Relations

$$R \in \mathcal{P}(A \times B) = \underline{Rel}(A, B)$$

$$R : A \leftrightarrow B \ , \ S : B \leftrightarrow C \ \rightsquigarrow \ (S \circ R) : A \leftrightarrow C$$

Directed graphs

$$R : A \leftrightarrow A$$

$$R^{\circ *} = Id_A \cup R \cup (R \circ R) \cup \dots \cup \underbrace{(R \circ \dots \circ R)}_{n \text{ times.}} \cup \dots$$

$x \, R^{\circ *} \, y$ iff There exists a path from x to y in (A, R).

Definition 90 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} \;=\; \bigcup \left\{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \right\} \;=\; \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 91 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s\ R^{\circ *}\ t$ *iff there exists a path with sourse* s *and target* t *in* R.

Say A is finite with $\#A = n \in \mathbb{N}$.

$$R^{\circ *} = \bigcup_{k=0}^{n} R^k = \mathrm{Id}_A \cup R \cup \cdots \cup \underbrace{(R \circ \cdots \circ R)}_{n \text{ times.}}$$

claim

In matrix form

$$\underline{\mathrm{mat}}(R) = I_n + M + M^2 + \cdots + M^n = \underline{\mathrm{mat}}(R^{\circ *})$$

$$(K+L)_{i,j} = K_{ij} \oplus L_{i,j}$$

$$(\mathrm{Id}+M)M = \mathrm{Id} + M + M^2$$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 = I_n \\ M_{k+1} = I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

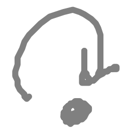
Preorders

Definition 92 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

Diagrammatically

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$



- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \ \&\ y \sqsubseteq z) \implies x \sqsubseteq z$$

satisfy an extra property:

ANTISYMMETRY

$x \leq y$ & $y \leq x \Rightarrow x = y$

+ preorder = partial order

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

$A \subseteq A$

$A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$

$n \mid n \ \forall n$ ✓

$a \mid b$ & $b \mid c \Rightarrow a \mid c$ ✓

$2 \mid -2$, $-2 \mid 2$ but $2 \neq -2$

<u>Claim</u> $R^{\circ *}$ is a preorder.

① $x R^{\circ *} x$ true because $Id \subseteq R^{\circ *} = \bigcup_{n \in \mathbb{N}} R^{\circ n}$

② $x R^{\circ *} y$ & $y R^{\circ *} z \overset{?}{\Longrightarrow} x R^{\circ *} z$

Assume $x R^{\circ *} y$ and $y R^{\circ *} z$. That is there are paths from x to y and from y to z. Joining them we get a path from x to z. Hence $x R^{\circ *} z$.

Theorem 93 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \ \&\ Q \textit{ is a preorder} \} \quad .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF: $R^{\circ *}$ is the least preorder containing R.

(ii) Show $\forall Q \subseteq A \times A .\ R \subseteq Q \ \&\ Q$ a preorder.

$x \, R^{\circ *} \, y \Rightarrow x \, Q \, y$

Assume $x \, R^{\circ *} \, y$; that is, there is a (finite) path from x to y!

$$x \xrightarrow{R} a_1 \xrightarrow{R} a_2 \xrightarrow{R} \cdots \xrightarrow{R} a_n \xrightarrow{R} y \quad \text{in } R$$

Then

$$x \, Q \, a_1 \, Q \, a_2 \cdots Q \, a_n \, Q \, y$$

$\Rightarrow x \, Q \, a_2$ because Q is transitive

$\Rightarrow x \, Q \, a_3$ ——— " ———

⋮

$\Rightarrow x \, Q \, a_n$ ——— " ———

$\Rightarrow x \, Q \, y$ ——— " ———

<u>Exercise</u>: do a proof by induction.

