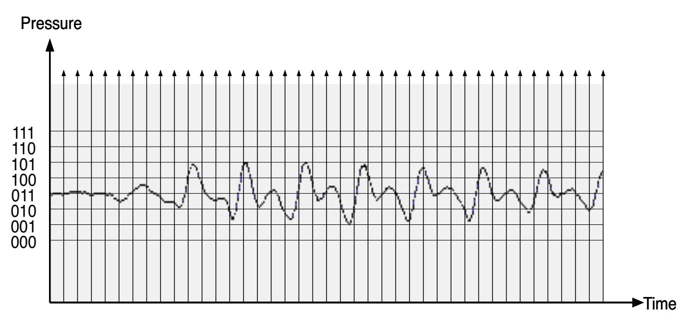


FIGURE 7. **A quantized speech signal. The first few sample values are 011,100,011,011,011,100,100,011...**

As a result of sampling and quantization, a continuous speech signal is translated into a series of digital values. This digital signal is then easily transmitted over a digital communication channel.

A similar approach can be used to digitize other types of continuous signals. To digitize two-dimensional signals, such as images, we define a two-dimensional path that covers the entire surface of the image over which the quantized samples are periodically collected. This also results in an output signal consisting of a series of digital values.

The time series of signal values can be represented as $x_1, x_2, \ldots$ where the x_i s are binary numbers. Any signal with n such values corresponds to a point in n-dimensional space. Each signal value typically represents a voltage. The corresponding *energy* is the proportional to the square of the signal voltage, so that, with an appropriate choice of units, we can write

$$\text{signal energy} = \sum_{i=1}^{n} x_i^2 \tag{EQ 14}$$

The expression on the right hand side is the square of the Euclidean distance of the point from the origin. We interpret this to mean that the energy associated with a signal point in a suitable n-dimensional space is the square of its distance from the origin.

9.6.2 A Gaussian Channel

Unlike ideal digital channels that carry only ‘0’ and ‘1’ symbols, in practice, channels are continuous, analog, and can be modelled as transporting real numbers from a transmitter to a receiver. A noiseless continuous channel has infinite information capacity because a single real number with its inherent infinite precision can encode an infinite amount of information. Of course, over a real channel, the presence of noise corrupts the transmitted signal and limits the channel’s information capacity.

The noise over many commonly used continuous channels is well-modelled as **white Gaussian noise**. The individual values of such an ideal noise signal are uncorrelated, independent, and with amplitudes drawn from a Gaussian distribution. The noise value is added to the transmitted signal value, and the received signal is their sum. A continuous channel subject to Gaussian noise is also called a **Gaussian channel**.

The use of a continuous channel to carry digital information and the impact of noise is demonstrated in Figure 8. Two signal amplitudes are designated as the levels corresponding to the symbol ‘0’ and the symbol ‘1’ respectively. The transmitter periodically places a ‘0’ or ‘1’ symbol for transmission on the channel. This transmission is subject to a random noise, so that the received signal, which is the sum of the channel and noise signals, differs from the transmitted signal. Moreover, due to channel noise, the received digital information may also differ from what the transmitter intended, as shown.

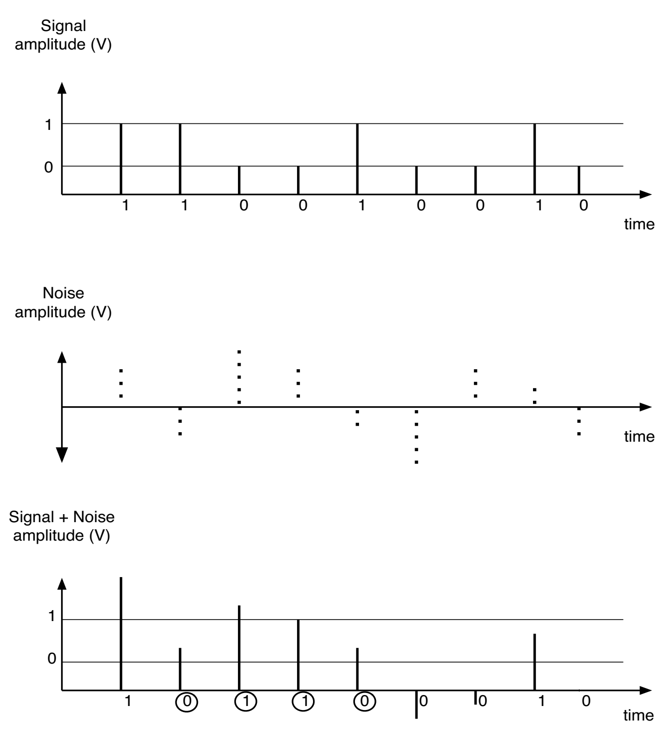


FIGURE 8. Carrying digital data on a continuous channel. The top figure shows the intended digit sequence and the corresponding signal amplitudes. The middle figure shows the random channel noise at each signal transmission time. The resulting corrupted signal is shown in the bottom figure, with errors circled.

It should be clear from Figure 8 that, if the noise amplitudes are kept unchanged, increasing signal amplitudes reduces the effect of noise. If signal levels, therefore, could be chosen to be sufficiently high, the effect of noise would be negligible and the channel would become a nearly noiseless channel. However, each transmission of a signal requires energy. To create an effectively noiseless channel may require the expenditure of large amounts of energy. If this energy is to come from a limited power source, such as the battery of a mobile phone, using large signal amplitudes would quickly drain the device battery. This is highly undesirable and a good reason to curtail signal levels.

A second reason to constrain signal levels has to do with signal transmission over a noisy wireless channel. In a wireless channel, two transmitters within range of the same receiver **interfere** with each other to their mutual detriment because both their transmissions are lost (each acts as a noise source to the other). If the signal amplitudes were large, a high-power transmitter would be heard across a wider geographical range and therefore would interfere with more other transmitters, actually reducing their overall channel capacity. Indeed, in the limit, if transmitters were to transmit with an arbitrarily large signal strength, only one transmitter anywhere in the universe should be permitted to be active at any given time in any frequency band to avoid interference at its intended receiver!

9.6.3 The Capacity of a Gaussian Channel

We now compute the capacity of a Gaussian channel. We will study the transmission and reception of a fixed-length message or block of n signal values, $x_1, x_2, \ldots, x_n$, where each signal value is a sequence of binary digits. An example of such a block with $n = 8$, corresponding to the signal in Figure 7, is 011, 100, 011, 011, 011, 100, 100, 011. Recall that the entire block of signal values can be represented by a single **signal point** in an n-dimensional space with energy $\sum_{i=1}^{n} x_i^2$.

To satisfy the Nyquist criterion, a source of a continuous signal whose highest frequency component is W (also called a signal with **band width** W) has to send at least $2W$ samples per second. Let T denote the time taken to transmit one block of signal values. In this time, the receiver receives $2WT$ samples. Therefore,

$$n = 2WT$$

Consider a *typical* block generated by a message source that selects symbols i.i.d. (see Section 9.5.1 on page 271). It can be shown that the average energy per signal value of a typical block, given by $\frac{1}{n}\sum_{i=1}^{n} x_i^2$, is almost surely some fixed value P, for a large enough block size n. The total energy of a typical block, therefore, is almost surely $2WPT$ Joules. Now, this is the energy associated with a point in n-dimensional space that is at a distance of $\sqrt{2WPT}$ from the origin. This means that the point corresponding to a typical block almost surely lies on the surface of an n-dimensional hypersphere of radius $\sqrt{2WPT}$.

Every transmitted sample is corrupted by noise, so in a time period T, there will also be $2WT$ noise values. Suppose that the average noise energy is given by N. Arguing along the lines above, we can show that, for a Gaussian noise source that generates 'typical' noise sequences, the set of $n = 2WT$ noise values will lie almost surely on the surface of a hypersphere of radius $\sqrt{2WNT}$. Geometrically, we view this to mean that each signal point sent by the transmitter gets distorted onto a point that lies the surface of a hypersphere with radius $\sqrt{2WNT}$ centered at the signal point.

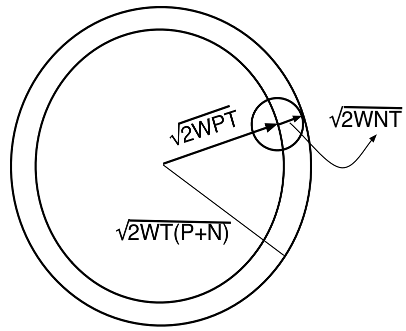


FIGURE 9. **Geometric view of a Gaussian channel. Signal points corresponding to message blocks lie on a hypersphere of radius $\sqrt{2WPT}$. Due to noise, each point is distorted onto a point that lies on the surface of a hypersphere centered on the signal point and with radius $\sqrt{2WNT}$. Points corresponding to noisy signal blocks lie within a hypersphere of radius $2WT(P+N)$.**

If two signal points are closer together than $2\sqrt{2WNT}$, a receiver will be unable to determine which signal point the transmitter intended. On the other hand, if the source chooses channel codewords so that every pair of signal points is spaced at least $2\sqrt{2WNT}$ units apart, then the receiver will almost surely be able to determine the intended signal despite the noise.

How many distinct signals can the transmitter choose to send in time *T* if they are to be decodable despite noise? This turns out to be a packing problem. We will use the mathematical fact that the volume of an *n*-dimensional hypersphere of radius *r* is proportional to r^n. Over time period *T*, the total received energy is $2WT(P+N)$, which corresponds to signals that lie on the surface of an *n*-dimensional hypersphere of radius $\sqrt{2WT(P+N)}$ and volume $(\sqrt{2WT(P+N)})^n$. Each signal should lie in the center of an *n*-dimensional hypersphere of radius $\sqrt{2WNT}$ and volume $(\sqrt{2WNT})^n$. Therefore, the total possible number of signals that can be sent in time *T* cannot be more than their ratio, which is $\frac{(\sqrt{2WT(P+N)})^n}{(\sqrt{2WNT})^n} = \left(1+\frac{P}{N}\right)^{n/2} = \left(1+\frac{P}{N}\right)^{WT}$. From Equation 9, this corresponds to an information carrying rate of at most $\frac{1}{T}\log\left(1+\frac{P}{N}\right)^{WT} = W\log\left(1+\frac{P}{N}\right)$ bits per second.

So far, we have denoted by *P* the average energy per signal value and by *N* the average energy per noise value. Over any time interval *T*, their ratio *P/N* is also the ratio of average signal power, given by *P/T*, to average noise power, *N/T*. Moreover, we can view the signal's band width *W* as a measure of the width of the continuous channel over which it is carried because the channel can carry any signal whose highest frequency component is *W* or smaller. With these substitutions, we can state the capacity of a Gaussian channel as:

$$\text{channel capacity} = \text{bandwidth} * \log(1+P/N) \quad \textbf{(EQ 15)}$$

Shannon showed that real codes over very long blocks of signal values can asymptotically achieve this capacity.

The ratio of the mean signal power to the mean noise power is often abbreviated as *SNR* and is measured in decibels (dB), where

$$SNR\ dB = 10\log_{10}\left(\frac{P}{N}\right) \quad \textbf{(EQ 16)}$$

A simple way to convert from dB values to normal values is to divide by ten and raising ten to the power of the resulting value. For example, 40dB corresponds to a ratio of $10^{40/10}$=10,000.

EXAMPLE 21: CAPACITY OF A GAUSSIAN CHANNEL

What is the channel capacity of a channel with band width 20 MHz if the SNR is 30dB?

Solution:

$P/N = 10^{30/10} = 1000$. So, the channel capacity is $20*10^6 * \log(1000+1) = 20*10^6*9.96 = 199.2$ Mbps. Another way to view this is that this channel has an inherent capacity of nearly 10 (bits/s)/Hz.

■

We note in passing that most computer networking practitioners refer to the information capacity of a potentially noisy communication channel as its 'bandwidth' measured in bits/second. This conflates many concepts. Strictly speaking, a channel of band width *W* Hz refers to a channel that can carry signals whose highest frequency component is smaller than *W* Hz. So, band width should be measured in Hz not bits/second. Second, the information carrying capacity of a noisy channel depends on the SNR of the carried signal. Without knowing the SNR, the channel's information carrying capacity (properly measured in bits/second) is unknown. However, for a nearly noiseless channel, such as an optical fiber, measuring information capacity in bits/second is reasonable.

