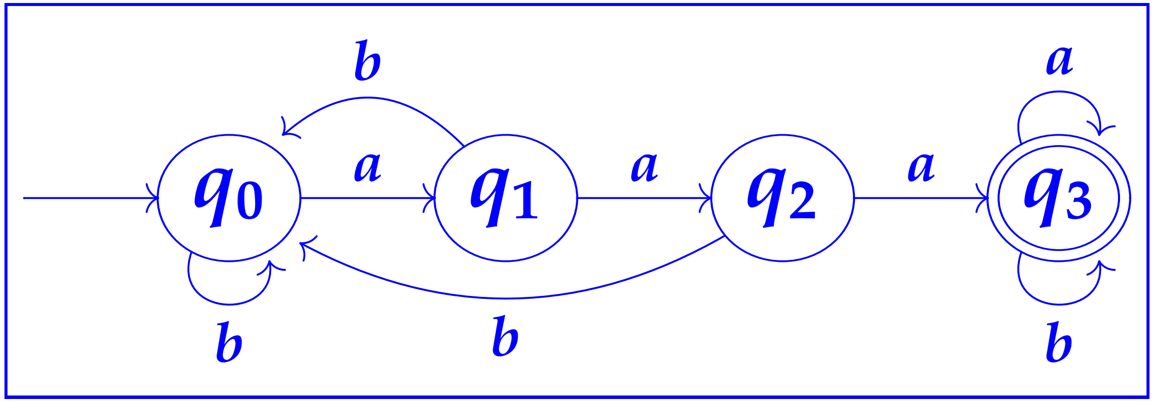
Finite Automata

Example of a finite automaton

$M \triangleq$



- set of states: $\{q_0, q_1, q_2, q_3\}$
- input alphabet: $\{a, b\}$
- transitions, labelled by input symbols: as indicated by the above directed graph
- start state: q_0
- accepting state(s): q_3

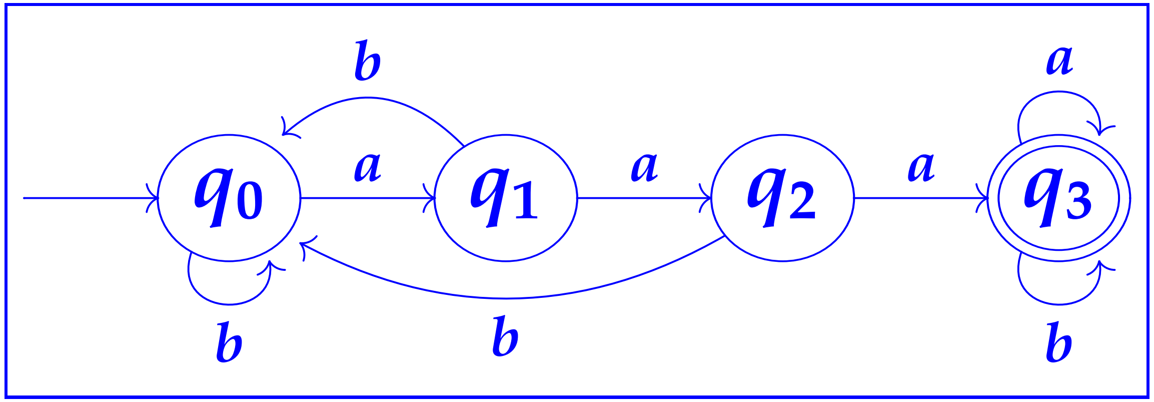
Language accepted by a finite automaton M

- Look at paths in the transition graph from the start state to *some* accepting state.
- Each such path gives a string of input symbols, namely the string of labels on each transition in the path.
- The set of all such strings is by definition **the language accepted by** M, written $L(M)$.

Notation: write $q \xrightarrow{u}{}^{*} q'$ to mean that in the automaton there is a path from state q to state q' whose labels form the string u.

(**N.B.** $q \xrightarrow{\varepsilon}{}^{*} q'$ means $q = q'$.)