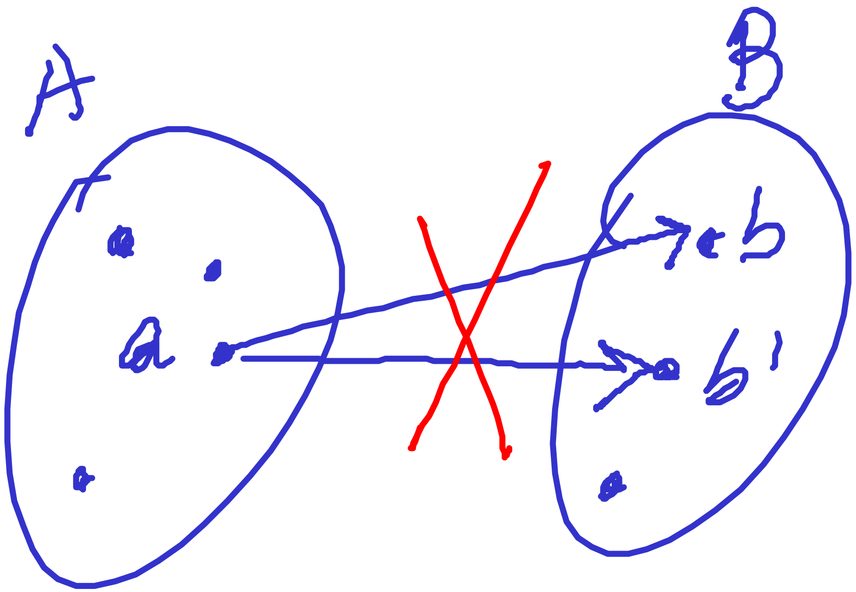
Partial functions

$PFun(A,B) \subseteq Rel(A,B)$

Definition 94 *A relation* $R : A \rightarrow\!\!\!\!+\!\!\!\!\rightarrow B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B. \ a \, R \, b_1 \ \& \ a \, R \, b_2 \implies b_1 = b_2 \ .$$

If $a R b$ then this b is unique.



An a in A may or may not be related to a b in B. But if it is then the b is uniquely determined by the a.

Say $f : A \to B$ a partial function from A to B

For $a \in A$, either a is related to some unique element of B in which case we call it $f(a)$; or a is not related to any element of B, in which case we say that f is <u>undefined</u> at a (and we write $f(a)\uparrow$).