The capacity of a wireless channel is greatly diminished due to interference that arises due to the simultaneous reception of two transmitted symbols at a receiver. Each transmitter appears as a source of noise to the other, decreasing the SNR. To take this into account, when talking about wireless channels, we often refer to the *SINR*, which is the Signal to (Interference + Noise) Ratio. This is demonstrated by the next example.

EXAMPLE 22: WIRELESS INTERFERENCE

Consider a WiFi (802.11b) wireless receiver that can receive symbols from either transmitter A with an SNR of 30 dB or transmitter B with an SNR of 25 dB, or from them both. What is the effective information capacity of the channel from A to the receiver and from B to the receiver when they are individually and simultaneously transmitting?

Solution:

A WiFi channel has a channel band width of 22MHz. When only A is transmitting $P/N = 10^{30/10} = 1000$. So, the channel capacity is $22*10^6 * \log(1001) = 22*10^6*9.96 = 219.12$ Mbps. Note that when using 802.11n encoding (orthogonal frequency division multiplexing or OFDM), the best achievable rate in practice is 54Mbps, which is a factor of four below the theoretical limit at this SNR.

When only B is transmitting, $P/N = 10^{25/10} = 316.2$. So, the channel capacity is $22*10^6 * \log(316.2) = 22*10^6*8.30 = 182.7$ Mbps.

When both A and B are transmitting, we can compute the SINR as follows. Let the noise power be *N*. Then, the signal power of transmitter A is 1000*N* and the signal power of transmitter B is 316.2*N*. When B's signal is viewed as noise, then the SINR is $\frac{1000N}{N + 316.2N} = 3.15$. So, the channel capacity is reduced to $22*10^6 * \log(3.15) = 36.4$ Mbps. Symmetrically, from B's perspective, the SINR is $\frac{316.2N}{N + 1000N} = 0.316$. This results in a negative value of $\log(P/N)$, which we interpret as a channel of zero information capacity: indeed, the signal is more than swamped by the noise.

■

9.7 Further Reading

Information theory was described by its creator in C. Shannon and W. Weaver, Mathematical Theory of Communication, University of Illinois Press, 1963. J. Pierce, An Introduction to Information Theory, Dover, 1981 is a concise, articulate, and lovely little book that presents the elements of information theory with little math and great insight. The standard text in this area is T. Cover and J.A. Thomas, Elements of Information Theory 2ed, Wiley, 2006. A more sophisticated mathematical treatment can be found in R. Gallager, Information Theory and Reliable Communication, Wiley, 1968.

9.8 Exercises

1 Entropy

What is the entropy of a random variable X whose probability mass function is given by $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$.

2 Entropy

What is the entropy of a source that generates equal-length messages of 100 symbols chosen uniformly randomly from an alphabet of 16 symbols?

3 Instantaneous codes

Consider the code shown below:

Message symbol	**Code**
‘a’	00
‘b’	001
‘c’	1
‘d’	11

What are the possible symbols that correspond to the code string ‘0011?’

4 Instantaneous codes

Is the following code instantaneous? Why or why not?

Message symbol	**Code**
‘a’	00
‘b’	01
‘c’	101
‘d’	100

5 Digit coding

What is the asymptotic limit in efficiency when coding digit sequences with binary strings?

6 Feasibility of a code

Your friend tells you that he has invented an instantaneous binary code for a set of 12 messages where the code lengths are 2, 2, 2, 3, 4, 6, 6, 6, 7, 7, 7, and 7 bits. Should you believe him?

7 Optimal codes

Consider a message source that generates message symbols with the following probabilities:

Message symbol	**Probability**
'a'	0.23
'b'	0.34
'c'	0.15
'd'	0.28

What is the expected length of the shortest instantaneous code to represent messages from this source?

8 Huffman codes

Give two Huffman codes for the message source of Exercise 7? What is its expected code length?

9 Huffman codes

Give an example where the Huffman code of Exercise 8 is not the optimal code for a specific message generated by the source.

10 Entropy rate

Suppose a source of Exercise 7 generates 100 independent symbols/second. What is its entropy rate? How many distinct messages can it generate in 100 seconds?

11 Typical messages

Consider a message source that independently generates '0' symbols with probability 0.9 and '1' symbols with probability 0.1. Suppose we define all messages with a probability of strictly lower than 10^{-10} to be 'atypical.' If all messages generated by the source are 12 symbols long, what fraction of its distinct messages are atypical? How many codes should be assigned to messages of length 50 symbols to ensure that the number of uncoded messages is vanishingly small?

12 A noiseless channel

Consider a noiseless channel of capacity 100 bits/second. How many symbols can it carry from the message source of Exercise 7?

13 Mutual information

Compute the mutual information for a channel coder that sends a '0' symbol with probability 0.2 and a '1' symbol with probability 0.8 over the channel of Figure 5.

14 Mutual information

Prove $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$.

15 Capacity of a binary symmetric channel

What is the capacity of a binary symmetric channel with an error probability of 0.001? Compare this to the channel capacity when the error probability is 0.

16 Capacity of a Gaussian channel

What is the channel capacity of a channel with band width 10 MHz if the SNR is 5dB? What should be the SNR in dB to achieve a channel capacity of 50 Mbps?

Solutions to Exercises

Chapter 1: Probability

1 Sample space

The sample space for CW is the discrete set {CWMIN, 2* CWMIN, 4* CWMIN, ...2^n*K*CWMIN}, where K is chosen so that 2^n*K*CWMIN < CWMAX. The sample space for backoff, given CW is a subset of the real line defined by [0, CW].

2 Interpretations of probability

An objective interpretation would be that we have a complete weather model that has an intersect source of randomness. Given this model and the current weather conditions, the model predicts that the probability of a snowstorm is 25%.

A frequentist approach would be to look at all prior days where today's weather conditions also held, and look at the number of such days where there was a snowstorm the next morning. We would see that 25% of the time, given the current weather, there was as snowstorm.

A subjective interpretation would be that an expert, who knew all the variables, would take 4:1 odds (or better) on a bet that it would snow tomorrow.

3 Conditional probability

(a) We have P(UDP) = 0.2, and P(UDP AND 100) = 0.1. So, P(100 | UDP) = 0.1/0.2 = 0.5.

(b) Here, P(UDP) = 0.5 and P(100|UDP) = 0.5. So, P(100 AND UDP) = 0.5*0.5 = 0.25.

4 Conditional probability again

Before you know the protocol type of a packet, the sample space is all possible packet lengths of all possible protocol types. After you know the protocol type, the sample space only include packet lengths for that protocol.

5 Bayes' rule

P(UDP|100) = (P(100|UDP)P(UDP))/P(100). We need P(100) = x. Then, P(UDP|100) = 0.5*0.2/x = 0.1/x.

6 Cumulative distribution function

(a) $F_D(i) = \sum_{j=1}^{i} \frac{1}{2^j} = 1-2^{-i}$.

(b) $f_{C(x)} = \frac{1}{x_2 - x_1}$, so $F_C(x) = \int_{x_1}^{x} \frac{1}{x_2 - x_1} dx = \frac{x - x_1}{x_2 - x_1}$.

7 Expectations

(a) $E[D] = \sum_{j=1}^{i} \frac{i}{2^j}$.

(b) By geometry, E[C] = $(x_2+x_1)/2$ (you can also derive this analytically).

8 Variance

$V[aX] = E[a^2X^2] - (E[aX])^2 = a^2(E[X^2] - (E[X])^2) = a^2V[X]$.

9 Moments

$M_\mu^3 = E((X-\mu)^3) = E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3) = M_0^3 - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 = M_0^3 - 3M_0^1 M_0^2 + 3\mu^2 M_0^1 - \mu^3$. The result follows from the fact that $M_0^1 = \mu$.

10 MGFs

$$\int_0^1 \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots\right) dx = x\Big|_0^1 + \frac{tx^2}{2!}\Big|_0^1 + \frac{t^2x^3}{3!}\Big|_0^1 + \frac{t^3x^4}{4!}\Big|_0^1 + \ldots$$

$$= 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots = \frac{1}{t}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) = \frac{1}{t}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1\right) = \frac{1}{t}(e^t - 1)$$

11 MGFs

To find the rth moment, we differentiate the MGF for the uniform distribution, i.e. $\frac{1}{t}(e^t - 1)$ r times and then set t to zero. Working directly from the series, we need to differentiate the expression $1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots$ r times and set t to 0. Note that all terms with powers of t smaller than r disappear when we differentiate this series r times. Moreover, all terms with powers of t greater than r disappear when we set t to zero after differentiation (why?). Therefore, the only term we need to consider the $t^r/(r+1)!$. It is clear the when we differentiate this r times, we get the term $r!/(r+1)!$, which reduces to $1/1+r$ as stated.

12 MGF of a sum of two variables

The MGF of the sum of two independent uniform random variables X_1 and X_2 is $\frac{1}{t^2}[e^t - 1]^2$, so, the MGF of (X-μ) is given by $\frac{e^{-\mu t}}{t^2}[e^t - 1]^2$. To find the variances need to differentiate this expression twice with respect to t and then set t to 0. Given the t in the denominator, it is convenient to rewrite the expression as

$\left(1-\mu t+\frac{\mu^2t^2}{2!}+\ldots\right)\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\ldots\right)\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\ldots\right)$ (we have divided e^t-1 by t in each of the second and third terms), where the ellipses refer to terms with third and higher powers of t, which will reduce to 0 when t is set to 0. In this product, we need only consider the coefficient of t^2, which is $\frac{\mu^2}{2!}+\frac{1}{3!}+\frac{1}{3!}-\frac{\mu}{2!}-\frac{\mu}{2!}+\frac{1}{2!2!}$. Differentiating the expression twice results in multiplying the coefficient by 2. Note that for the sum of two uniform standard random variables, $\mu = 1$, so that when we set t to zero, we obtain $E((X-\mu)^2) = V(X) = 2\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{6}-\frac{1}{2}-\frac{1}{2}+\frac{1}{4}\right) = \frac{1}{6}$. As a check, note that the variance of each variable is 1/12, so that the variance of the sum is the sum of the variances, as we found.

13 MGF of a normal distribution

The MGF of $a+bX$ is $e^{at}M(bt) = e^{at}e^{\mu bt+\frac{1}{2}\sigma^2(bt)^2} = e^{(a+\mu b)t+\frac{1}{2}(\sigma^2b^2)t^2}$. Set $a = \frac{-\mu}{\sigma}$ and $b = \frac{1}{\sigma}$. Then,

$e^{(a+\mu b)t+\frac{1}{2}(\sigma^2b^2)t^2} = e^{\left(\frac{-\mu}{\sigma}+\frac{\mu}{\sigma}\right)+\frac{1}{2}\left(\frac{\sigma^2}{\sigma^2}\right)t^2} = e^{\frac{t^2}{2}}$, which is the MGF of a $N(0,1)$ variable.

14 Bernoulli distribution

Consider the event E defined as 'Room X is making an outgoing call during the busy hour.' Clearly, P(E) =p = 1/6. The probability of 5 simultaneous calls is $\binom{20}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{15} = 0.129$ and of 15 simultaneous calls is

$\binom{20}{15}\left(\frac{1}{6}\right)^{15}\left(\frac{5}{6}\right)^{5} = 1.33*10^{-8}$.

15 Geometric distribution

Packet and ack transmissions are geometrically distributed with parameter p=0.9. So the expected number of packet transmissions is $1/p$ = 1.11 and the expected number of ack transmissions is also 1.11. These are independent events, so the expected number of data transmissions for successful packet+ack transfer = 1.11+1.11 = 2.22.

16 Poisson distribution

(a) Using the binomial distribution, the value is $\binom{10}{8}(0.1^8)(0.9^2)$ = .36*10^{-6}. For the Poisson approximation, λ= 1, so the value is $P(X=8) = e^{-1}\left(\frac{1^8}{8!}\right)$ = 9.12*10^{-6}. (b) Using the binomial distribution, the value is $\binom{100}{8}(0.1^8)(0.9^{92})$ = .114. For the Poisson approximation, λ= 10, so the value is $P(X=8) = e^{-10}\left(\frac{10^8}{8!}\right)$ = .112. It is clear that as n increases, the approximation greatly improves.