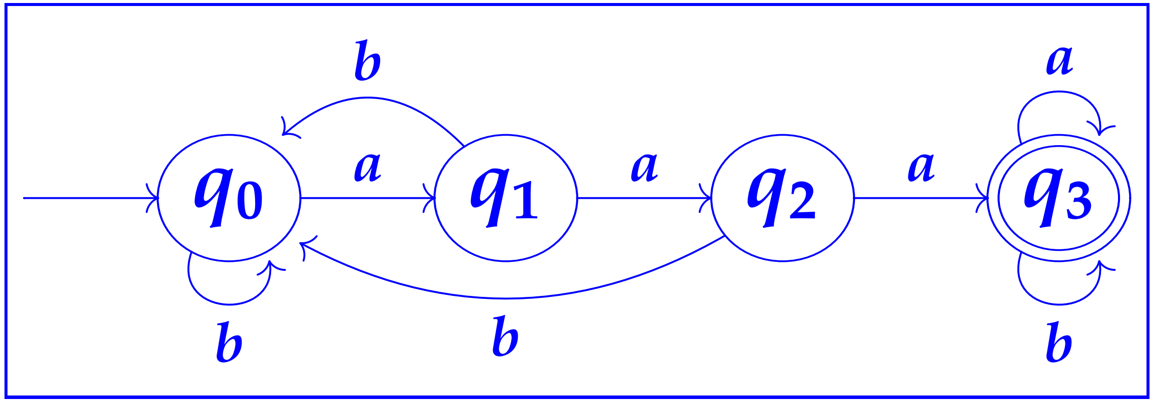
Example of an accepted language

$M \triangleq$ 

For example

- $aaab \in L(M)$, because $q_0 \xrightarrow{aaab}{}^{*}\ q_3$
- $abaa \notin L(M)$, because $\forall q(q_0 \xrightarrow{abaa}{}^{*}\ q \ \Leftrightarrow\ q = q_2)$

Example of an accepted language

$M \triangleq$ 

Claim:

$$L(M) = L((a|b)^*aaa(a|b)^*)$$

set of all strings matching the regular expression $(a|b)^*aaa(a|b)^*$

(q_i (for $i = 0, 1, 2$) represents the state in the process of reading a string in which the last i symbols read were all as)

Non-deterministic finite automaton (NFA)

is by definition a 5-tuple $M = (Q, \Sigma, \Delta, s, F)$, where:

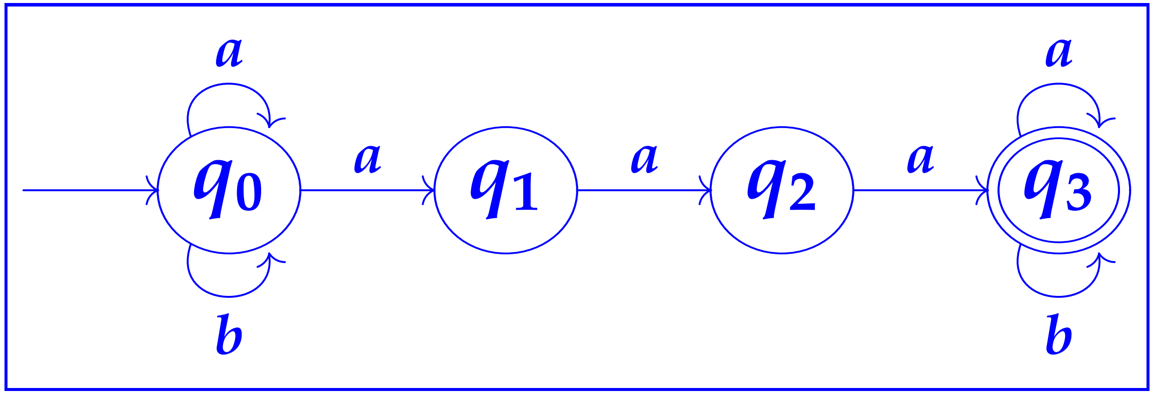
- Q is a finite set (of **states**)
- Σ is a finite set (the alphabet of **input symbols**)
- Δ is a subset of $Q \times \Sigma \times Q$ (the **transition relation**)
- s is an element of Q (the **start state**)
- F is a subset of Q (the **accepting states**)

Notation: write "$q \xrightarrow{a} q'$ in M" to mean $(q, a, q') \in \Delta$.

Example of an NFA

Input alphabet: $\{a, b\}$.

States, transitions, start state, and accepting states as shown:



For example $\{q \mid q_1 \xrightarrow{a} q\} = \{q_2\}$

$$\{q \mid q_1 \xrightarrow{b} q\} = \emptyset$$

$$\{q \mid q_0 \xrightarrow{a} q\} = \{q_0, q_1\}.$$

The language accepted by this automaton is the same as for the automaton on Slide 44, namely $\{u \in \{a, b\}^* \mid u \text{ contains three consecutive } a\text{'s}\}$.