To define

$$f : A \to B$$

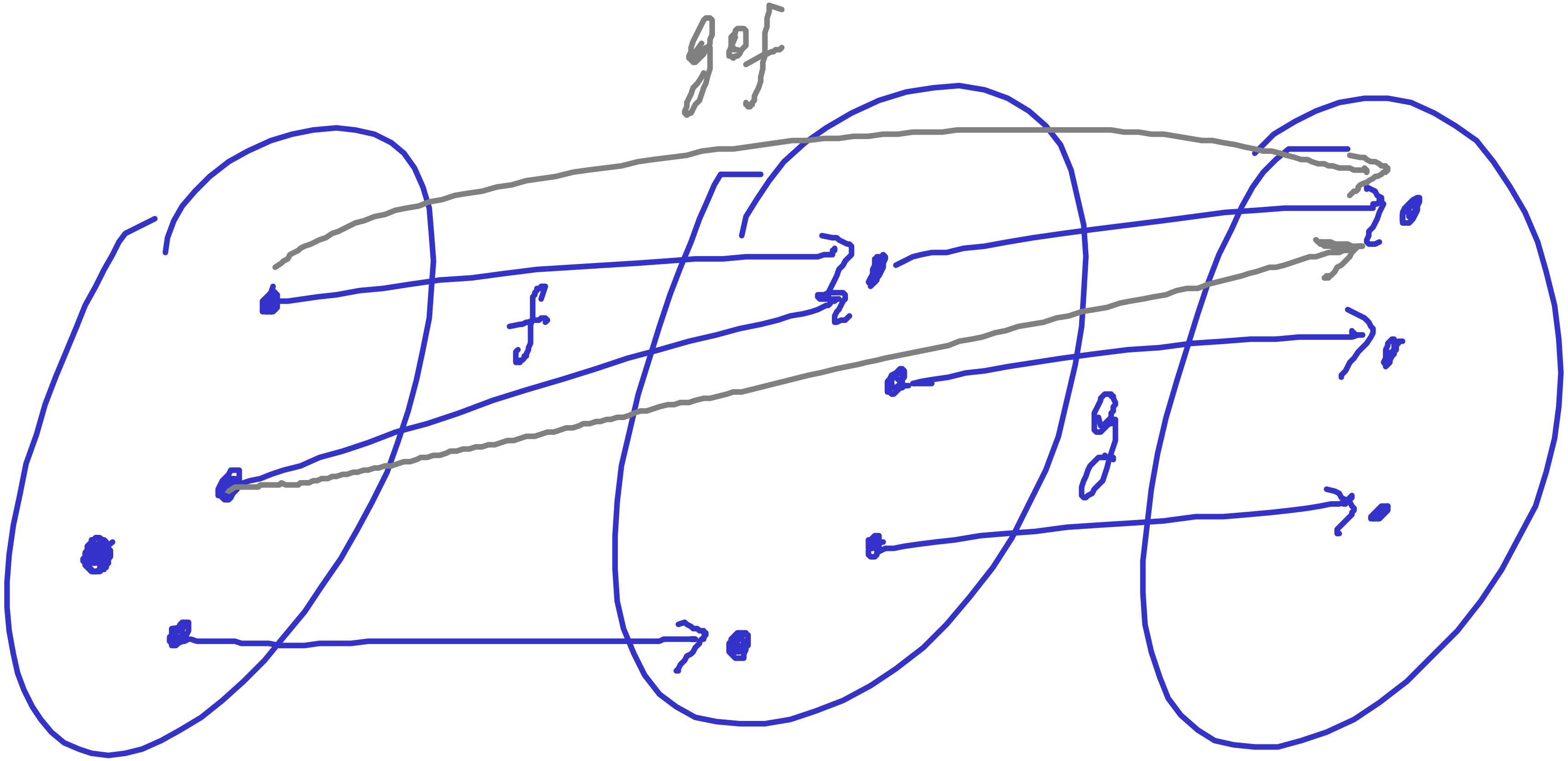
we typically give

① a domain of definition $D \subseteq A$, and

② a rule (or mapping) that to each a in D associates a unique! $b = f(a)$ in B, also written as

$$a \longmapsto b = f(a)$$

Theorem 95 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*



Extends division with remainder to integers.

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{Z}$:

- for $n \geq 0$ and $m > 0$,
  $$(n, m) \mapsto \big(\, \mathrm{quo}(n, m)\,,\; \mathrm{rem}(n, m)\,\big)$$
- for $n \geq 0$ and $m < 0$,
  $$(n, m) \mapsto \big(\, -\mathrm{quo}(n, -m)\,,\; \mathrm{rem}(n, -m)\,\big)$$
- for $n \leq 0$ and $m > 0$,
  $$(n, m) \mapsto \big(\, -\mathrm{quo}(-n, m) - 1\,,\; \mathrm{rem}(m - \mathrm{rem}(-n, m), m)\,\big)$$
- for $n \leq 0$ and $m < 0$,
  $$(n, m) \mapsto \big(\, \mathrm{quo}(-n, -m) + 1\,,\; \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m)\,\big)$$

Its domain of definition is $\{\, (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \,\}$.

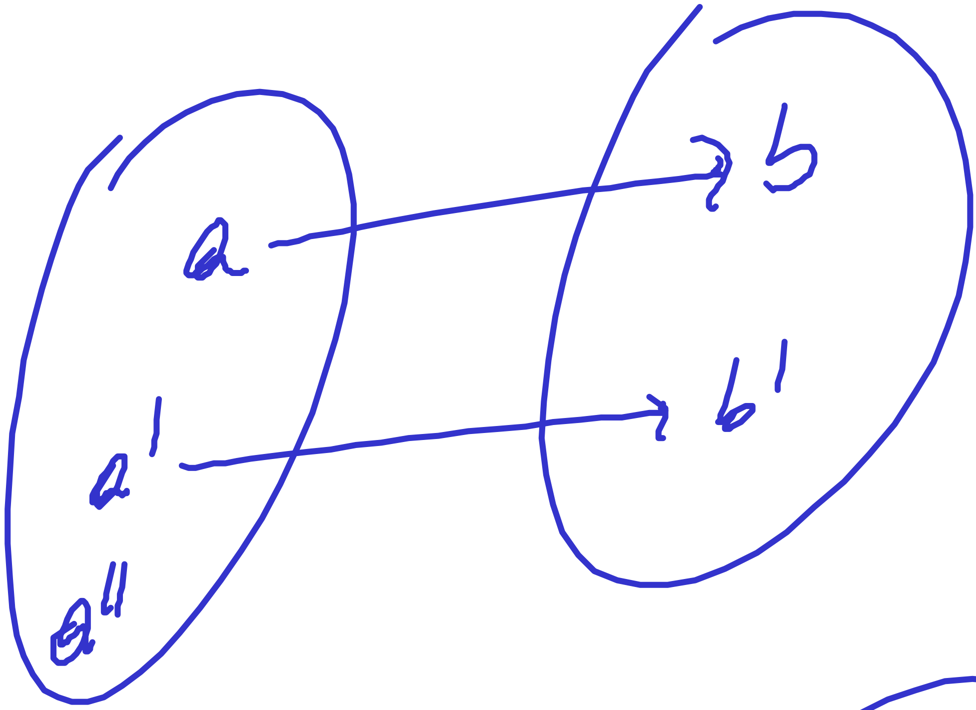
Proposition 96 *For all finite sets* A *and* B,