17 Gaussian distribution

Consider the cumulative distribution of $Y = F_Y(y)$ =

$P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{(y-b)}{a}\right) = F_X\left(\frac{(y-b)}{a}\right)$ if a > 0. Then, $f_Y(y) = F'_Y(y) =$

$$F_X'\left(\frac{(y-b)}{a}\right) = \frac{1}{a} f_X\left(\frac{(y-b)}{a}\right) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{\left(\left(\frac{y-b}{a}\right)-\mu\right)^2}{2\sigma^2}} = \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}} = \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}}.$$

Comparing with the standard definition of a Gaussian, we see that the parameters of Y are ($a\mu + b$, $(a\sigma)^2$). A similar calculation holds if $a < 0$.

18 Exponential distribution

We have $1/\lambda = 5$. We need to compute 1-F(15) = $1-(1-e^{-\lambda x}) = e^{\frac{-15}{5}} = e^{-3} = 4.98$ %.

19 Exponential distribution

Because the exponential distribution is memoryless, the expected waiting time is the same, i.e. 200 seconds, no matter how long your break for icecream. Isn't that nice?

20 Power law

x	$f_{power\_law}(x)$	$f_{exponential}(x)$
1	1	0.27
5	0.04	$9.07*10^{-5}$
10	0.01	$4.1*10^{-9}$
50	$4*10^{-4}$	$7.44*10^{-44}$
100	$1*10^{-4}$	$2.76*10^{-87}$

It should now be obvious why a power-law distribution is called 'heavy-tailed'!

21 Markov's inequality

(a) We need 1-$F(10) = e^{-20}$= $2.06*10^{-9}$. (b) The mean of this distribution is 1/2. So, $P(X \ge 10) \le \frac{0.5}{10} = 0.05$. It is clear that the bound is very loose.

22 Joint probability distribution

(a) p_X = {0.5, 0.5}; p_Y = {0.2, 0.8}; p_Z = {0.3, 0.7}; p_{XY} = {0.1, 0.4, 0.1, 0.4}; p_{XZ} = {0.15, 0.35, 0.15, 0.35}; p_{YZ} = {0.1, 0.1, 0.2, 0.6}

(b) X and Y are independent because $p_{XY} = p_X p_Y$. X and Z are independent because $p_{XZ} = p_X p_Z$.

(c) P(X=0|Z=1) = P(X=0 AND Z=1)/P(Z=1) = 0.35/0.7 = 0.5.

Chapter 2: Statistics

23 Means

To minimize $\sum_{i=1}^{n}(x_i - x^*)^2$, we differentiate the expression with respect to x^* and set this value to 0. We find that $\frac{d}{dx^*}\sum_{i=1}^{n}(x_i - x^*)^2 = \sum_{i=1}^{n}-2(x_i - x^*) = 0$, so that $\sum x_i - \sum x^* = 0$. Rewriting $\sum x^*$ as nx^*, we get the desired result.

24 Means

$$\frac{1}{n}\left(\sum_{i=1}^{n}(x_i - \mu)^2 - n(\bar{x} - \mu)^2\right) = \frac{1}{n}\left(\sum_{i=1}^{n}(x_i^2 + \mu^2 - 2x_i\mu) - n(\bar{x}^2 + \mu^2 - 2\bar{x}\mu)\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}x_i^2 + \mu^2 - 2x_i\mu - \bar{x}^2 - \mu^2 + 2\bar{x}\mu\right) = \frac{1}{n}\left(\sum_{i=1}^{n}x_i^2 - 2x_i\mu - \bar{x}^2 + 2\bar{x}\mu\right) = \frac{1}{n}\left(\sum_{i=1}^{n}x_i^2 - \bar{x}^2 + \sum_i 2\bar{x}\mu - 2x_i\mu\right)$$. But

$\sum_i (2\bar{x}\mu - 2x_i\mu) = 2n\bar{x}\mu - 2n\bar{x}\mu = 0$, hence the desired result.

25 Confidence intervals (normal distribution)

The sample mean is 61.11. We compute $\sum_{i=1}^{n}(x_i - \bar{x})^2$ as 936647.76. Therefore, the variance of the sampling distribution of the mean is estimated as 936647.76/(17*16) = 3443.55 and the standard deviation of this distribution is estimated as its square root, i.e., 58.68. Using the value of ±1.96σ for the 95% confidence interval, the 95% confidence interval is 61.11±115.02. The very large interval is due to the outlier value.

26 Confidence intervals (t distribution)

We simply substitute the value of ±2.12σ to obtain the interval as 61.11±124.40.

27 Hypothesis testing: comparing the mean to a constant

The mean is 2.46%. We compute the variance as 0.0076% and the standard deviation as 0.87%. We could use the t distribution to test the hypothesis, but it is clear by inspection that 2% lies within 1 standard deviation of the mean, so we cannot reject the null hypothesis. For completeness' sake, the confidence interval for the t distribution with 9 degrees of freedom (at the 95% level) is 2%±2.262*0.87%.

28 Chi-squared test

The critical value of n_1 is when the chi-squared value is $X = (n_1\text{-}42)^2/42 + (100\text{-}n_1\text{-}58)^2/58 = 3.84$. Solving, we get $n_1 > 51.67$. So, an value greater than or equal to 52 will result in the hypothesis being rejected.

29 Fitting a distribution and chi-squared test

The total number of time periods is 28+56+...+5 = 1193. The total number of arrivals is (28*2)+(56*3)+...+(5*16) = 8917. Therefore, the mean number of packets arriving in 1ms is 8917/1203 = 7.47. This is the best estimate of the mean of a Poisson distribution. We use this to generate the probability of a certain number of arrivals in each 1ms time period using the Poisson distribution. This probability multiplied by the total number of time periods is the expected count for that number of arrivals, and this is shown below.

Number of packet arrivals	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Count	28	56	105	126	146	164	165	120	103	73	54	23	16	9	5
Expected count	19	47	88	132	164	175	164	136	102	69	43	25	13	7	3

The chi-squared value is computed as $(28-19)^2/21 + (56-47)^2/47.... + (5-3)^2/3 = 19.98$. Since we estimated one parameter from the sample, the degrees of freedom = 15-1-1 = 13. From the chi-squared table, with 13 degrees of freedom, at the 95% confidence level, the critical value is 22.36. Therefore, we cannot reject the hypothesis that the sample is well-described by a Poisson distribution at this confidence level.

30 Independence, Regression, and Correlation

(a) If number of peers were independent of the number of peers, then, as the uplink capacity changed, the number of peers should remain roughly constant and equal to the population mean, whose best estimate is the sample mean. Therefore, the expected value of the number of peers is 50+31+...+49/10 = 40.4.

(b) The chi-squared variate is $(50-40.4)^2/40.4 + (31-40.4)^2/40.4 + ... + (49-40.4)^2/40.4 = 27.93$. Because we estimated one parameter from the data set (i.e., the mean), we have 10-1-1 = 8 degrees of freedom. We find that at the 95% confidence level, the critical value of the chi-squared distribution with 8 degrees of freedom is 15.51. Therefore, we can reject the hypothesis that the number of peers is independent of the uplink capacity with 95% confidence. The critical value at the 99.9% level is 25.125, so we can reject the hypothesis even at the 99.9% level.

(c) We use the equation $b = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$ to find $b = 0.21$.

(d) Using $r = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left(\sum (x_i - \bar{x})^2\right)\left(\sum (y_i - \bar{y})^2\right)}}$, we find $r = 0.952$, which is close to 1. Therefore, we can state that the two variables are well-represented by a linear relationship, which indicates dependence, rather than independence.

(e) The portion of variability in the number of peers by the uplink capacity is $r^2 = 90.1\%$.

31 Correlation coefficient

For convenience, we use the following notation:

$$X^2 = \sum (x_i - \bar{x})^2$$

$$Y^2 = \sum (y_i - \bar{y})^2$$

$$XY = \sum (x_i - \bar{x})(y_i - \bar{y})$$

$S^2 = \Sigma(y_i - a - bx_i)^2$. Ignoring the summation symbol, we can rewrite the summand as

$$\left(y_i - \left(\bar{y} - \bar{x}\frac{XY}{X^2}\right) - x_i\frac{XY}{X^2}\right)^2 = \left((y_i - \bar{y}) - \frac{(x_i - \bar{x})XY}{X^2}\right)^2 = Y^2 + \frac{(XY)^2X^2}{X^2X^2} - 2\frac{(XY)^2}{X^2} = Y^2 - \frac{(XY)^2}{X^2} = Y^2 - Y^2\left(\frac{(XY)^2}{Y^2}\right)$$

$= Y^2(1 - r^2)$, as desired. To understand the third step, recall the presence of the summation symbol.

32 Single Factor ANOVA

Here I = 3 and J = 10. We compute $\overline{Y_{1.}}$ = 55.53, $\overline{Y_{2.}}$ = 55.94, $\overline{Y_{3.}}$ = 55.95. This allows us to compute SSW = 3102.29 and SSB = 1.15. The F statistic is therefore (1.15/ 2)/(3102.29/27) = 0.0050. Looking up the F table we find that with (3, 27) degrees of freedom, the critical F value even at the 5% confidence level is 2.96. The computed statistic is far below this value. Therefore, the null hypothesis cannot be rejected.

Chapter 3: Linear Algebra

1 Transpose

$$\begin{bmatrix} 4 & 7 & 3 \\ 0 & 82 & -2 \\ -3 & 12 & 2 \end{bmatrix}$$

2 Matrix multiplications

$$\begin{bmatrix} -44 & 118 & -54 \\ 59 & 14 & -24 \\ -40 & 20 & 40 \end{bmatrix}$$

3 Exponentiation

The proof is by induction. The base case is for *k=2*, where by direct computation, we find that

$\begin{bmatrix} a_{11} & \dots & 0 \\ \dots & a_{ii} & \dots \\ 0 & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} a_{11} & \dots & 0 \\ \dots & a_{ii} & \dots \\ 0 & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & \dots & 0 \\ \dots & a_{ii}^2 & \dots \\ 0 & \dots & a_{nn}^2 \end{bmatrix}$. The inductive assumption is that

$A^k = \begin{bmatrix} a_{11} & \dots & 0 \\ \dots & a_{ii} & \dots \\ 0 & \dots & a_{nn} \end{bmatrix}^k = \begin{bmatrix} a_{11}^k & \dots & 0 \\ \dots & a_{ii}^k & \dots \\ 0 & \dots & a_{nn}^k \end{bmatrix}$. Then, we compute the $k+1^{\text{th}}$ power of A as

$A^kA = \begin{bmatrix} a_{11}^k & \dots & 0 \\ \dots & a_{ii}^k & \dots \\ 0 & \dots & a_{nn}^k \end{bmatrix}\begin{bmatrix} a_{11} & \dots & 0 \\ \dots & a_{ii} & \dots \\ 0 & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^{k+1} & \dots & 0 \\ \dots & a_{ii}^{k+1} & \dots \\ 0 & \dots & a_{nn}^{k+1} \end{bmatrix}$. QED.

4 Linear combination of scalars

The linear combination is 10*0.5 + 5*0.4 + 2*0.25 + -4*0.25 = 5 + 2 + .5 - 1 = 6.5

5 Linear combination of vectors

The first element of the linear combination is given by 1*0.5 + 3*0.4 + 7*0.25 + 2*0.25 = 0.5 + 1.2 + 1.75 + 0.5 = 3.95. Computing the other elements similarly, we obtain the solution [3.95 5.8 6.2 6.15].

6 Linear independence and rank.

The implicitly defined coefficient matrix is given by

$\begin{bmatrix} 12 & 2 & -4 \\ 2 & 2 & -24 \\ 2.5 & 0 & 5 \end{bmatrix}$. If the vectors are independent, then the rank of this matrix will be 3, so that Gaussian elimination would result in no equations being reduced to the trivial form 0=0. We proceed with Gaussian elimination as follows: Equation 3 does not contain the second variable, so we remove the second variable from the second equation by subtracting the first row from the second row, to get $\begin{bmatrix} 12 & 2 & -4 \\ -10 & 0 & -20 \\ 2.5 & 0 & 5 \end{bmatrix}$. It is clear that the second row is the third row multiplied by -4, so that if we add 4 times the third row to the second row, we get $\begin{bmatrix} 12 & 2 & -4 \\ 0 & 0 & 0 \\ 2.5 & 0 & 5 \end{bmatrix}$. Can we reduce any of the remaining equations to the form 0=0? It is clear that the first row is not a multiple of the third row, because the second element of the third row is 0, and the second element of the first row is not. Hence, the rank of the co-efficient matrix is 2, which is smaller than 3, so that the three vectors are *not* independent.