Deterministic finite automaton (DFA)

A **deterministic finite automaton** (DFA) is an NFA $M = (Q, \Sigma, \Delta, s, F)$ with the property that for each state $q \in Q$ and each input symbol $a \in \Sigma_M$, there is a unique state $q' \in Q$ satisfying $q \xrightarrow{a} q'$.

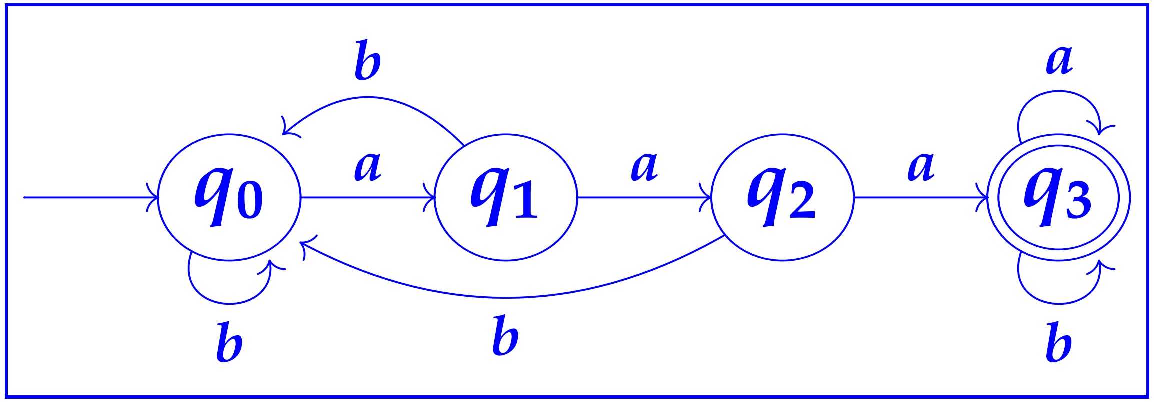
In a DFA $\Delta \subseteq Q \times \Sigma \times Q$ is the graph of a function $Q \times \Sigma \to Q$, which we write as δ and call the **next-state function**.

Thus for each (state, input symbol)-pair (q, a), $\delta(q, a)$ is the unique state that can be reached from q by a transition labelled a:

$$\forall q' (q \xrightarrow{a} q' \Leftrightarrow q' = \delta(q, a))$$

Example of a DFA

with input alphabet $\{a, b\}$

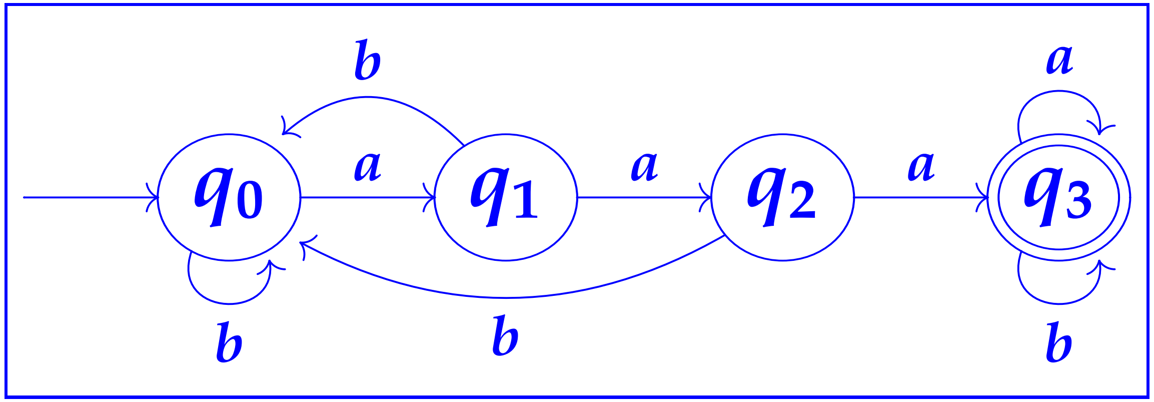
$M \triangleq$ 

next-state function:

