L108: Category theory and logic
Exercise sheet 5

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Finite products

Let $\mathbb{C}$ be a category with finite products.

1. For morphisms $f : A \to B$, $h : B \to X$, $k : B \to Y$ in $\mathbb{C}$, show that

$$\langle h, k \rangle \circ f = \langle h \circ f, k \circ f \rangle.$$

   For morphisms $f : A \to B$, $g : A \to C$, $h : B \to X$, $l : C \to Y$, show that

$$(h \times l) \circ \langle f, g \rangle = \langle h \circ f, l \circ g \rangle.$$

   Solution. By the univeral property of the product, $m = \langle h \circ f, k \circ f \rangle : A \to X \times Y$ is the unique morphism with $\pi_1 \circ m = h \circ f$ and $\pi_2 \circ m = k \circ f$. $\langle h, k \rangle \circ f$ also has this property we can deduce that the two are equal.

   For the second part, we again show equality by showing that the left hand side satisfies the universal property that uniquely determines the right hand side. Thus, we have to show that $\pi_1 \circ (h \times l) \circ \langle f, g \rangle = h \circ f$ and $\pi_2 \circ (h \times l) \circ \langle f, g \rangle = l \circ g$. For the first equality we can argue

$$\pi_1 \circ (h \times l) \circ \langle f, g \rangle = h \circ \pi_1 \circ \langle f, g \rangle = h \circ f,$$

   and the other equation follows analogously.

2. For $A \in \mathrm{obj}(\mathbb{C})$, we define the *diagonal map* δ_A by $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$. Show that $\delta_B \circ f = (f \times f) \circ \delta_A$ for $f : A \to B$.

   Solution. Using the equations shown in the first exercise, we can argue

$$\delta_B \circ f = \langle \mathrm{id}_B, \mathrm{id}_B \rangle \circ f = \langle f, f \rangle = (f \times f) \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = (f \times f) \circ \delta_A.$$

3. Show that the mappings $(A, B) \mapsto A \times B$ and $(f, g) \mapsto f \times g$ define a functor $P : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

   Solution. We have to show that P is compatible with composition and identities.

   For composition, assume $A \xrightarrow{f} C \xrightarrow{h} E$ and $B \xrightarrow{g} D \xrightarrow{k} F$ in $\mathbb{C}$, i.e. $(A, B) \xrightarrow{(f,g)} (C, D) \xrightarrow{(h,k)} (E, F)$ in $\mathbb{C} \times \mathbb{C}$.

We argue:

$$P(h,k) \circ P(f,g) = (h \times k) \circ (f \times g) = (h \times k) \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle =$$
$$\langle h \circ f \circ \pi_1, k \circ g \circ \pi_2 \rangle = (h \circ f) \times (k \circ g) = P(h \circ f, k \circ g) = P((h,k) \circ (g,f))$$

Preservation of identities is shown as follows (since $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B)$):

$$P(\mathrm{id}_{(A,B)}) = \mathrm{id}_A \times \mathrm{id}_B = \langle \pi_1, \pi_2 \rangle = \mathrm{id}_{A \times B},$$

where the last equation follows from the universal property of $A \times B$.

4. Composing the functor P with the diagonal functor $\delta_{\mathbb{C}} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, we obtain the functor

$$\begin{aligned} D : \mathbb{C} &\to \mathbb{C} \\ A &\mapsto A \times A \\ f &\mapsto f \times f \end{aligned}$$

Show that the diagonal mappings δ_A for $A \in \mathbb{C}$ are the components of a natural transformation $\delta : \mathrm{id}_{\mathbb{C}} \to D$. (In particular, draw the relevant naturality square)

Define natural transformations $\lambda, \rho : D \to \mathrm{id}_{\mathbb{C}}$ whose components are the projections π_1 and π_2, respectively. (Again, verify naturality and draw the relevant square)

Solution. First of all we observe that the diagonal maps have the correct types: the component of a natural transformation $\delta : \mathrm{id}_{\mathbb{C}} \to D$ at $A \in \mathrm{obj}(\mathbb{C})$ has to have type $A = \mathrm{id}_{\mathbb{C}}(A) \to D(A) = A \times A$, which is the case for δ_A as defined before. So to see that the δ_A give rise to a natural transformation, we have to check naturality. Let $f : A \to B$. The naturality square of δ and f is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta_A"'] & B \arrow[d, "\delta_B"] \\ A \times A \arrow[r, "f \times f"'] & B \times B \end{tikzcd}$$

which commutes because of exercise 2.

For the second part, it is again easy to see that the (say first) projection maps $\pi_1 : A \times A \to A$ have the correct type for the components of a natural transformation $\lambda : D \to \mathrm{id}_{\mathbb{C}}$. The naturality square for $f : A \to B$ is

$$\begin{tikzcd} A \times A \arrow[r, "f \times f"] \arrow[d, "\pi_1"'] & B \times B \arrow[d, "\pi_1"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and this square commutes since $(f \times f) = \langle f \circ \pi_1, f \circ \pi_2 \rangle$.