7 Basis and dimension

Two of the three vectors are linearly independent, so we have two vectors in the basis and a generated vector space of dimension 2. One possible basis is simply the two vectors themselves, that is, {[12 2 -4], [2.5 0 5]}. We can get another basis by multiplying either vector by any scalar. For example, we can multiply the first vector by 0.5 to get another basis as {[6 1 -2], [2.5 0 5]}.

8 Gaussian elimination

Noticing that the third equation has a zero in the second column, we will eliminate the second variable in the firs row as well, by subtracting twice the second row from the first row, to obtain $\begin{bmatrix} 22 & 0 & -16 & 9 \\ -8 & 2 & 4 & -2 \\ 10 & 0 & 4 & 1 \end{bmatrix}$. We can eliminate the third variable from the first row by multiplying the third row by 4 and adding it to the first row, to get $\begin{bmatrix} 62 & 0 & 0 & 13 \\ -8 & 2 & 4 & -2 \\ 10 & 0 & 4 & 1 \end{bmatrix}$. We can read off x_1 = 13/62 = 0.2096. Substituting in the third row, we find 2.096 + 4x_3 = 1, so that x_3 = (1-2.096)/4 = -0.274. Substituting these in the first row of the original equation, we find 6*0.2096 + 4*x_2 -8*-0.274 = 5, so that x_2 = 0.3876.

9 Rank

Consider the *i*th row of a non-zero diagonal matrix. Its diagonal element is a_{ii} which is not 0, but all other elements in the *i*th column are 0. Therefore, there is no way to obtain the *i*th row as a linear combination of the other rows. Since this is true for all *i*, the rows are all linearly independent, and the rank of the matrix is *n*. Note that the rows are therefore a basis of the corresponding vector space.

10 Determinant

Expanding by the second column, we find the determinant to be 8*(4*2 - 3*(-3)) - (-2)*(4*12-7*(-3)) = 8*(8+9) +2*(48+21) = 8*17+2*69 = 274.

11 Inverse

We already know the determinant of the matrix is 274 (see Exercise 10). The co-factor C_{11} is given by (-1)$^{1+1}$(8*2-(-2)*(12)) = 16+24 = 40. C_{21}= (-1)$^{1+2}$(0*2 -(-2)*(-3)) = -(-6)= 6. Computing the other co-factors similarly, we obtain the inverse as $\frac{1}{274}\begin{bmatrix} 40 & 6 & 24 \\ 22 & 17 & -69 \\ -38 & 8 & 32 \end{bmatrix}$.

12 Matrix as a transformation

Let the angle made by the vector from (0,0) to (*x*, *y*) be *t* and let its length be *r*. Then, we can write *x* and *y* as

$$x = r\cos(t)$$
$$y = r\sin(t)$$

Let the rotated vector join the origin to the point (*X*,*Y*). We expand:

$$X = r\cos(t+p) = r(\cos(t)\cos(p) - \sin(t)\sin(p)) = x*\cos(p) - y*\sin(p)$$

$$Y = r\sin(t+p) = r(\sin(t)\cos(p) + \cos(t)\sin(p)) = y*\cos(p) + x*\sin(p)$$

We can write this as

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos(p) & -\sin(p) \\ \sin(p) & \cos(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so that the rotation matrix is $\begin{bmatrix} \cos(p) & -\sin(p) \\ \sin(p) & \cos(p) \end{bmatrix}$.

13 Composing transformations

We compute the composition as

$$\begin{bmatrix} \cos(p) & -\sin(p) \\ \sin(p) & \cos(p) \end{bmatrix} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} = \begin{bmatrix} \cos(p)\cos(t) - \sin(p)\sin(t) & -\cos(p)\sin(t) - \sin(p)\cos(t) \\ \sin(p)\cos(t) + \cos(p)\sin(t) & -\sin(p)\sin(t) + \cos(p)\cos(t) \end{bmatrix} =$$

$\begin{bmatrix} \cos(p+t) & -\sin(p+t) \\ \sin(p+t) & \cos(p+t) \end{bmatrix}$, which we recognize as a rotation by a total of *t*+*p* degrees, as expected.

14 Eigenvalues and eigenvectors

The characteristic equation is $\begin{vmatrix} 1-\lambda & 9 \\ 4 & 1-\lambda \end{vmatrix} = 0$, so that $(1-\lambda)^2 - 36 = 0$, and we get $\lambda = -5, 7$ as the eigenvalues.

We compute the eigenvector corresponding to the value -5 by solving the equation

$\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-5) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This gives us the equations $x_1 + 9x_2 = -5x_1$; $4x_1 + x_2 = -5x_2$. Either one can be solved to get $x_2 = -(2x_1)/3$, corresponding to an eigenvector family of scalar multiples of $[\ 1\ -2/3]^T$.

We compute the eigenvector corresponding to the value 7 by solving the equation

$\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 7 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This gives us the equations $x_1 + 9x_2 = 7x_1$; $4x_1 + x_2 = 7x_2$. Either one can be solved to get $x_2 = 2x_1/3$, corresponding to an eigenvector family of scalar multiples of $[\ 1\ 2/3]^T$.

15 Computing $A^n x$

From Exercise 14, we know that the eigenvectors of $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \frac{-2}{3} \end{bmatrix}$. We recognize the vector $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$ can be written as $4\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} + 4\begin{bmatrix} 1 \\ \frac{-2}{3} \end{bmatrix}$. Hence,

$$\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}^5 \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}^5 \left(4\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} + 4\begin{bmatrix} 1 \\ \frac{-2}{3} \end{bmatrix} \right) = 4\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}^5 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} + 4\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}^5 \begin{bmatrix} 1 \\ \frac{-2}{3} \end{bmatrix} = 4\left(-5^5 \begin{bmatrix} 1 \\ \frac{-2}{3} \end{bmatrix} + 7^5 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right) = \begin{bmatrix} 54728 \\ 53152 \end{bmatrix}$$

16 Finding eigenvalues

The matrix is symmetric, so its eigenvalues are real. From the Gerschgorin circle theorem, the eigenvalues lie in the intersection of the real intervals [4-1.5 4+1.5], [6-1.3 6+1.3], [5-0.8 5+0.8] = {[2.5 5.5], [4.7 7.3], [4.2 5.8]} = [2.5 7.3].

17 Power method

We start with the initial vector $x_0 = \begin{bmatrix}1\\1\end{bmatrix}$. Applying the matrix once, we get $x_1 = \begin{bmatrix}10\\5\end{bmatrix}$. The Rayleigh ratio evaluates to([1 1] * $\begin{bmatrix}10\\5\end{bmatrix}$)/([1 1] * $\begin{bmatrix}1\\1\end{bmatrix}$) = 15/2 = 7.5. Repeating, we get $x_2 = \begin{bmatrix}55\\45\end{bmatrix}$ and the ratio evaluates to 775/125 = 6.2. After one more iteration, we get $x_3 = \begin{bmatrix}460\\265\end{bmatrix}$, and the ratio evaluates to 37225/5050 = 7.37. For the fourth iteration, we get $x_4 = \begin{bmatrix}2845\\2105\end{bmatrix}$, and the ratio evaluates to 1866525/281825 = 6.622. We see that the series slowly converges to the dominant eigenvalue of 7.

To compute the dominant eigenvalue, we start with $x_1 = \begin{bmatrix}10\\5\end{bmatrix}$, which we rescale to $x_1 = \begin{bmatrix}1\\0.5\end{bmatrix}$. Then, $x_2 = \begin{bmatrix}1 & 9\\4 & 1\end{bmatrix}\begin{bmatrix}1\\0.5\end{bmatrix} = \begin{bmatrix}5.5\\4.5\end{bmatrix}$, which we rescale to $\begin{bmatrix}1\\0.818\end{bmatrix}$. Thus, $x_3 = \begin{bmatrix}1 & 9\\4 & 1\end{bmatrix}\begin{bmatrix}1\\0.818\end{bmatrix} = \begin{bmatrix}8.362\\4.818\end{bmatrix}$, which we rescale to $\begin{bmatrix}1\\0.576\end{bmatrix}$.Finally, $x_4 = \begin{bmatrix}1 & 9\\4 & 1\end{bmatrix}\begin{bmatrix}1\\0.576\end{bmatrix} = \begin{bmatrix}6.184\\4.576\end{bmatrix}$, which we rescale to $\begin{bmatrix}1\\0.734\end{bmatrix}$ and is the estimate of the dominant eigenvector. Compare this to the true value of $x = \begin{bmatrix}1\\0.66\end{bmatrix}$.

18 Diagonalization

This is the matrix with the same eigenvalues as the given matrix, i.e., $\begin{bmatrix}-5 & 0\\0 & 7\end{bmatrix}$.

19 Stochastic matrix

The matrix is left- (or column-) stochastic but not right- (or row-) stochastic because its columns add to 1.0, but its rows do not.

20 State transitions

The initial state vector is $[0.5\ 0.5\ 0]^T$. After one time step, the state vector is

$\begin{bmatrix}0.25 & 0.1 & 0\\0.5 & 0.9 & 0\\0.25 & 0 & 1.0\end{bmatrix}\begin{bmatrix}0.5\\0.5\\0\end{bmatrix} = \begin{bmatrix}0.175\\0.7\\0.125\end{bmatrix}$. After another time step, the state vector is $\begin{bmatrix}0.25 & 0.1 & 0\\0.5 & 0.9 & 0\\0.25 & 0 & 1.0\end{bmatrix}\begin{bmatrix}0.175\\0.7\\0.125\end{bmatrix} = \begin{bmatrix}0.11375\\0.7175\\0.16875\end{bmatrix}$. Therefore, the probability of being in state 1 after two time steps is 0.11375, and of being in state 2 after two time steps is 0.7175.

Chapter 4: Optimization

1 Modelling

This problem has many solutions. Here is one possible

<u>Control variables</u> (these are from the statement of the problem):

- *x*i : starting point of flight i
- *di* : duration of flight *i*, $d_i \geq 15$
- The cost of ticket for the *i*th flight, *t*i.

Note that the number of passengers in a flight is not a control parameter, because it is related to the cost of a ticket - once the cost of the ticket is determined, the number of passengers cannot be independently controlled.

Fixed parameters:

- The possible take-off locations, *V* .
- The cost of chase vehicle (gas needed, maintenance, etc.) per kilometer of travel.
- Location of the roads, *R*.
- The cost of the natural gas, *g*, per minute of flight.
- Pilot's wages, *w*, per minute of flight.

Input parameters:

- The wind speed and direction for flight *i*

Transfer functions:

- Where a balloon lands, as a function of starting point, wind speed and direction, and flight duration.
- The number of passengers p as a function of cost of a ticket.
- A function that, given the cost of every process in the business, computes the cost of flight i.

Output parameters:

- For flight *i*, the distance of the balloon landing spot from a road
- Number of passengers for flight *i*, p_i
- The cost of flight i, denoted f_i

Objective function:

- Maximize $p_i t_i - f_i$

Empirically estimating the transfer functions:

- For each starting point, for each wind speed and direction, empirically determine the landing spot.
- The number of passengers for a cost can be determined by trial and error or by doing a market study.
- The cost of flight can be determined empirically as a linear combination of the xed parameters.

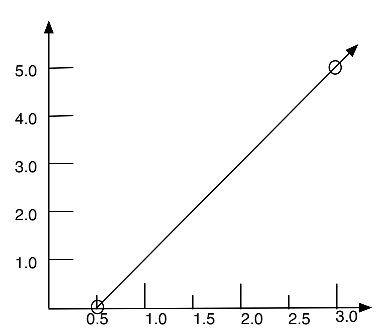
2 Optimizing a function of two variables

The figure below shows the plot for the curve $2x_1 - x_2 = 1$ subject to $x_1 \geq 0$ and $x_2 \geq 0$. The system does not impose any constraint on how much x_1 and x_2 can grow, so the maximum value is unbounded. We know that the minimal value is at a vertex, in this case we only have one (0:5; 0). If we evaluate the function in (0:5; 0) and at a randomly chosen point, say (3; 5) we get:

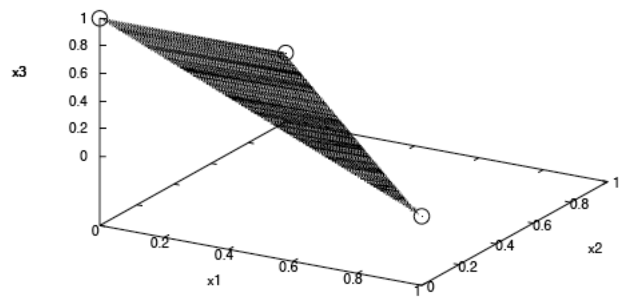
$O(0{:}5; 0) = 5$

$O(3; 5) = 15$

Using this information, we know that the minimum value of O, given the constraints, is 5 at (0:5; 0). There is no maximum value, since O is unbounded in the space dened by the constraints.