δ	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_3	q_0
q_3	q_3	q_3

Example of an NFA

with input alphabet $\{a, b, c\}$

$M \triangleq$ 

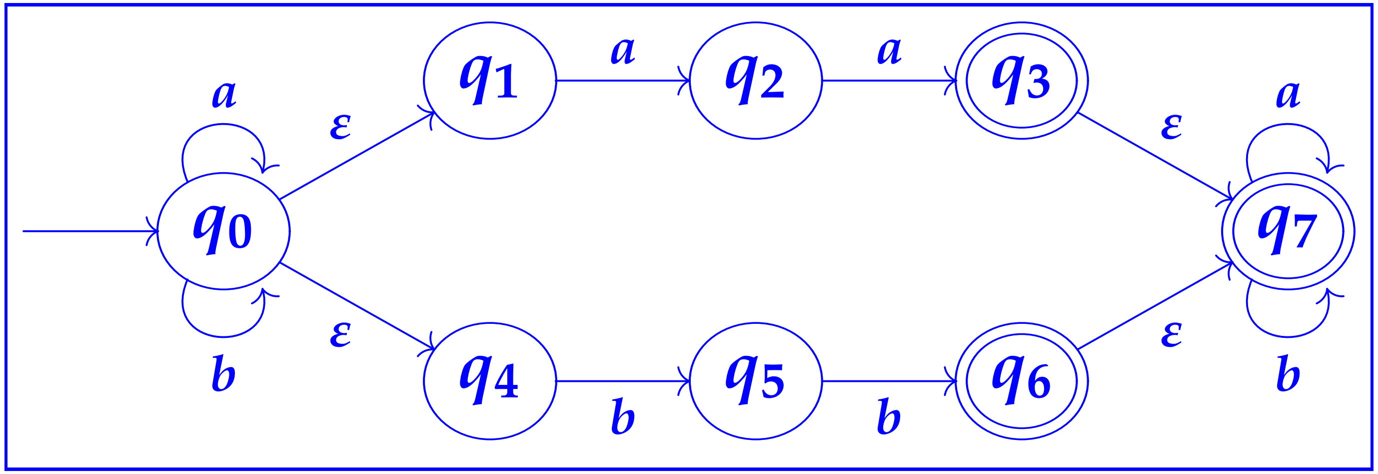
M is non-deterministic, because for example $\{q \mid q_0 \xrightarrow{c} q\} = \emptyset$.

An **NFA with ε-transitions** (NFA$^\varepsilon$)

$$M = (Q, \Sigma, \Delta, s, F, T)$$

is an NFA $(Q, \Sigma, \Delta, s, F)$ together with a subset $T \subseteq Q \times Q$, called the **ε-transition relation**.

Example:



Notation: write "$q \xrightarrow{\varepsilon} q'$ in M" to mean $(q, q') \in T$.

(**N.B.** for NFA$^\varepsilon$s, we always assume $\varepsilon \notin \Sigma$.)

Language accepted by an NFA$^\varepsilon$

$M = (Q, \Sigma, \Delta, s, F, T)$

- Look at paths in the transition graph (including ε-transitions) from start state to *some* accepting state.
- Each such path gives a string in Σ^*, namely the string of non-ε labels that occur along the path.
- The set of all such strings is by definition **the language accepted by** M, written $L(M)$.

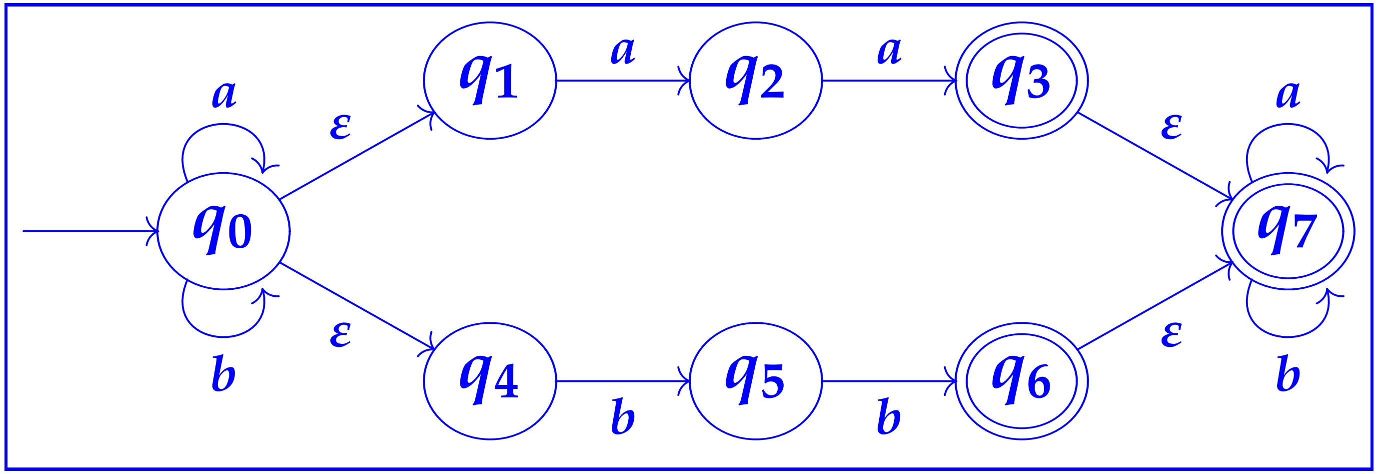
Notation: write $q \stackrel{u}{\Rightarrow} q'$ to mean that there is a path in M from state q to state q' whose non-ε labels form the string $u \in \Sigma^*$.