Slice categories

Let $\mathbb{C}$ be a category, and $I \in \mathrm{obj}(\mathbb{C})$. The *slice category* $\mathbb{C}/I$ is the category whose

- **objects** are pairs $(A \in \mathrm{obj}(\mathbb{C}), F : A \to I)$, and whose
- **morphisms** from (A, f) to (B, g) are morphisms $h : A \to B$ in $\mathbb{C}$ such that $g \circ h = f$.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

- Composition and identities are given by composition and identities in $\mathbb{C}$.

5. Show that $\mathbb{C}/I$ has a terminal object. Give a sufficient condition for the existence of initial objects in $\mathbb{C}/I$.

   Solution. The terminal object in $\mathbb{C}/I$ is given by (I, id_I), since for every other object (A, f) there is a unique map that can be put at the place of the dashed arrow in

$$\begin{tikzcd} A \arrow[rr, dashed] \arrow[dr, "f"'] & & I \arrow[dl, "\mathrm{id}"] \\ & I & \end{tikzcd}$$

   such that the triangle commutes – f itself.

   A sufficient condition for the existence of an initial object in $\mathbb{C}/I$ is that $\mathbb{C}$ has an initial object 0. In this case, the initial object of $\mathbb{C}/I$ is given by $(0, i)$ where i is the unique map of type $0 \to I$.

6. Let M be a set. Show that $\mathbf{Set}/M$ has binary (and thus finite) products.

   Solution. A product of (A, f) and (B, g) in $\mathbf{Set}/M$ is given by $(P, f \circ \pi_1 \circ i)$, where $P \subseteq A \times B$ is given by $P = \{(a, b) \mid f(a) = g(b)\}$, and $i : P \hookrightarrow A \times B$ is the inclusion (observe that P is defined in such a way that $f \circ \pi_1 \circ i = g \circ \pi_2 \circ i$, so we could equivalently have used the right hand side in the definition).

   The projection maps are $\pi_1 \circ i : (P, f \circ \pi_1 \circ i) \to (A, f)$ and $\pi_2 \circ i : (P, g \circ \pi_2 \circ i) \to (B, g)$.

   Given an object (X, s) of $\mathbf{Set}/M$ and morphisms $h : (X, s) \to (A, f)$, $k : (X, s) \to (B, g)$, we know that $fh = gk$, which implies that $\langle f, g\rangle : X \to A \times B$ factors through $i : P \to A \times B$, i.e., there exists $l : X \to P$ (unique since i is a monomorphism) such that $i \circ l = \langle f, g\rangle$. It is easy to see that this l constitutes a morphism of type $(X, s) \to (P, f \circ \pi_1 \circ i)$ satisfying the desired property (being uniquely determined by $\pi_1 \circ i \circ l = h$ and $\pi_2 \circ i \circ l = k$).

7. Give a characterization of products in $\mathbb{C}/I$, using a concept from the previous exercise sheet.

   Solution. A product of (A, f) and (B, g) in $\mathbb{C}/I$ is the same as a *pullback* of the span $A \xrightarrow{f} I \xleftarrow{g} B$ in $\mathbb{C}$.

Coproducts

Coproducts are the dual concept to products. In other words, coproducts in $\mathbb{C}$ are products in $\mathbb{C}^{\mathsf{op}}$. Concretely, a coproduct of objects A, B of a category $\mathbb{C}$ is an object $A + B$[1] together with *injection* maps $\sigma_1 : A \to A + B$, $\sigma_2 : B \to A + B$ such that for all objects Y and morphisms $f : A \to Y$, $g : B \to Y$, there exists a unique $[f, g] : A + B \to Y$ with $[f, g] \circ \sigma_1 = f$ and $[f, g] \circ \sigma_2 = g$.

$$\begin{tikzcd} A \arrow[r, "\sigma_1"] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\sigma_2"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

8. The *disjoint union* of sets A, B is defined by $A \uplus B = A \times \{0\} \cup B \times \{1\}$. Show that the disjoint union of two sets is a coproduct in **Set**.

   Solution. The injection maps are defined by $\sigma_1(a) = (a, 0)$ and $\sigma_2(b) = (b, 1)$. Given $f : A \to Y$ and $g : B \to Y$, we define $[f, g] : A \uplus B \to Y$ by

   $$[f, g](x, i) = \begin{cases} f(x) & \text{if } i = 0 \\ g(x) & \text{otherwise} \end{cases}$$

   Then it is clear that $[f, g] \circ \sigma_1 = f$ and $[f, g] \circ \sigma_2 = g$. To show uniqueness, let $h : A \uplus B \to Y$ such that $h \circ \sigma_1 = f$ and $h \circ \sigma_2 = g$. To show that $h = [f, g]$ we have to show that $h(x, i) = [f, g](x, i)$ for all $(x, i) \in A \uplus B$. Let $(x, i) \in A \uplus B$. Then either $(x, i) = (a, 0)$ with $a \in A$, or $(x, i) = (b, 1)$ with $b \in B$. In the first case, we have $(x, i) = (a, 0) = \sigma_1(a)$ and thus $h(x, i) = h(a, 0) = h(\sigma_1(a)) = f(a) = [f, g](a, 0) = [f, g](x, i)$. In the second case we have $h(x, i) = h(b, 1) = h(\sigma_2(b)) = g(b) = [f, g](b, 1) = [f, g](x, i)$. Together we can conclude $h = [f, g]$.