3 Optimizing a function of three variables



The figure above shows the plot for the plane $x_1 + x_2 + x_3 = 1$ for x_1, x_2, x_3 non-negative. The resulting polyhedron serves to find the optimal values of O. The optimal value has to be at a vertex, so we evaluate the value of the objective function at the three vertices.

$O(1; 0; 0) = 5$

$O(0; 1; 0) = 2$

$O(0; 0; 1) = -1$

Clearly, the maximum value is reached at point (1; 0; 0) and the minimum value is reached at point (0; 0; 1).

4 Network flow

We will consider the problem when there is only one source node *s*. Otherwise, if we have many sources, we can always create a new source with unbounded capacity to transfer to all the sources, which would each have limited capacity. Similarly, we can unify all the sinks to form a single sink *t*.

Let $G = (V, E)$ be the graph, and let f_{ij} be the flow between nodes v_i and v_j. Let c_{ij}be the capacity of link from v_i to v_j. The classical problem is stated as:

$$O = \sum_i f_{si} \quad \text{subject to}$$

$$\sum_i f_{ij} = \sum_k f_{jk} \quad \forall j \notin \{s, t\}$$

$$f_{ij} \le c_{ij} \quad \forall i, j$$

We can interpret the 'capacity of a warehouse' in two ways. One way to interpret it is that no more than cap_j flow can go through warehouse *j*. To model this, we add the following constraint:

$$\sum_i f_{ij} \le cap_j \quad \forall j$$

A more complex interpretation of the constraint is that each warehouse has a limited storage capacity. This would allow the ingress flow to exceed the egress flow for a limited duration of time. Specifically, if the storage capacity of warehouse *j* is B_j, then, denoting the flow on link v_i-v_j by $f_{ij}(t)$,

$$\sum_i \int_{t_i}^{t_2} f_{ij}(t)dt \le \sum_k \int_{t_i}^{t_2} f_{jk}(t)dt + B_j \quad \forall j \notin \{s, t\}, \forall t_i, \forall t_2$$

so that the ingress flows to any node *j*, integrated over all possible time periods, never exceeds the egress flows, integrated over the same time period, taking into account the possibility of storing B_j units in the warehouse.

5 Integer linear programming

Let the variable x_{ijh} indicate whether or not user i can schedule a job on machine h at time period j. Let the cost and benefit of assigning machine h to user i at time period j be c_{ijh} and g_{ijh} respectively. Then, the function to optimize is

$$O = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{h=1}^{k}(g_{ijh} - c_{ijh})x_{ijh}$$

The obvious constraints are that $x_{ijh} \in \{0, 1\}$, which makes the problem an ILP. In addition, we express the constraint that at each time slot a machine can be assigned to at most one user using the constraint:

$$\sum_{i=1}^{n} x_{ijh} \le 1 \quad \forall j, h$$

6 Weighted bipartite matching

The standard bipartite matching allows us to place only one ball in one urn, so we modify the elements of the bipartite graph as follows: given m urns in M indexed by i, we create a new set M' that contains $2m$ elements, labelled m'_{i1} and m'_{i2}. Now, create links from K to M' where the payoff on the link from ball k_j to urn m'_{i1} and m'_{i2} is the same as the payoff from ball k_j to urn m_i in the original problem, i.e., p_{ji}. The solution to the weighted bipartite matching problem in M' trivially gives us the solution to the problem in M.

7 Dynamic programming

Let $D(i,j)$ denote the numbers of errors in a match that between the first i characters in S and all possible substrings formed from the first j characters in L. Let $err(a,b) = 1$ if $a = b$ and 0 otherwise.

Suppose $S(i) = D(j)$. Then, $D(i, j) = D(i\text{-}1, j\text{ -}1)$.

Otherwise, we can compute $D(i, j)$ as the smallest of scores computed from one of three actions:

(a) Substituting $S(i)$ in $L(j)$ with a penalty of 1 added to $D(i\text{-}1, j\text{ -}1)$.

(b) Deleting the $L(j)$th character, so that we are matching the first i characters of S with the first j-1 characters of L, which costs $D(i, j\text{-}1)$ + a penalty of 1.

(c) Inserting a character at the jth position in L to make it match the character at the ith position of S. This costs the same as matching the first i-1 characters in S with the first j characters in L, i.e., $D(i\text{-}1, j)$ plus a added penalty of 1.

We can rewrite this as:

If $S(i) = D(j)$ then $D(i, j) = D(i\text{-}1, j\text{ -}1)$

else

$D(i, j) = \min(D(i\text{-}1, j\text{ -}1) +1, D(i, j\text{-}1) +1, D(i\text{-}1, j) +1)$

Note that in all cases, $D(i, j)$ depends on a smaller index of either i or j or both, which creates an optimal substructure with reusable results.

If we start with $i = 1$, j=1, we can memoize the $|L||S|$ entries and compute scores in time proportional to $|L||S|$. We set $D(i, 0) = i$ and $D(0, j) = j$ as boundary conditions. The string associated with each memoized position is the best match for that position, and is kept track of in the table depending on which of the three actions above were chosen to compute that position.

8 Lagrangian optimization

Both functions are continuous and twice-differentiable. We define the Lagrangian $F(x, y, \lambda) = x^3 + 2y + \lambda(x^2 + y^2 - 1)$. Setting $\nabla F = 0$, we get

$$\frac{\partial F}{\partial x} = 3x^2 + 2\lambda x = 0 \qquad \textbf{(EQ 1)}$$

$$\frac{\partial F}{\partial y} = 2 + 2\lambda y = 0 \qquad \textbf{(EQ 2)}$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 1 = 0 \qquad \textbf{(EQ 3)}$$

Solving (1), we get two solutions for x, denoted x_1 and x_2 :

$$x_1 = 0, x_2 = -\frac{2\lambda}{3}$$

Corresponding to x_1 we solve (3) to find $y_{11} = 1$, $y_{12} = -1$ and put these in (2) to get $\lambda_{11} = -1$, $\lambda_{12} = 1$. The extermal values of $z = x^3 + 2y$ for this solution of x therefore are 2 and -2, achieved at the points (0,1) and (0,-1).

Corresponding to x_2 we find from (3) that $\frac{4}{9}\lambda^2 + y^2 = 1$. Substituting $\lambda = -\frac{1}{y}$ from (2) and solving for y, we find that y is complex, so that there are no real points (x,y) satisfying (3). Therefore, the only viable extremal points are the two found above, which correspond to a constrained maximum and constrained minimum respectively.

9 Hill climbing

We start with K random points and compute the optimal value reached at each point. If we have K unique results, we return the best point. Otherwise, we eliminate the repeated results, say r of them, and start again with r points and repeat the process (remembering those results already computed). When we reach K different points the algorithm finishes and returns the global optimum. Note that we could iterate infinitely before finding the K local optima. However, without making any additional assumptions about the space, we cannot guarantees a better method to find the global optimum.

Chapter 5: Transform domain techniques

1 Complex arithmetic

$$\begin{aligned}
& e^{-j\frac{\pi}{2}} + e^{j\frac{\pi}{2}} \\
&= \left(\left(\cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right) + \left(\cos\left(\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right)\right) \\
&= 2\cos\left(\frac{\pi}{2}\right) \\
&= 0
\end{aligned}$$

2 Phase angle

This is given by $\operatorname{atan}\left(\frac{1}{1}\right) = \frac{\pi}{4}$.

3 Discrete convolution

$z(5) = \sum_{\tau=-\infty}^{\infty} x(\tau)y(5-\tau)$. This reduces to computing products $x(a).y(b)$ where $a+b = 5$. These are the pairs (1, 9), (3, 5), (5, 4), (2, 7), (5, 1), (8, 3), whose products are 9, 15, 20, 14, 5, 24 and whose sum = $z(5)$ = 87.

4 Signals

Temperature readings from a digital thermometer.

5 Complex exponential

The projection is obtained by setting the real value to 0, so that the curve is given by the expression $j5e^t\sin(3t)$. This curve lies entirely in the complex (Im-t) plane. It corresponds to a sinusoid of frequency 3Hz whose amplitude increases exponentially with time. At time 0, it has an amplitude of 5, at time 1/3 an amplitude of 5e, at time 2/3 an amplitude of $5e^2$, and, in general, at time $3k$, an amplitude of $5e^k$.

6 Linearity

$H(k_1x_1 + k_2x_2) = \left(\frac{5d(k_1x_1)}{dt}+1\right)+\left(\frac{5d(k_2x_2)}{dt}+1\right) = \left(\frac{5k_1dx_1}{dt}+1\right)+\left(\frac{5k_2dx_2}{dt}+1\right) = H(k_1x_1)+H(k_2x_2)$, so the system is linear.

7 LTI system

Any sinusoid can be written as the sum of equal and opposite complex exponentials. A complex exponential input to an LTI system results in a complex exponential output. Since the system is LTI, a sinusoidal input will result in an output that is the sum of equal and opposite complex exponentials, which sum to a real sinusoid that is potentially a scaled and phase-shifted version of the input, but with the same frequency.

8 Natural response

Compute the natural response of the LTI system given by $2\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dt}+15y(t) = 32x(t)$.

Solution: The natural response is given by the differential equation $(2D^2+11D+15)y(t) = 0$. This can be factored as $2((D+3)(D+2.5))y(t) = 0$. Thus, the natural response is given by $c_1e^{-3t}+c_2e^{-2.5t}$, where the two constants can be determined from the initial conditions $y(0)$ and $\dot{y}(0)$.

9 Natural response

The natural response is given by the differential equation $2D^2+1 = 0$, whose factorization is $\left(D-\frac{j}{\sqrt{2}}\right)\left(D+\frac{j}{\sqrt{2}}\right)$.

The system is therefore given by $c_1e^{\frac{-jt}{\sqrt{2}}}+c_2e^{\frac{jt}{\sqrt{2}}} = y(t)$. Setting $y(0) = 0$, we get $c_1+c_2 = 0$. Setting $\dot{y}(0) = 1$, we get $\frac{-jc_1}{\sqrt{2}}+\frac{jc_2}{\sqrt{2}} = 1$, which we can rewrite as $c_1-c_2 = j\sqrt{2}$. Solving, we get $c_1 = -\frac{j}{\sqrt{2}}$, $c_2 = \frac{j}{\sqrt{2}}$, so that the natural response is $-\frac{j}{\sqrt{2}}e^{\frac{-jt}{\sqrt{2}}}+\frac{j}{\sqrt{2}}e^{\frac{jt}{\sqrt{2}}} = y(t)$. The frequency of this signal is $\frac{1}{\sqrt{2}}$ Hz.

10 Stability

The signal reduces to the complex sinusoid $j\sqrt{2}\sin\left(\frac{t}{\sqrt{2}}\right)$ whose real value is always zero, so that they system is stable.

