L108: Category theory and logic
Exercise sheet 6

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Functor categories

1. Consider functors $F, G : \mathbb{A} \to \mathbb{B}$, and a natural transformation $\eta : F \to G$. Show that η is an isomorphism in the functor category $\mathbb{B}^{\mathbb{A}}$ if and only if all its components η_A for $A \in \mathrm{obj}(\mathbb{A})$ are isomorphisms in $\mathbb{B}$.

2. Consider categories $\mathbb{A}, \mathbb{B}, \mathbb{C}$, functors $F, G : \mathbb{A} \to \mathbb{B}$, $H, K : \mathbb{B} \to \mathbb{C}$, and natural transformations $\eta : F \to G$ and $\theta : H \to K$.

   (a) Using H and η, define a natural transformation of type $H \circ F \to H \circ G$.

   (b) Using θ and G, define a natural transformation of type $H \circ G \to K \circ G$.

   (c) Define a natural transformation of type $H \circ F \to K \circ G$

   These three construction are known as *whiskering* or *horizontal composition*. The word *horizontal* is used, since one imagines the functors and natural transformations arranged in this pattern:

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbb{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathbb{C} \end{tikzcd}$$

   Following the same image, the composition of natural transformations in the functor categories $\mathbb{B}^{\mathbb{A}}$ and $\mathbb{C}^{\mathbb{B}}$ may be called *vertical* composition.

3. Using the constructions of the previous exercise, define a functor of type $\mathbb{B}^{\mathbb{A}} \to \mathbb{C}^{\mathbb{A}}$ representing horizontal composition with K.

4. In the same way, define a functor of type $\mathbb{C}^{\mathbb{B}} \to \mathbb{C}^{\mathbb{A}}$ representing horizontal composition with F.

Adjunctions via unit and counit

In the lecture, we first defined adjunctions as pairs $F : \mathbb{A} \to \mathbb{B}$, $U : \mathbb{B} \to \mathbb{A}$ of functors together with a family

$$\varphi_{A,B} : \mathbb{B}(FA, B) \xrightarrow{\cong} \mathbb{A}(A, UB)$$

of bjections which are natural in A, B. We later stated that the information contained in φ can be equivalently expressed by a pair

$$\eta : \mathrm{id}_{\mathbb{A}} \to U \circ F, \qquad \varepsilon : F \circ U \to \mathrm{id}_{\mathbb{B}}$$

of natural transformations[1], subject to the condition that the triangles

$$\begin{tikzcd} FA \arrow[d, "F\eta_A"'] \arrow[dr, "\mathrm{id}_{FA}"] & \\ FUFA \arrow[r, "\varepsilon_{FA}"'] & FA \end{tikzcd} \qquad \begin{tikzcd} UB \arrow[r, "\eta_{UB}"] \arrow[dr, "\mathrm{id}_{UB}"'] & UFUB \arrow[d, "U\varepsilon_B"] \\ & UB \end{tikzcd} \tag{†}$$

commute for all $A \in \mathrm{obj}(\mathbb{A})$ and $B \in \mathrm{obj}(\mathbb{B})$; and sketched a proof that these two presentations are indeed equivalent, i.e. that φ and η, ε are interdefinable. The following exercises fill in some of the details omitted in the proof.

For the following exercises, we fix a pair $F : \mathbb{A} \to \mathbb{B}$, $U : \mathbb{B} \to \mathbb{A}$ of functors.

5. Given a natural family $\varphi_{A,B} : \mathbb{B}(FA, B) \xrightarrow{\cong} \mathbb{A}(A, UB)$ of bijections, we define η and ε by

$$\begin{aligned} \eta_A &= \varphi(\mathrm{id}_{FA}) && \text{for } A \in \mathrm{obj}(\mathbb{A}) \\ \varepsilon_B &= \varphi^{-1}(\mathrm{id}_{UB}) && \text{for } B \in \mathrm{obj}(\mathbb{B}) \end{aligned}$$

(as in the lecture, we often omit subscripts of φ)

Show that these definition do indeed give rise to natural transformations, and that these satisfy the axioms (†).

6. Conversely, given $\eta : \mathrm{id}_{\mathbb{A}} \to U \circ F$, $\varepsilon : F \circ U \to \mathrm{id}_{\mathbb{B}}$, Show that the definition $\varphi(f) := Uf \circ \eta_A$ for $f : FA \to B$ gives rise to a natural family $\varphi_{A,B} : \mathbb{B}(FA, B) \xrightarrow{\cong} \mathbb{A}(A, UB)$ of bijections.

7. Show that the constructions of 5. and 6. are mutually inverse.

Examples of adjunctions

The following exercises are most easily solved by using the theorem characterizing the existence of left adjoints given in the lecture (and the dual theorem in the case of right adjoints).

8. Show that the forgetful functor U : **Preord** $\to$ **Set** which assigns to each preorder $(D, \leq)$ its underlying set D has a left adjoint, and give an explicit definition of this functor.

9. Show that the forgetful funcor U : **Preord** $\to$ **Set** has a right adjoint, and give a definition.

10. Let $\mathbb{C}$ be a cartesian closed category, and let $B \in \mathrm{obj}(\mathbb{C})$. Show that the functor

$$\begin{array}{rccc} \mathrm{Pr}_B : & \mathbb{C} & \to & \mathbb{C} \\ & A & \mapsto & A \times B \\ & f & \mapsto & f \times \mathrm{id}_B \end{array}$$

has a *right* adjoint.

[1] η is called the *unit* of the adjunction, and ε the *counit*

11. A *pointed set* is a pair (X, x) where X is a set, and $x \in X$ is an element. A morphism of pointed sets $f : (X, x) \to (Y, y)$ is a function $f : X \to Y$ such that $f(x) = y$. Pointed sets and their morphisms form a category denoted by $\mathbf{Set}_*$.

    There is an evident forgetful functor $U : \mathbf{Set}_* \to \mathbf{Set}$ forgetting the designated element. Show that this forgetful functor has a left adjoint.

Monads

The adjunctions considered in the previous examples give rise to monads that are known from functional programming.

12. Describe explicitly the monad arising from the adjunction $F \dashv U : \mathbf{Mon} \to \mathbf{Set}$ between sets and monoids.

    This monad is called the *list monad*.

13. Describe explicitly the monad arising from the adjunction in exercise 11.

    This is Haskell's *Maybe monad*.

14. Describe explicitly the monad arising from the adjunction in exercise 10. For this, assume that $\mathbb{C} = \mathbf{Set}$.

    This monad is known as the *state monad*, B is viewed as a set of abstract states that the program can be in.