9. Show that **Preord** has binary coproducts.

   Solution. A coproduct of preorders $(D, \leq)$ and $(E, \leq)$ is given by $(D \uplus E, \leq)$, where the order on $D \uplus E$ is defined by

   $$\begin{array}{ll} (d, 0) \leq (d', 0) & \text{iff } d \leq d' \\ (d, 0) \leq (e, 1) & \text{never} \\ (e, 1) \leq (d, 0) & \text{never} \\ (e, 1) \leq (e', 1) & \text{iff } e \leq e' \end{array}$$

   The injection mappings and the map $[f, g]$ for $f : (D, \leq) \to (Y, \leq)$ and $g : (E, \leq) \to (Y, \leq)$ are defined as in **Set**. It remains to show that σ_1 and σ_2 are monotonic, and $[f, g]$ is monotonic for monotonic f and g. Monotonicity of σ_1, σ_2 is immediate. For monotonicity of $[f, g]$, let $(x, i), (y, j) \in A \uplus B$ such that $(x, i) \leq (y, j)$. Then by the definition of the order on $A \uplus B$, we have either both x and y in D, or both in E. W.l.o.g. assume $x, y \in D$ Then we have $x \leq y$, and by monotonicity of f we can deduce

   $$[f, g](x, i) = f(x) \leq f(y) = [f, g](y, j)$$

   which shows monotonicity of $[f, g]$.

[1] In many (older) books, coproducts are also denoted by $A \coprod B$.

Cartesian closed categories

Let $\mathbb{C}$ be a cartesian closed category.

10. For fixed $C \in \mathrm{obj}(\mathbb{C})$, show that the assignment $B \mapsto C^B$ gives rise to a functor of type $\mathbb{C}^{\mathsf{op}} \to \mathbb{C}$. For this you have to construct the *morphism part* of the functor, i.e. define a function of type

$$\mathbb{C}(A, B) \to \mathbb{C}(C^B, C^A)$$

for all $A, B \in \mathrm{obj}(\mathbb{C})$, and then you have to verify the functor axioms.

Solution. The morphism part is given by

$$(f : A \to B) \mapsto (\Lambda(\varepsilon_C^B \circ (\mathrm{id}_{C^B} \times f)) : C^B \to C^A)$$

To show functoriality, we have to verify the equations

1. $\Lambda(\varepsilon) = \mathrm{id}$
2. $\Lambda(\varepsilon \circ (\mathrm{id} \times (f \circ g)) = \Lambda(\varepsilon \circ (\mathrm{id} \times g)) \circ \Lambda(\varepsilon \circ (\mathrm{id} \times f))$

(I omit subscripts for brevity, you are welcome to fill them in to see that everything checks out.) We make use of the fact that $\Lambda h = k$ if and only if $h = \varepsilon \circ (k \times \mathrm{id})$. With this, the first equation follows immediately.

For the second, taking k to be the right hand side of the equation, the following sequence of rewritings of $\varepsilon \circ (k \times \mathrm{id})$ gives the argument of the $\Lambda(-)$ on the left:

$$\begin{aligned}
&\varepsilon \circ ((\Lambda(\varepsilon \circ (\mathrm{id} \times g)) \circ \Lambda(\varepsilon \circ (\mathrm{id} \times f))) \times \mathrm{id}) \\
&= \varepsilon \circ (\Lambda(\varepsilon \circ (\mathrm{id} \times g)) \times \mathrm{id}) \circ (\Lambda(\varepsilon \circ (\mathrm{id} \times f)) \times \mathrm{id}) \\
&= \varepsilon \circ (\mathrm{id} \times g) \circ (\Lambda(\varepsilon \circ (\mathrm{id} \times f)) \times \mathrm{id}) \\
&= \varepsilon \circ (\Lambda(\varepsilon \circ (\mathrm{id} \times f)) \times g) \\
&= \varepsilon \circ (\Lambda(\varepsilon \circ (\mathrm{id} \times f)) \times \mathrm{id}) \circ (\mathrm{id} \times g) \\
&= \varepsilon \circ (\mathrm{id} \times f) \circ (\mathrm{id} \times g) \\
&= \varepsilon \circ (\mathrm{id} \times (f \circ g))
\end{aligned}$$

Here, we used twice the principle that $\varepsilon \circ (\Lambda(f) \times \mathrm{id}) = f$

10'. Optional: Extending the previous exercise, show that the assignment $(B, C) \mapsto C^B$ gives rise to a functor of type $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$.

Solution. I'll come back to this if I find time ...