11 Fourier series

Since the series is infinite, we can choose to center one of the pulses around the origin and compute the Fourier coefficients in the range $-T_0/2$ to $T_0/2$. The kth coefficient of the Fourier series corresponding to this function is given by $c_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jk\omega_0t}dt$. In this range, the function is $1+t$ in the range $[-\tau, 0]$, $1-t$ in the range $[0,\tau]$ and 0 elsewhere. For convenience, let $a = -jk\omega_0$. Then, the integral reduces to

$$c_k = \frac{1}{T_0}\left(\int_{-\tau}^{0}(1+t)e^{at}dt+\int_{0}^{\tau}(1-t)e^{at}dt\right)$$

$$= \frac{1}{T_0}\left(\int_{-\tau}^{0}e^{at}dt+\int_{-\tau}^{0}te^{at}dt+\int_{0}^{\tau}e^{at}dt-\int_{0}^{\tau}te^{at}dt\right)$$

$$= \frac{1}{T_0}\left(\int_{-\tau}^{\tau}e^{at}dt+\int_{-\tau}^{0}te^{at}dt-\int_{0}^{\tau}te^{at}dt\right)$$

We can solve this as $\frac{1}{T_0}\left(\frac{1}{a}e^{at}\Big|_{-\tau}^{\tau} + \frac{ate^{at}-e^{at}}{a^2}\Big|_{-\tau}^{0} - \frac{ate^{at}-e^{at}}{a^2}\Big|_{0}^{\tau}\right)$, which reduces to

$$\frac{1}{aT_0}\left((e^{a\tau}-e^{-a\tau}) + \left(\frac{e^{-a\tau}-1}{a} + \tau e^{-a\tau}\right) - \left(\frac{e^{a\tau}-1}{a} - \tau e^{a\tau}\right)\right)$$
$$= \left(\frac{(a-1)}{a^2T_0}(e^{a\tau}-e^{-a\tau}) + \frac{\tau}{aT_0}(e^{a\tau}+e^{-a\tau})\right) .$$

12 Fourier series

The fundamental frequency $\omega_0 = \frac{2\pi}{10}$. The third coefficient is the value of $X(\omega) = \frac{\tau\omega_0}{2\pi}\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$

for the value $\omega = 3\omega_0 = \frac{6\pi}{10} = 0.6\pi$. This is given by $\frac{1}{10}\frac{\sin(0.3\pi)}{0.3\pi}$ = 0.085.

13 Fourier transform

Since the function is non-zero only in the range [0,1], the transform is given by

$$X(j\omega) = \int_0^1 (1-t)e^{-j\omega t}dt = \int_0^1 e^{-j\omega t}dt - \int_0^1 te^{-j\omega t}dt$$. This reduces to

$$\frac{e^{-j\omega t}}{-j\omega}\Bigg|_0^1 + \frac{(-j\omega)te^{-j\omega t}-e^{-j\omega t}}{\omega^2}\Bigg|_0^1 = \frac{e^{-j\omega}-1}{-j\omega} + \frac{(-j\omega)e^{-j\omega}-e^{-j\omega}+1}{\omega^2} .$$

14 Inverse Fourier transform

The inverse transform is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))e^{j\omega t}d\omega = \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+\omega_0)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega$$. Applying Equation 15 twice, the integral reduces to $\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$, which from Equation 5 is simply $\cos(\omega_0 t)$.

15 Computing the Fourier transform

Using the time-shift and linearity properties, and the standard transforms, this is given by

$$(\pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)))e^{j\omega t_0} + (j\pi(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)))e^{-j\omega t_0} .$$

16 Laplace transform

We use Euler's formula to rewrite the signal as $u(t)\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)$. By definition,

$$X(s) = \int_{-\infty}^{\infty} u(t)\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)e^{-st}dt = \frac{\left(\int_0^{\infty} e^{j\omega_0 t}e^{-st}dt - \int_0^{\infty} e^{-j\omega_0 t}e^{-st}dt\right)}{2j} = \frac{1}{2j}\left(\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right) = \frac{\omega_0}{s^2+\omega_0^2}$$, with the region of convergence is $Re(s) > 0$. The poles are at $s = \pm j\omega_0$ and the transform is either always 0, if ω_0 is zero, or never zero, otherwise.

17 Laplace transform

From the previous exercise and the time-shifting property of the Laplace transform, this is given by $\frac{e^{st_0}\omega_0}{s^2+\omega_0^2}$.

18 Solving a system using the Laplace transform

From Table 4 the Laplace transform of the transfer function $H(s)$ is given by $\frac{s}{s^2+\omega_0^2}$, *Re(s) > 0*. Moreover, because $x(t) = e^{-t}u(t)$, from Table 4 $X(s) = \frac{1}{s+1}$, *Re(s) > 1*. Therefore, the transform of the system response $Y(s) = \left(\frac{1}{s+1}\right)\left(\frac{s}{s^2+\omega_0^2}\right) = \frac{s}{(s+1)(s+j\omega_0)(s-j\omega_0)}$, *Re(s) > 1*. Expanding by partial fractions, we get

$$Y(s) = \frac{\left(\frac{-1}{1+\omega_0^2}\right)}{(1+s)} + \frac{\left(\frac{1}{2(1-j\omega_0)}\right)}{(s+j\omega_0)} + \frac{\left(\frac{1}{2(1+j\omega_0)}\right)}{(s-j\omega_0)}$$

. This allows us to write the the time evolution of the system as

$$y(t) = \frac{-e^{-t}}{1+\omega_0^2} + \frac{e^{-j\omega_0 t}}{2(1-j\omega_0)} + \frac{e^{j\omega_0 t}}{2(1+j\omega_0)}.$$

19 Discrete-time Fourier transform

The transform is given by $\frac{1}{1-0.5e^{-j\omega T}}$.

20 Discrete-time-and-frequency Fourier transform

The fourth Fourier value,with $k = 3$, $X\left[j\frac{6\pi}{9}\right]$ is given by $\frac{1}{9}\sum_{n=0}^{8} x[nT]e^{-\frac{3j2\pi n}{9}}$ =

$$\frac{1}{9}\left(1e^{-j\frac{6\pi}{9}} + 2e^{-j\frac{12\pi}{9}} + 3e^{-j\frac{18\pi}{9}} + \ldots + 2e^{-j\frac{42\pi}{9}} + 1e^{-j\frac{48\pi}{9}}\right)$$

21 Z transform

The transform is given by $\sum_{k=0}^{\infty} kz^{-k}$. Assuming that the series converges (we'll get to that later), denote the sum by S. Clearly, $S = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots$. Therefore,

$$Sz = 1 + \frac{2}{z} + \frac{3}{z^2} + \ldots = \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) + \left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\right) = \frac{1}{1-z^{-1}} + S$$

. Thus, $S(z-1) = \frac{1}{(1-z^{-1})}$, so that $S = \frac{1}{(1-z^{-1})(z-1)} = \frac{z^{-1}}{(1-z^{-1})^2}$. Now, this series only converges when $|z^{-1}| < 1$ or $|z| > 1$. In this region of convergence, the operations on the sum are valid.

22 Z transform

From Example 26, the Z transform of the function is $\frac{1}{1-e^{-a}z^{-1}}$. Therefore, from the time-shift rule, the desired transform is $\frac{z^{-k_0}}{1-e^{-a}z^{-1}}$

Chapter 6: Queueing theory

1 Little’s law

(a) The mean waiting time is 180 min, and the arrival rate is 0.2 patients/minute. Thus, the mean number of patients is their product = 180*0.2 = 36. (b) We do not have enough information to determine the maximum size of the waiting room! We know we need at least 36 spaces, but it’s possible that a burst of a hundred patients may arrive, for example, due to an incident of mass food poisoning. But, as a rule of thumb, some small integer multiple of the mean, such as three or four times the mean, ought to be enough. In real life, we are forced to work with such ‘fudge factors’ because it is often too difficult or too expensive to determine the exact arrival process, which, in any case, may abruptly change over time.

2 A stochastic process

At time 0, $P[X_0=10] = 1.0$.

At time 1, $P[X_1 = 9] = 0.2$; $P[X_1 = 10] = 0.6$; $P[X_1 = 11] = 0.2$.

At time 2, $P[X_2 = 8] = 0.2(0.2) = 0.04$; $P[X_2 = 9] = 0.2(0.6) + 0.6(0.2) = 0.24$; $P[X_2 = 10] = 0.2(0.2) + 0.6(0.6) + 0.2 (0.2) = 0.44$, and, by symmetry, $P[X_2 = 11] = 0.24$; $P[X_2 = 12] = 0.04$.

3 Markov process

The process is Markovian, because the probability of moving from stair *i* to stairs *i-1, i,* and *i+1* do not depend on how the person reached stair *i*.

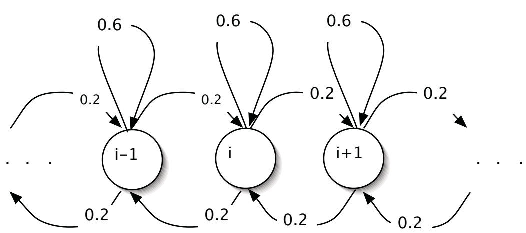
4 Homogeneity

The transition probabilities are time-independent, and therefore the process is homogeneous.

5 Representation

(a)

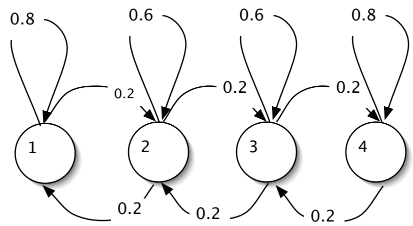
$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0.2 & 0.6 & 0.2 & 0 & \dots & \dots \\ \dots & 0 & 0.2 & 0.6 & 0.2 & 0 & \dots \\ \dots & \dots & 0 & 0.2 & 0.6 & 0.2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$



(b) The rows need to sum to 1, because at each time step, the process has to move to *some* state. The columns do not need to sum to 1 (think of a star-shaped state transition diagram with *N* states surrounding state 0, where state 0 has 1/*N* probability of going to any other state, and every state returns to state 0 with probability 1).

(c) We need to assume the boundary conditions. Suppose the at stair 1, the probability of staying at the same stair is 0.8, and at stair 4, the probability of staying at the same stair is also 0.8. Then, the transition matrix and state transition diagram are as shown below.

$$\begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.2 & 0.6 & 0.2 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$



6 Reducibility

The chain is irreducible because every state can be reached from every other state.

7 Recurrence

State 1 is recurrent because the chain is finite and irreducible. f_1^1 is the probability that the process first returns to state 1 after one time step, and this is clearly 0.8. f_1^2 is the probability that the process first returns to state 1 after two time steps, and this is 0.2 * 0.2 = 0.04. f_1^3 is the probability that the process first returns to state 1 after three time steps. This can happen after a transition to state 2, a self loop in state 2, and then back. Thus, the value is 0.2*0.6*0.2 = 0.024.

8 Periodicity

The chain is not periodic because of the self-loop in every state. A trivial chain with period N is a ring with N states, with the transition probability of going from state i to state $(i+1)$ mod N = 1.

9 Ergodicity

No state in the chain is non-ergodic because the chain is finite aperiodic and irreducible.

10 Stationary probability

From Theorem 2, because the chain is ergodic, we obtain:

$$\begin{aligned}
\pi_1 &= 0.8\pi_1 + 0.2\pi_2 \\
\pi_2 &= 0.2\pi_1 + 0.6\pi_2 + 0.2\pi_3 \\
\pi_3 &= 0.2\pi_2 + 0.6\pi_3 + 0.2\pi_4 \\
\pi_4 &= 0.2\pi_3 + 0.8\pi_4 \\
1 &= \pi_1 + \pi_2 + \pi_3 + \pi_4
\end{aligned}$$

This can be easily solved to obtain $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0.25$. (If you choose other assumptions for the boundary states, your computation will differ).

11 Residence times

$p_{11} = p_{44} = 0.8$, so the residence times in these states is 1/(1-0.8) = 1/0.2 = 5. $p_{22} = p_{33} = 0.6$, so the residence times in these states is 1/0.4 = 2.5.

12 Stationary probability of a birth-death-process

(a) Similarities: both are graphs with each node corresponding to a discrete state. Differences: the notation on an edge is the transition rate, not transition probability. The sum of rates leaving a node does not add up to 1, but total ingress rate matches total egress rate at each node.

(b) $$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -6 & 4 & 0 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

(c) We have:

$-2P_0 + 2P_1 = 0$

$2P_0 - 6P_1 + 4P_2 = 0$

$4P_1 - 6P_2 + 2P_3 = 0$

$2P_2 - 2P_3 = 0$

$P_0 + P_1 + P_2 + P_3 = 1$

This yields: $P_0 = P_1 = P_2 = P_3 = 0.25$.

13 Poisson process

Consider a pure-death process, i.e. a birth-death process whose birth rates are zero. Clearly, the inter-departure times are nothing more than the residence times in each state. But we know that the residence times in a homogeneous continuous-time Markov chain are exponentially distributed (see 6.3.2 on page 180). QED.

14 Stationary probabilities of a birth-death process

We see that in this chain, $\lambda_i = \mu_{i+1}$ so immediately we get $P_0 = P_1 = P_2 = P_3$. By summing them to 1, we can see that they are all 0.25.

15 M/M/1 queue

It is not M/M/1 because the state-transition rates are state-dependent.

