L11: Algebraic Path Problems with applications to Internet Routing
Lecture 08

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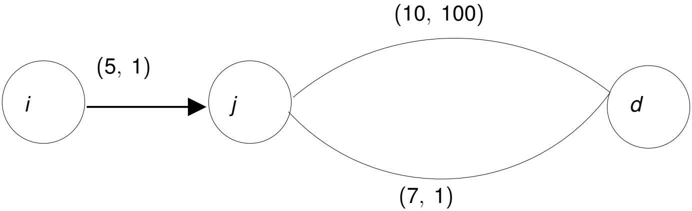
Examples

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

name	LD	LC	LK
min_plus	Yes	Yes	No
max_min	Yes	No	No

name	definition	LD
Widest Shortest-paths	$\text{min\_plus} \vec{\times} \text{max\_min}$	Yes
Shortest Widest-paths	$\text{max\_min} \vec{\times} \text{min\_plus}$	No

Shorest widest paths

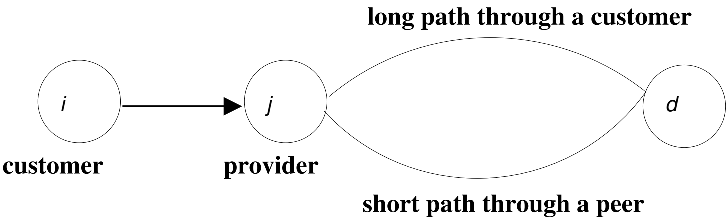


- node j prefers $(10,\ 100)$ over $(7,\ 1)$.
- node i prefers $(5,\ 2)$ over $(5,\ 101)$.

$$(5,\ 1) \otimes ((10,\ 100) \oplus (7,\ 1)) = (5,\ 1) \otimes (10,\ 100) = (5,\ 101)$$

$$((5,\ 1) \otimes (10,\ 101)) \oplus ((5,\ 1) \otimes (7,\ 1)) = (5,\ 101) \oplus (5,\ 2) = (5,\ 2)$$

Something similar from inter-domain routing in the global Internet



- *j* prefers long path though one of its customers
- *i* prefers the shorter path

Solving (some) equations

If $\mathbf{A}^*$ exists , then $\mathbf{L} = \mathbf{A}^*$ solves the equation

$$\mathbf{L} = \mathbf{AL} \oplus \mathbf{I}$$

and $\mathbf{R} = \mathbf{A}^*$ solves the equation

$$\mathbf{R} = \mathbf{RA} \oplus \mathbf{I}.$$

Towards a "non classical" theory of algebraic path problems ...

If we weaken the axioms of the semiring (drop distributivity, for example), could it be that we can find examples where $\mathbf{A}^*$, $\mathbf{L}$, and $\mathbf{R}$ exist, but are all distinct?

Health warning : matrix multiplication over structures lacking distributivity is not associative!

Left-Local Optimality

Say that $\mathbf{L}$ is a left locally-optimal solution when

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{L}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{L}(q,\ j)$$

- $\mathbf{L}(i,\ j)$ is the best possible value given the values $\mathbf{L}(q,\ j)$, for all out-neighbors q of source i.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees from** i (think Bellman-Ford).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees to** i.
- Works well with hop-by-hop forwarding from i.

Right-Local Optimality

Say that $\mathbf{R}$ is a right locally-optimal solution when

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{R}(i,\ q) \otimes \mathbf{A}(q,\ j)$$

- $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all in-neighbors q of destination j.
- Rows $\mathbf{L}(i,\ \_)$ represents **out-trees <u>from</u>** i (think Dijkstra).
- Columns $\mathbf{L}(\_,\ i)$ represents **in-trees <u>to</u>** i.

With and Without Distributivity

With distributivity

For (bounded) semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

$$\mathbf{A}^* = \mathbf{L} = \mathbf{R}$$

Without distributivity

It may be that $\mathbf{A}^*$, $\mathbf{L}$, and $\mathbf{R}$ exists but are all distinct.