$$\#\underbrace{(A \rightharpoonup B)}_{\substack{\text{the set of all partial functions} \\ \text{from } A \text{ to } B\text{, also denoted} \\ \underline{\mathrm{PFun}}(A,B)\,.}} \;=\; (\#B+1)^{\#A} \;.$$

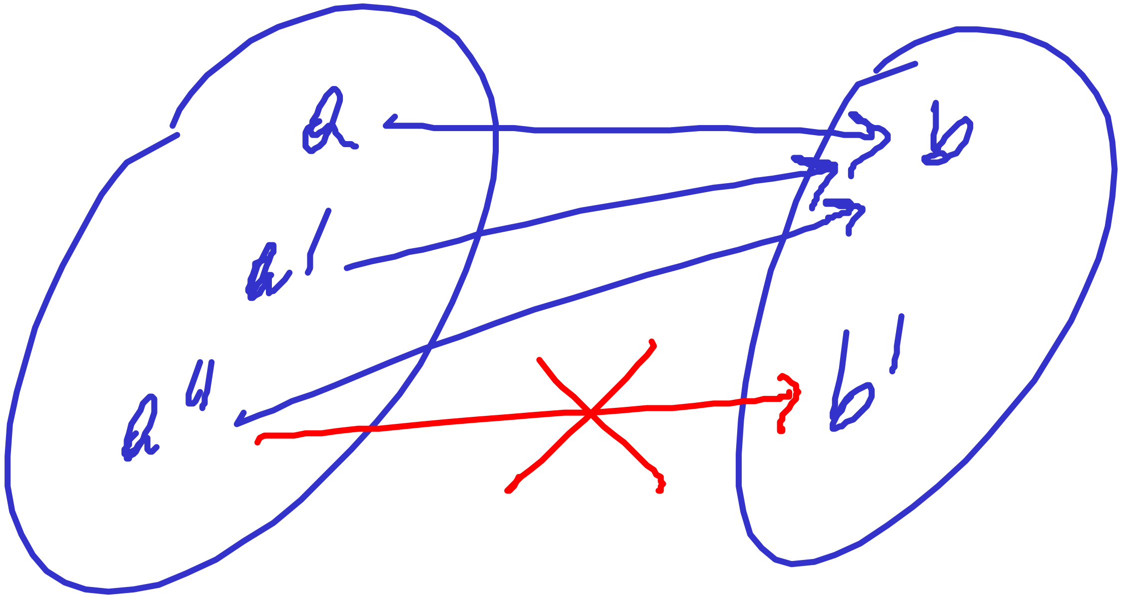
PROOF IDEA:

$$\begin{array}{cccc}
\uparrow & \uparrow & & \\
b_n & b_n & & \\
\vdots & \vdots & & \\
b_2 & b_2 & & \\
b_1 & b_1 & & \vdots \\
\uparrow & \uparrow & & \uparrow \\
\hline
a_1 & a_2 & \cdots & a_m
\end{array}$$

$$(n+1)\cdot(n+1)\cdot \qquad \cdot(n+1) = (n+1)^m$$



a
a'
a''
b
b'



a
a'
a''
b
b'

Functions (or maps)

Definition 97 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$$\underline{\mathrm{Fun}}(A,B) \subseteq \underline{\mathrm{PFun}}(A,B) \subseteq \underline{\mathrm{Rel}}(A,B)$$