16 M/M/1 queue

(a) The packet length is 250 bytes = 2,000 bits, so that the link service rate of 1,000,000 bits/sec = 500 packets/sec. Therefore, the utilization is 450/500 = 0.9. When the link queue has 1 packet, it is in state j=2, because one packet is being served at that time. Thus, we need $P_2 = 0.9^2 * 0.1 = 0.081$. For the queue having two packets, we compute $P_3 = 0.9^3 * 0.1 = 0.0729$. For 10 packets in the queue, we compute $P_{11} = 0.9^{11} * 0.1 = 0.031$. (Compare these with values in Example 20 where the load is 0.8).

(b) The mean number of packets in the system is 0.9/1-0.9 = 9. Of these, 8 are expected to be in the queue.

(c) The mean waiting time is (1/500)/(1-0.9) = 0.002/0.1 = 0.02 s = 20 milliseconds.

17 Responsive (M/M/∞) server

The ratio is:

$$\frac{e^{-\rho}\rho^j\frac{1}{j!}}{\rho^j(1-\rho)} = \frac{e^{-\rho}}{j!(1-\rho)} = \frac{1}{j!(1-\rho)e^{\rho}} = \frac{C}{j!}$$

, where C is a constant with respect to j. Therefore, for an M/M/∞ queue the probability of being in state j diminishes proportional to $j!$ compared to being in state j for an M/M/1 queue. Clearly, this favors much lower queue lengths for the M/M/∞ queue.

18 M/M/1/K server

Packet losses happen when there is an arrival and the system is in state j=11. This is upper bounded by P_{11}, which is given by

$$P_{11} = \frac{1-\rho}{1-\rho^{K+1}}\rho^j = \frac{0.1}{1-0.9^{12}}0.9^{11} = 0.0437 \quad .$$

19 M/D/1 queue

(a) The mean number of customers in the system for such a queue is given by

$$\rho + \frac{\rho^2}{2(1-\rho)} = 0.9 + \frac{0.9^2}{2(0.1)} = 4.95$$

, which is roughly half the size of an equivalently loaded M/M/1 queue (from Exercise 17).

(b) The ratio is $\dfrac{\rho + \dfrac{\rho^2}{2(1-\rho)}}{\dfrac{\rho}{1-\rho}} = 1 - \dfrac{\rho}{2}$. This tends to 0.5 as the utilization tends to 1.

(c) Under heavy loads, the mean waiting time for an M/D/1 queue is half that of a similarly loaded M/M/1 queue.

Chapter 7: Game theory

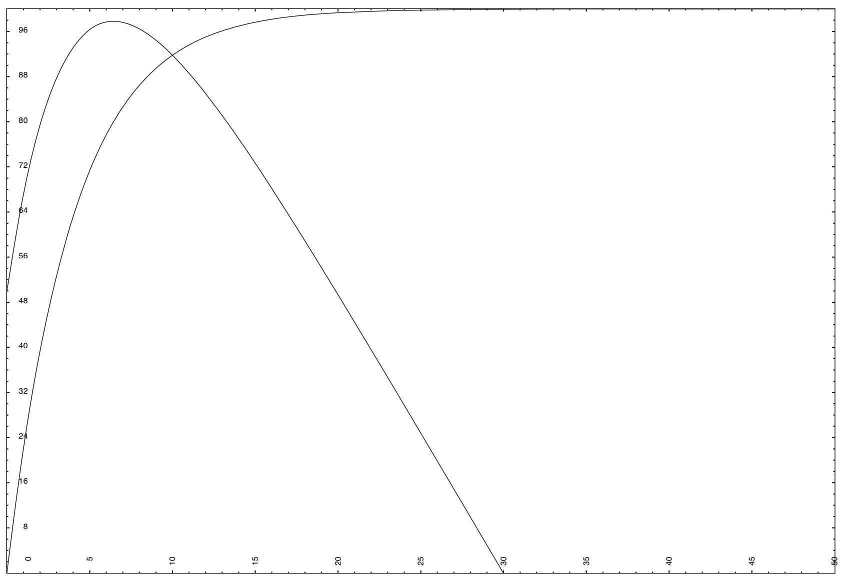
1 Preferences

Denote apple = A, banana = B, carrot = C, peach = P. We are free to choose utilities as we wish, so let U(A)=0, U(C) = 1. Then, U(B) =.7 and U(P) =.9, so you prefer peaches to bananas. (b) Let P(win B) = p. Then, $.7p + 1(1-p) = .9$, so $.3p = .1$, so $p = 0.33$.

2 Utility functions

Your net utility from transferring x GB is $100(1-e^{-0.25x})$ if $x<10$ and $100(1-e^{-0.25x}) - 5(x-10)$ otherwise. The plot of these two functions is shown below:

It is clear that the maximum occurs at x=10 for a value of approximately 92. So, your utility is maximized by transferring exactly 10GB/month.



3 Pure and mixed strategies

The only possible first actions are: play corner, play middle, and play center. Depending on which move is played, the second player would have response, and depending on that response the first player would have a response etc. A pure strategy for each player is each valid response to the prior move (whether or not it is rational). A mixed strategy would play one of the pure strategies (i.e the entire sequence) with some probability. It turns out that in tic-tac-toe, with two expert players, a tie is guaranteed with a pure strategy, but a mixed strategy (depending over what you mix) could lose when played against an optimal strategy. So, it never makes sense to mix. In general, every component of a mixed strategy must be a potentially winning strategy. Otherwise, the mixed strategy would improve by discarding a component that can never win.

4 Zero-sum game

No, because utilities are only unique to a affine transformation.

5 Representation

	L	M	H
Y	(1,a-1)	(2,a-2)	(3,a-3)
N	(0,0)	(0,0)	(0,0)

6 Representation

We need to prove two things (a) if information sets are permitted every normal form game can be represented in extensive form (b) if information sets are permitted every extensive-form game can be represented in normal form. To prove (a): given a normal form game with n players, simply draw a tree of depth n, where all moves by the first player are associated with a node with an edge leading from the root to that node, and all nodes are in the same information set. Then, from each such node, draw an edge for each possible move for the second player, and place each set of nodes in the same information set. Repeat for each successive player, and label the leaves with the payoff from the corresponding array element. To prove (b): given the extensive form game, form paths from the root to each leaf. Decompose the path into moves by each of the players and find all possible moves by each player each time it is allowed to make a move. Let S_i^t denote the set of moves that player i can move on its t turn. Then the strategy space for player i is the cross product of these sets. Finally, the normal form is an n-dimensional matrix with the ith dimension indexed by the strategy space of the ith player, and the corresponding element having the payoff for these strategies.

7 Best response

The best response depends on the value of a. For each of the strategies of the ISP, i.e., L, M, and H, the best response is Y if *a-price* > 0, otherwise it is N.

8 Dominant strategy

If you attend, your payoff is your utility for either pass or fail, but if you miss, your payoff is your utility for fail. Assuming that utility(pass) > utility(fail), your payoff for attending is as good as or better than the payoff for not attending. So, your dominant strategy is to attend.

9 Bayesian game

It is easy to verify that no matter the type of the Column player (strong or weak signal), the best response for Row if Column plays S is D and if Column plays D is S. Therefore, knowing the type of the Column player does not help Row, and the game does not have a dominant strategy for Row.

10 Repeated game

The one shot payoff is -3 for each, so the repeated payoff is $-3^* \sum_{i=0}^{\infty} 0.6^i = -3/.4 = -7.5$.

11 Dominant strategy equilibrium

It is dominant for both players to send rather than wait. In equilibrium, they always send right away so their packets always collide, and in fact, no progress is made, so that delays are actually infinite. This game illustrates the aphorism: haste makes waste. The EDCA protocol allows higher priority (delay sensitive) stations to wait for a shorter time than lower-priority stations before accessing the medium, therefore making it more probable that they would get access to medium and experience a shorter delay.

12 Iterated deletion

Consider the following game, where we only show the payoffs for Row:

	C1	C2
R1	0	0
R2	1	-1
R3	-2	2

Neither R2 nor R3 dominate R1. However any mixed strategy of R2 and R3 that plays R3 with a probability greater than 2/3 dominates R1. Therefore, we can delete R1 from the game.

13 Maximin

In Example 10, Row can get as low as -1 with S, but at least 0 with D, so its maximin strategy is D. Column is assured 1 with S, so its maximin strategy is S, and the equilibrium is DS.

In Example 14, Row maximizes its minimum payoff with S. The game is symmetric, so the maximin equilibrium is SS.

14 Maximin in a zero-sum game

In Figure 3, note that when p is smaller than 0.5, the column player can play pure strategy C1 to reduce Row's payoff below 2.5. Similarly, if p is greater than 0.5, Column can use a pure strategy C2 to reduce Row's payoff. For any value of p, Column can play a mixture qC1 + (1-q) C2 to give Row a payoff of $q(p+2) + (1-q)(4-3p)$. To make this smaller than 2.5, we set $q(p+2) + (1-q)(4-3p) < 2.5$, or $q > (3-6p)/(4-8p)$. For instance, if $p=0$, $q > 3/4$, and if $p=1$, $q>3/4$. (The inequality is not valid when $p= 0.5$.)

15 Nash equilibrium

Let the row player play pH + (1-p)T. Then, its payoff, given Column's mixed strategy, is $p(q-(1-q))+(1-p)(-q+(1-q)) = 4pq -2q -2p +1 =(1-2p)(1-2q)$. If $q < 0.5$, p should be 0, otherwise p should be 1. Intuitively, if the Column player is more likely to play T, then Row should play T for sure and *vice versa*.

16 Correlated equilibrium

Consider an external agency that tells the players to play pDS + (1-p)SD. When Row is told to play D, it knows that it will get a payoff of -1 if it deviates. Similarly, when told to play S, it will get 0 if it deviates (instead of 1). So, it will not deviate, independent of the value of p. By symmetry, the same analysis holds for Column, and therefore we have a correlated equilibrium. The external agency can arrange for any desired payoffs to Row and Column by adjusting p.

17 Price discrimination

Assume that the valuations of each player are $v_1,...,v_n$ for minimum quantities of $q_1,...,q_n$. The scheme is essentially to charge v_i for q_i adjusting for the fact that player i could buy multiples of q_j $j<i$ if that minimizes its total cost.

18 VCG mechanism

(a) The overall function is $(20+40+80)(1-e^{-0.5x}) - 20x = 140(1-e^{-0.5x}) - 20x$.

(b) The types are the only unknowns in the utility functions, i.e. 20, 40, and 80 respectively.

(c) The optimal social choice comes from maximizing the function in (a). Setting $f(x) = 140(1-e^{-0.5x}) - 20x$, solve for $f'(x^*)=0$, so that $x^* = 2.5055$.

(d) To compute x^{-1}, we maximize $(120(1-e^{-0.5x}) - 20x)$ to get 2.197. Similarly, $x^{-2} = 1.832$, and $x^{-3} = 0.8109$. Thus, $p_1 = v_2(x^{-1}) + v_3(x^{-1}) - (v_2(x^*) + v_3(x^*)) = (40+80)(1-e^{-0.5*2.197}) - (40+80)(1-e^{-0.5*2.5055}) = 120*(e^{-1.25275-}e^{-1.0985}) = -5.718$.

Similarly, $p_2 = v_1(x^{-2}) + v_3(x^{-2}) - (v_1(x^*) + v_3(x^*)) = 100(e^{-1.25275-}e^{-0.5*1.832}) = -11.439$,

$p_3 = v_1(x^{-3}) + v_2(x^{-3}) - (v_1(x^*) + v_2(x^*)) = 60(e^{-0.5*0.8109}-e^{-1.25275})= 60*(e^{-1.25275-}e^{-0.4055}) = -22.857$.

(e) No, the budget is not balanced: the CIO has to pay each department.

Chapter 8: Control Theory

1 A bandwidth management system

Solution:

The plant is the congested link. The command is the desired maximum percentage of P2P traffic on this link. The control input is the number of P2P connections that are reset at a particular time. The disturbance is the intrinsic fluctuations in the number of P2P connections. The output is the percentage of P2P connections on the link.

2 Effort and flow

Solution: The effort is the number of connections that are to be reset at a point in time. The flow is the number of connections that are actually reset per unit time.