An **NFA with ε-transitions** (NFA$^\varepsilon$)

$$M = (Q, \Sigma, \Delta, s, F, T)$$

is an NFA $(Q, \Sigma, \Delta, s, F)$ together with a subset $T \subseteq Q \times Q$, called the **ε-transition relation**.

Example:

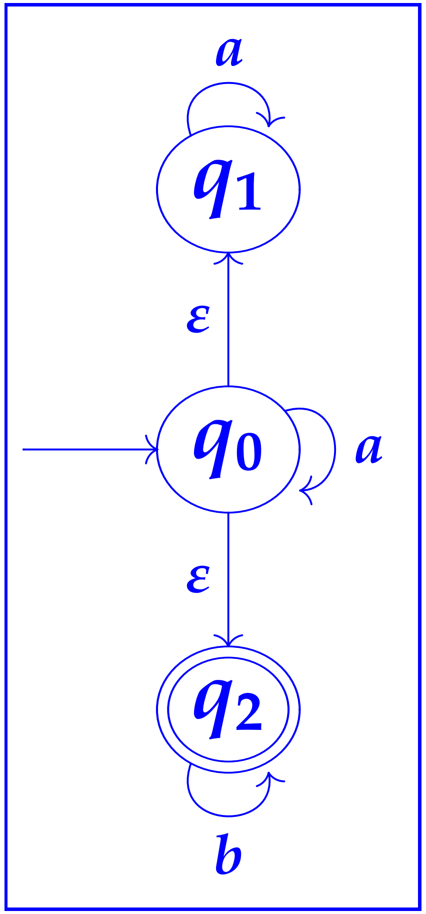


For this NFA$^\varepsilon$ we have, e.g.: $q_0 \overset{aa}{\Rightarrow} q_2$, $q_0 \overset{aa}{\Rightarrow} q_3$ and $q_0 \overset{aa}{\Rightarrow} q_7$.

In fact the language of accepted strings is equal to the set of strings matching the regular expression $(a|b)^*(aa|bb)(a|b)^*$.

Example of the subset construction

M



next-state function for PM

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{q_0\}$	$\{q_0, q_1, q_2\}$	$\{q_2\}$
$\{q_1\}$	$\{q_1\}$	$\emptyset$
$\{q_2\}$	$\emptyset$	$\{q_2\}$
$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$	$\{q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_2\}$
$\{q_1, q_2\}$	$\{q_1\}$	$\{q_2\}$
$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_2\}$

Theorem. For each NFA$^\varepsilon$ $M = (Q, \Sigma, \Delta, s, F, T)$ there is a DFA $PM = (\mathcal{P}(Q), \Sigma, \delta, s', F')$ accepting exactly the same strings as M, i.e. with $L(PM) = L(M)$.

Definition of PM:

- set of states is the powerset $\mathcal{P}(Q) = \{S \mid S \subseteq Q\}$ of the set Q of states of M
- same input alphabet Σ as for M
- next-state function maps each $(S, a) \in \mathcal{P}(Q) \times \Sigma$ to $\delta(S, a) \triangleq \{q' \in Q \mid \exists q \in S.\ q \stackrel{a}{\Rightarrow} q' \text{ in } M\}$
- start state is $s' \triangleq \{q' \in Q \mid s \stackrel{\varepsilon}{\Rightarrow} q'\}$
- subset of accepting sates is $F' \triangleq \{S \in \mathcal{P}(Q) \mid S \cap F \neq \emptyset\}$

To prove the theorem we show that $L(M) \subseteq L(PM)$ and $L(PM) \subseteq L(M)$.