Back and Forth

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I} \quad \Longleftrightarrow \quad \mathbf{L}^T = (\mathbf{L}^T \otimes^T \mathbf{A}^T) \oplus \mathbf{I}$$

where $\otimes^T$ is matrix multiplication defined with $a \otimes^T b = b \otimes a$

(Distributed) Bellman-Ford can compute left-local solutions[1]

$$\begin{aligned} \mathbf{A}^{[0]} &= \mathbf{I} \\ \mathbf{A}^{[k+1]} &= (\mathbf{A} \otimes \mathbf{A}^{k}) \oplus \mathbf{I}, \end{aligned}$$

- Bellman-ford algorithm must be modified to ensure only loop-free paths are inspected.
- $(S, \oplus, \overline{0})$ is a commutative, idempotent, and selective monoid,
- $(S, \otimes, \overline{1})$ is a monoid,
- $\overline{0}$ is the annihilator for $\otimes$,
- $\overline{1}$ is the annihilator for $\oplus$,
- Left strictly inflationarity, L.S.INF : $\forall a, b : a \neq \overline{0} \implies a < a \otimes b$
- Here $a \leq b \equiv a = a \oplus b$.

Convergence to a unique left-local solution is guaranteed. Currently no polynomial bound is known on the number of iterations required.

[1] See dissertation of Alexander Gurney

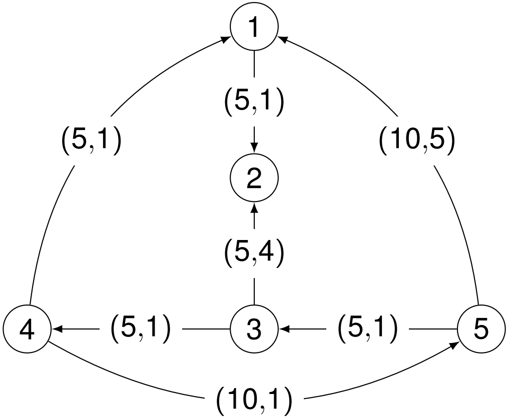
Right version ...

$$
\begin{aligned}
{}^{[0]}\mathbf{A} &= \mathbf{I} \\
{}^{[k+1]}\mathbf{A} &= ({}^{k}\mathbf{A} \otimes \mathbf{A}) \oplus \mathbf{I},
\end{aligned}
$$

- (algorithm must be modified to ensure only loop-free paths are inspected)
- $(S, \oplus, \overline{0})$ is a commutative, idempotent, and selective monoid,
- $(S, \otimes, \overline{1})$ is a monoid,
- $\overline{0}$ is the annihilator for $\otimes$,
- $\overline{1}$ is the annihilator for $\oplus$,
- Right strictly inflationarity, R.S.INF : $\forall a, b : a \neq \overline{0} \implies a < b \otimes a$
- Here $a \leq b \equiv a = a \oplus b$.

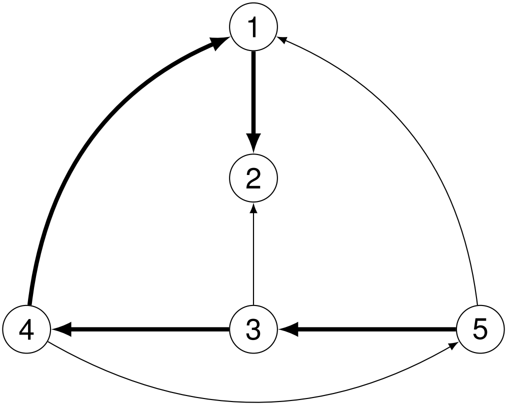
Convergence to a unique right-local solution is guaranteed. Currently no polynomial bound is known on the number of iterations required.

Example



(bandwidth, distance) with lexicographic order (bandwidth first).

Left-locally optimal paths to node 2



Right-locally optimal paths to node 2