L108: Category theory and logic
Exercise sheet 2

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1. List all possible functors of types $\mathbf{1} \to \mathbf{Span}$ and $\mathbf{2} \to \mathbf{Span}$ where $\mathbf{1}$ is the category with one object and only the identity morphism, and $\mathbf{2}$ is the category with two objects and one non-identity morphism between them ($\mathbf{Span}$ is defined on the first exercise sheet).

   What are functors of type $\mathbf{1} \to \mathbb{C}$ and $\mathbf{2} \to \mathbb{C}$ for an arbitrary category $\mathbb{C}$?

2. Let Σ be a set, and let $\Sigma/\mathbf{Mon}$ be the category where

   - *objects* are pairs $((M, \cdot, e), f)$ where $(M, \cdot, e)$ is a monoid and $f : \Sigma \to M$ is a function.
   - *morphisms* from $((M, \cdot, e), f)$ to $((N, \cdot, e), g)$ are monoid homomorphisms $h : (M, \cdot, e) \to (N, \cdot, e)$ such that the triangle

   $$\begin{tikzcd} \Sigma \arrow[d, "f"'] \arrow[dr, "g"] & \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

     commutes, i.e. $h \circ f = g$.

   Show that the pair $((\Sigma^*, \cdot, \varepsilon), i)$ is an initial object in $\Sigma/\mathbf{Mon}$, where $(\Sigma^*, \cdot, \varepsilon)$ is the monoid of lists over Σ with concatenation as multiplication and the empty list ε as unit, and $i : \Sigma \to \Sigma^*$ is the function that sends each element $s \in \Sigma$ to the corresponding list $[s]$ of length one.

3. Define a functor $\mathrm{List} : \mathbf{Set} \to \mathbf{Mon}$ whose object part is given by

   $$\mathrm{List}(A) = A^* \qquad \text{(the monoid of lists on } A\text{)}.$$

   Prove that your definition is well defined (i.e. verify the axioms in the definition of functor).

4. Given a set A, a *finite multiset* on A is a function $m : A \to \mathbb{N}$ with $m(a) = 0$ everywhere except for a finite number of $a \in A$. Define $F(A)$ to be the set of finite multisets on A.

   (a) Define a structure of commutative monoid on $F(A)$, using the additive monoid structure on $\mathbb{N}$ (in this case, it is more suggestive to write the monoid operation as addition, not as multiplication).

   (b) Using this monoid structure, define a functor $F : \mathbf{Set} \to \mathbf{Mon}$.

5. Given a set A, we can define a monoid $(PA, \cup, \emptyset)$, where PA is the power set (the set of all subsets) of A, the monoid operation is given by union $\cup$, with the empty set $\emptyset$ as unit element.

   Define a functor $P : \mathbf{Set} \to \mathbf{Mon}$ whose object part is $P(A) = (PA, \cup, \emptyset)$.

6. Define natural transformations $\eta : \mathrm{List} \to F$ and $\theta : F \to P$.

7. The functor $N : \mathbf{Set} \to \mathbf{Mon}$ is defined by

$$\begin{aligned} N : \mathbf{Set} &\to \mathbf{Mon} \\ A &\mapsto (\mathbb{N}, +, 0) \\ f &\mapsto \mathrm{id}_{\mathbb{N}} \end{aligned}$$

   (This is the *constant functor with value* $(\mathbb{N}, +, 0)$.)

   Define natural transformations of type $\mathrm{List} \to N$ and $F \to N$ using the length of a list and the 'size' of a multiset.