Suppose $a_1 a_2 \cdots a_n \in L(M)$ [pp 60, 61]

So have

$$s \xRightarrow{a_1} q_1 \xRightarrow{a_2} \cdots \xRightarrow{a_n} q_n \in F \quad \text{in } M$$

Suppose $a_1 a_2 \cdots a_n \in L(M)$

So have

$$s \xRightarrow{a_1} q_1 \xRightarrow{a_2} \cdots \xRightarrow{a_n} q_n \in F \quad \text{in } M$$

$$\begin{array}{ccc} s & & q_1 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ S' & \xrightarrow{a_1} & S_1 \\ & & \rotatebox{90}{$=$} \\ & & \delta(S', a_1) \end{array}$$

Suppose $a_1 a_2 \cdots a_n \in L(M)$

So have

$$
\begin{tikzcd}
s \arrow[r, Rightarrow, "a_1"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\in$}"] & q_1 \arrow[r, Rightarrow, "a_2"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\in$}"] & \cdots \arrow[r, Rightarrow, "a_n"] & q_n \in F \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\in$}"] & \text{in } M \\
S' \arrow[r, "a_1"] & S_1 \arrow[r, "a_2"] & \cdots \arrow[r, "a_n"] & S_n & \text{in } PM
\end{tikzcd}
$$

Suppose $a_1 a_2 \cdots a_n \in L(M)$

So have

$$\begin{tikzcd}
s \arrow[r, Rightarrow, "a_1"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & q_1 \arrow[r, Rightarrow, "a_2"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \cdots \arrow[r, Rightarrow, "a_n"] & q_n \in F \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \text{in } M \\
S' \arrow[r, "a_1"] & S_1 \arrow[r, "a_2"] & \cdots \arrow[r, "a_n"] & S_n & \text{in } PM
\end{tikzcd}$$

So $S_n \in F'$ &

Suppose $a_1 a_2 \cdots a_n \in L(M)$

So have

$$s \xRightarrow{a_1} q_1 \xRightarrow{a_2} \cdots \xRightarrow{a_n} q_n \in F \quad \text{in } M$$

$$s' \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n \in F' \quad \text{in } PM$$

So $a_1 a_2 \cdots a_n \in L(PM)$

Suppose $a_1 a_2 \cdots a_n \in L(PM)$

so have

$$s' \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots s_{n-1} \xrightarrow{a_n} s_n \in F' \text{ in } PM$$

Suppose $a_1 a_2 \cdots a_n \in L(PM)$

so have

$$s' \xrightarrow{a_1} S_1 \xrightarrow{a_2} \cdots S_{n-1} \xrightarrow{a_n} S_n \in F' \text{ in } PM$$

$$\Downarrow$$

$$q_n \in F$$

Suppose $a_1 a_2 \cdots a_n \in L(PM)$

so have

$$\begin{array}{cccccccc}
S' & \xrightarrow{a_1} & S_1 & \xrightarrow{a_2} & \cdots & S_{n-1} & \xrightarrow{a_n} & S_n \in F' \text{ in } PM \\
\rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} & & & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
q_0 & \overset{a_1}{\Longrightarrow} & q_1 & \overset{a_2}{\Longrightarrow} & \cdots & q_{n-1} & \overset{a_n}{\Longrightarrow} & q_n \in F \text{ in } M
\end{array}$$

Suppose $a_1a_2\cdots a_n \in L(PM)$

so have

$$\begin{tikzcd}
S' \arrow[r, "a_1"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & S_1 \arrow[r, "a_2"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & \cdots \; S_{n-1} \arrow[r, "a_n"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & S_n \in F' \text{ in } PM \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] \\
q_0 \arrow[r, Rightarrow, "a_1"] & q_1 \arrow[r, Rightarrow, "a_2"] & \cdots \; q_{n-1} \arrow[r, Rightarrow, "a_n"] & q_n \in F \text{ in } M \\
S \arrow[u, Rightarrow, "\varepsilon"'] & & &
\end{tikzcd}$$

Suppose $a_1 a_2 \cdots a_n \in L(PM)$

So have

$$
\begin{tikzcd}
s' \arrow[r, "a_1"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & S_1 \arrow[r, "a_2"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \cdots\ S_{n-1} \arrow[r, "a_n"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & S_n \in F' \text{ in } PM \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
q_0 \arrow[r, Rightarrow, "a_1"] & q_1 \arrow[r, Rightarrow, "a_2"] & \cdots\ q_{n-1} \arrow[r, Rightarrow, "a_n"] & q_n \in F \text{ in } M \\
S \arrow[u, Rightarrow, "\varepsilon"'] & & &
\end{tikzcd}
$$

So $a_1 a_2 \cdots a_n \in L(M)$