3 State space representation

Solution:

The natural state variable for this system is $x(t)$ = the current fraction of P2P traffic. The state evolution is given by

$$\dot{x} = \begin{Bmatrix} -\frac{u}{T} + w \text{ if } (x > r) \\ w \text{ otherwise} \end{Bmatrix}$$

4 Transfer function

Solution: The output y is related to the input by

$\dot{y} = -\frac{u}{T} + w$. Taking the Laplace transform of both sides, we get $Y(s) = \frac{-U(s)}{T} + W(s)$. Ignoring W, we have

$\frac{Y(s)}{U(s)} = G(s) = -\frac{1}{T}$, which is the desired transfer function.

5 First order system

Solution: A first order system reaches the 63% mark at 2τ ,so $\tau = 1$. The asymptotic value is K, so K=4.25. The transfer function is $G(s) = \frac{K}{1+\tau s} = \frac{4.25}{1+s}$.

6 Second order system

Solution: We have $Y(s) = \frac{K}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\varsigma s}{\omega_n} + 1\right)} = \frac{K}{s\left(\frac{s^2}{\omega_n^2} + \frac{2s}{\omega_n} + 1\right)} = \frac{K}{s\left(\frac{s}{\omega_n} + 1\right)^2}$. We use partial fraction expansion to write this as $Y(s) = K\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{(s+\omega_n)}\right]$ The solution is obtained by finding the inverse Laplace transform term by term, using Table 4 on page 152.

7 Proportional mode control

Solution: The system pole is at -(loop gain).

8 Integral mode control

Solution: The impulse response is $Y = \frac{K_i}{(s^2 + K_i)} = \sqrt{K_i}\left(\frac{\sqrt{K_i}}{s^2 + (\sqrt{K_i})^2}\right)$. Taking the inverse Laplace transform, this is given by $\sqrt{K_i}\sin\sqrt{K_i}t$.

9 Multiple mode control

Solution: With this control, we have $U = \left(\frac{K_i}{s} + K_d s\right)E = \left(\frac{K_i}{s} + K_d s\right)(R - Y)$ and

$Y = \frac{1}{s}(U - W) = \frac{1}{s}\left(\left(\frac{K_i}{s} + K_d s\right)(R - Y) - W\right)$. We can rearrange this as $Y = \left(\frac{K_i + K_d s^2}{K_i + s^2(K_d + 1)}\right)(R - W)$. For the impulse response, we set R=1 and W = 0. The inverse transform is given, after some rearranging, by

$$\left(\frac{K_d + K_i}{\sqrt{K_i(K_d + 1)}}\right)\sin\sqrt{\frac{K_i}{K_d + 1}}t \ .$$

10 Stability

Solution: The roots are -0.10812, -0.72122 + j0.61911, -0.72122 - j0.61911, 1.44195 + j1.15457, 1.44195- j1.15457. Because two roots have a real component in the right half of the complex s plane, the system is BIBO unstable.

11 Matrix exponential

Solution: We have $e^{\dot{A}t} = 0 + \boldsymbol{A} + \frac{2\boldsymbol{A}^2 t}{2!} + \frac{3\boldsymbol{A}^3 t^2}{3!} + \ldots = \boldsymbol{A} + \boldsymbol{A}^2 t + \frac{\boldsymbol{A}^3 t^2}{2!} + \frac{\boldsymbol{A}^4 t^3}{3!}$ and

$\boldsymbol{A}e^{\boldsymbol{A}t} = A\left(e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \ldots\right) = \boldsymbol{A} + \boldsymbol{A}^2 t + \frac{\boldsymbol{A}^3 t^2}{2!} + \frac{\boldsymbol{A}^4 t^3}{3!} + \ldots$. Both terms are equal term by term, and therefore the infinite sums are also equal, proving that $e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \ldots$ satisfies $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.

12 Matrix exponential

Solution: Because $\boldsymbol{A}$ is diagonal, $\boldsymbol{A}^r = \begin{bmatrix} a_{11}^r & 0 & 0 & 0 \\ 0 & a_{22}^r & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & a_{nn}^r \end{bmatrix}$. So, $e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \ldots$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 1^r \end{bmatrix} + \begin{bmatrix} a_{11}t & 0 & 0 & 0 \\ 0 & a_{22}t & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & a_{nn}t \end{bmatrix} + \begin{bmatrix} \frac{a_{11}^2}{2}t^2 & 0 & 0 & 0 \\ 0 & \frac{a_{22}^2}{2}t^2 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \frac{a_{nn}^2}{2}t^2 \end{bmatrix} + \begin{bmatrix} \frac{a_{11}^3}{3!}t^3 & 0 & 0 & 0 \\ 0 & \frac{a_{22}^3}{3!}t^3 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \frac{a_{nn}^3}{3!}t^3 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} \sum_i \frac{(a_{11}t)^i}{i!} & 0 & 0 & 0 \\ 0 & \sum_i \frac{(a_{22}t)^i}{i!} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \sum_i \frac{(a_{nn}t)^i}{i!} \end{bmatrix} = \begin{bmatrix} e^{a_{11}t} & 0 & 0 & 0 \\ 0 & e^{a_{22}t} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & e^{a_{nnt}} \end{bmatrix}$$

. Therefore, $e^{\boldsymbol{A}t}$ for $\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ =

$$\begin{bmatrix} 20.08 & 0 & 0 \\ 0 & 0.0183 & 0 \\ 0 & 0 & 0.367 \end{bmatrix} .$$

13 Partial fraction expansion

Solution: Let $\frac{s}{(s+3)(s+5)} = \frac{a_1}{s+3} + \frac{a_2}{s+5}$. $a_1 = \lim_{s \to -3} (s+3)\frac{s}{(s+3)(s+5)} = \lim_{s \to -3} \frac{s}{(s+5)} = -1.5$.

$$a_2 = \lim_{s \to -5} (s+5)\frac{s}{(s+3)(s+5)} = \lim_{s \to -5} \frac{s}{(s+3)} = 2.5$$

14 Partial fraction expansion

Solution: Using the quadratic formula, we find $s = 2 \pm j5$. Therefore, we write the fraction as

$f(s) = \frac{1}{(s-(2+j5))(s-(2-j5))} = \frac{a_1}{(s-(2+j5))} + \frac{a_2}{(s-(2-j5))}$. Then,

$$a_1 = \lim_{s \to (2+j5)} (s-(2+j5))\left(\frac{1}{(s-(2+j5))(s-(2-j5))}\right) = \frac{1}{(2+j5-(2-j5))} = \frac{1}{j10}$$ and $a_2 = \frac{1}{-j10}$, so that

$$f(s) = \frac{1}{j10(s-(2+j5))} - \frac{1}{j10(s-(2-j5))}$$.

Chapter 9: Information Theory

1 Entropy

Solution: The entropy is $-(0.25\log 0.25 + 0.25\log 0.25 + 0.25\log 0.25 + 0.125\log 0.125 + 0.125\log 0.125) = 2.25 bits$.

2 Entropy

Solution: Since all messages are equally likely, the entropy is given by log(number of distinct messages) = log(16^{100}) = 100 log 16 = 400 bits.

3 Instantaneous codes

Solution: 'acc','ad', and 'bd.'

4 Instantaneous codes

Solution: The codewords lie at the leaves of a binary tree, so this code is instantaneous.

5 Digit coding

Solution: The number of distinct strings represented by *n* decimal digits is 10^n . This can be represented by a binary string of length $\log 10^n$. The mean number of bits per digit is given by $\frac{\log 10^n}{n} = \log 10 = 3.32$, which is also the asymptotic limit.

6 Feasibility of a code

Solution: From the Kraft inequality, $\sum_i 2^{-l_i} \le 1$. Here, we have

$$\sum_i 2^{-l_i} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{128} + \frac{1}{128} + \frac{1}{128} + \frac{1}{128} = 1.015625 > 1$$. This violates the Kraft inequality, so you should disbelieve your friend.

7 Optimal codes

Solution: The source entropy is 1.94 bits, which is also the expected length of the shortest instantaneous code.

8 Huffman codes

Solution: Two possible Huffman codes are 'a': '00', 'b':'01', 'c': '10', 'd':'11'. 'a': '01', 'b':'00', 'c': '10', 'd':'11'. The expected code length 2 bits, which is less than (entropy + 1) bits, because the entropy is 1.94 bits.

9 Huffman codes

Solution: Consider the source message 'aabbaaabb.' Since there are no 'c' or 'd' symbols in the message, we could use, for this message, the code 'a':'0', 'b':'1', which has a shorter encoding than the Huffman encoding.

10 Entropy rate

Solution: Each symbol has an entropy of 1.94 bits. In one second, the source generates 100 independent symbols, so its entropy rate is 194 bits/second. In 100 seconds, it generates an entropy of 19,400 bits, which corresponds to $2^{19,400}$ distinct messages.

11 Typical messages

Solution: The number of distinct messages with 12 symbols is $2^{12} = 4096$. For a message to be atypical, it must have at least 11 ‘1’ symbols. The total number of such messages is $\binom{12}{11} + \binom{12}{12} = 12 + 1 = 13$. So, the fraction of atypical messages is 13/4096 = 3.17* 10^{-3}.

The entropy per symbol is 0.469 bits, so the entropy of a set of messages of length 50 symbols is 23.35. The size of the typical set is $2^{23.35} = 10,691,789$ messages and this is the number of codes that need to be assigned to messages of length 50 symbols to ensure that the number of uncoded messages is vanishingly small.

12 A noiseless channel

Solution: Each symbol from this source has an entropy of 1.94 bits. So, the channel can carry 100/1.94 = 51.55 symbols/second.

13 Mutual information

Solution: The probability of each symbol on the channel is given by:

	***P*(*X*)**
***X*=0**	0.2
***X*=1**	0.8

Therefore, *H*(*X*) = -(0.2 log 0.2 + 0.8 log 0.8) = 0.72. To compute *H*(*X*|*Y*), we first need to know the distribution of *Y*. From Table 3 on page 273, we find this to be:

	***P*(*Y*)**
***Y*=0**	0.028
***Y*=1**	0.972

From Table 4 on page 274, the conditional distribution of *X* given *Y* = 0 is

	***P*(*X*|*Y*=0)**
***X*=0**	0.7142
***X*=1**	0.2857

which has an entropy of 0.863 bits.

From Table 5 on page 274, the conditional distribution of *X* given *Y* = 1 is

	***P*(*X*|*Y*=1)**
***X*=0**	0.185
***X*=1**	0.815

which has an entropy of 0.691. We multiply these conditional entropies by the probability of *Y* being 0 or 1 respectively, to compute *H*(*X*|*Y*) as 0.028* 0.863 + 0.972*0.691 = 0.695. Therefore, the mutual information is *I*(*X;Y*) = 0.72 - 0.695 = 0.024 bits/symbol.

14 Mutual information

$$I(X;Y) = \sum_X \sum_Y P(xy) \log \frac{P(xy)}{P(x)P(y)}$$ (by definition)

$$= \sum_X \sum_Y P(xy) \log \frac{P(x|y)}{P(x)}$$ (by definition of conditional probability $P(x|y) = \frac{P(xy)}{P(y)}$)

$$= \sum_X \sum_Y P(xy) \log P(x|y) - \sum_X \sum_Y P(xy) \log P(x)$$ (expanding log (*a*/*b*) as log *a* - log *b*)

$$= -H(X|Y) - \sum_X \left(\sum_Y P(xy) \log P(x) \right)$$ (from definition of $H(X|Y)$)

$$= -H(X|Y) - \sum_X P(x) \log P(x)$$ (summing $P(xy)$ over Y gives us $P(x)$)

$$= H(X) - H(X|Y) .$$

The symmetric result is obtained by converting $\frac{P(xy)}{P(x)P(y)}$ to $\frac{P(y|x)}{P(x)}$ in step 2 and proceeding along the same lines.

15 Capacity of a binary symmetric channel

Solution: C = 1 + $e$log$e$ + (1-e)log(1-e) bits/symbol. We have e = 0.01, so C = 1 + 0.001 log 0.001 + 0.999 log 0.999 = 0.988 bits/symbol. This is 0.012 bits/symbol lower than the channel capacity of the noiseless channel, whose capacity is 1 bit/symbol.

16 Capacity of a Gaussian channel

Solution: P/N = $10^{5/10} = 3.162$. So, the channel capacity is $10*10^6$ * log(1+3.162) = $10*10^6*2.06$ = 20.6 Mbps. To achieve a capacity of 50 Mbps, we set $50*10^6 = 10*10^6$ *log(1+SNR), so that log(1+SNR) = 5, and SNR = 31. This corresponds to a dB value of $10*\log_{10}(31)$ = 14.9 dB.