In ML,
datatype
(α,β) disjunion
= one of α | two of β

For a set X, $\{\ell\} \times X$ is a set of elements of X tagged by ℓ

Disjoint unions

Definition 75 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B = \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1,a)\big) \vee \big(\exists b \in B.\, x = (2,b)\big) \,.$$

Proposition 76 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$

PROOF IDEA: $\#A = m$, $A = \{a_1, \ldots, a_m\}$

$\#B = n$, $B = \{b_1 \text{---} b_n\}$

$A \uplus B$

$$\underbrace{\underbrace{(1,a_1), \ldots, (1,a_m)}_{m}, \underbrace{(2,b_1), \ldots, (2,b_n)}_{n}}_{m+n}$$

sets of ordered pairs.

Relations

domain (or source)

Definition 77 *A* (binary) relation R *from a set* A *to a set* B

codomain (or target)

$$\mathrm{R} : \mathrm{A} \longrightarrow\!\!\!\!\!+ \mathrm{B} \quad \textit{or} \quad \mathrm{R} \in \mathrm{Rel}(\mathrm{A}, \mathrm{B}) \;,$$

$$\mathrm{Rel}(\mathrm{A}, \mathrm{B}) = \mathcal{P}(\mathrm{A} \times \mathrm{B})$$

is

$$\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$$

Notation 78 *One typically writes* $\mathrm{a} \,\mathrm{R}\, \mathrm{b}$ *for* $(\mathrm{a}, \mathrm{b}) \in \mathrm{R}$*.*

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

Examples:

- Empty relation.
  $\emptyset : A \rightarrow\!\!\!\!\!+\!\!\!\rightarrow B$ $\qquad$ ($a \; \emptyset \; b \iff \mathbf{false}$)

- Full relation.
  $(A \times B) : A \rightarrow\!\!\!\!\!+\!\!\!\rightarrow B$ $\qquad$ ($a \; (A \times B) \; b \iff \mathbf{true}$)

- Identity (or equality) relation.
  $I_A = \{ (a, a) \mid a \in A \} : A \rightarrow\!\!\!\!\!+\!\!\!\rightarrow A$ $\qquad$ ($a \; I_A \; a' \iff a = a'$)

- Integer square root.
  $R_2 = \{ (m, n) \mid m = n^2 \} : \mathbb{N} \rightarrow\!\!\!\!\!+\!\!\!\rightarrow \mathbb{Z}$ $\qquad$ ($m \; R_2 \; n \iff m = n^2$)

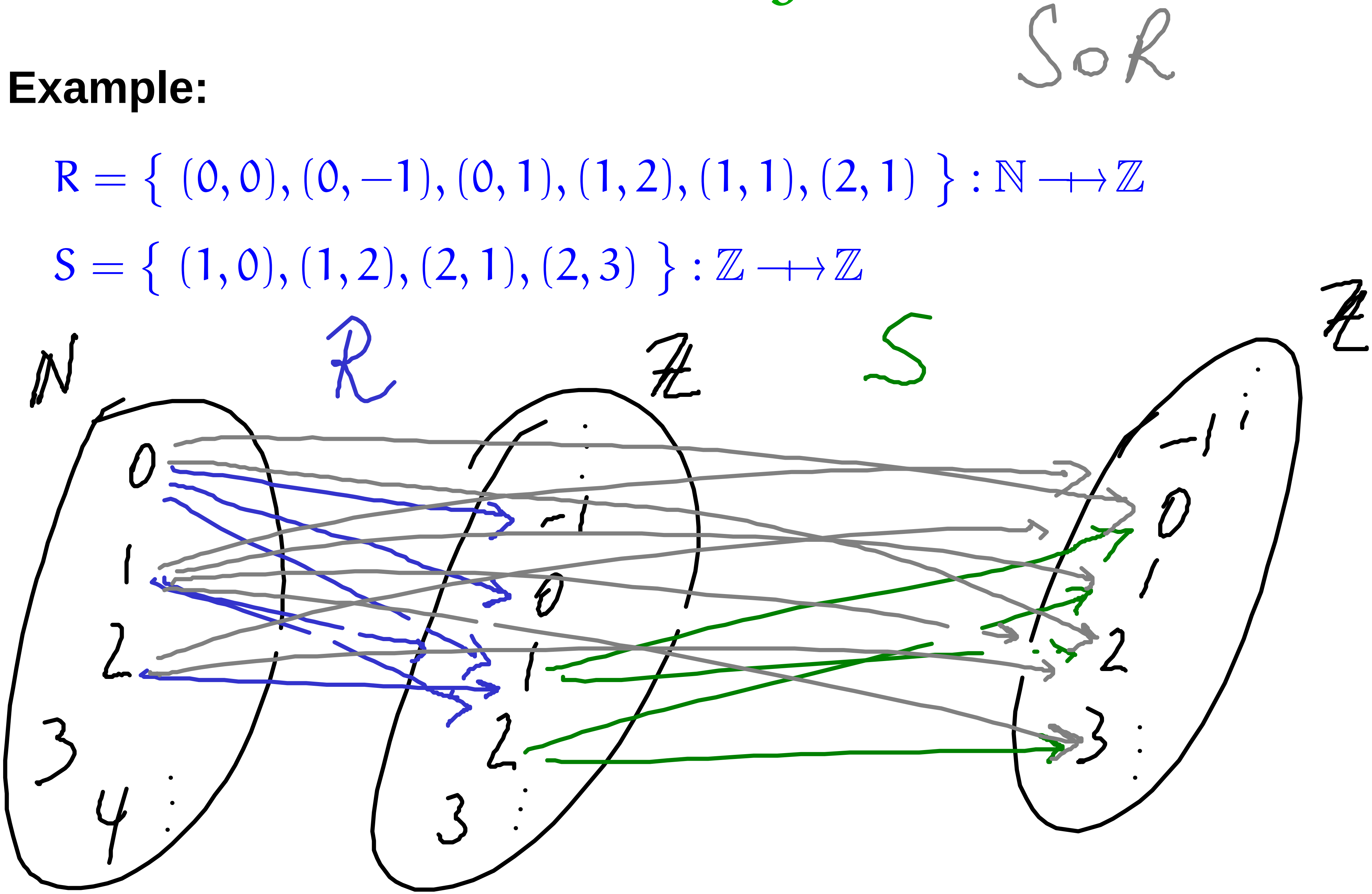
eg. $4 \; R_2 \; 2$, $4 \; R_2 \, (-2)$

Internal diagrams

Example:

$$R = \{\ (0,0),(0,-1),(0,1),(1,2),(1,1),(2,1)\ \} : \mathbb{N} \nrightarrow \mathbb{Z}$$

$$S = \{\ (1,0),(1,2),(2,1),(2,3)\ \} : \mathbb{Z} \nrightarrow \mathbb{Z}$$



Relational composition

Given

$$R : A \leftrightarrow B \quad , \quad S : B \leftrightarrow C$$

we define

$$S \circ R : A \leftrightarrow C$$

notation

$$R ; S : A \leftrightarrow C$$

by

$$a (S \circ R) c \iff \exists b \in B . \; a R b \; \& \; b S c$$

It is unambiguous to write $T \circ S \circ R$

Theorem 79 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \rightarrow\!\!\!\!+ B$, $S : B \rightarrow\!\!\!\!+ C$, *and* $T : C \rightarrow\!\!\!\!+ D$,

  $$(T \circ S) \circ R = T \circ (S \circ R)$$

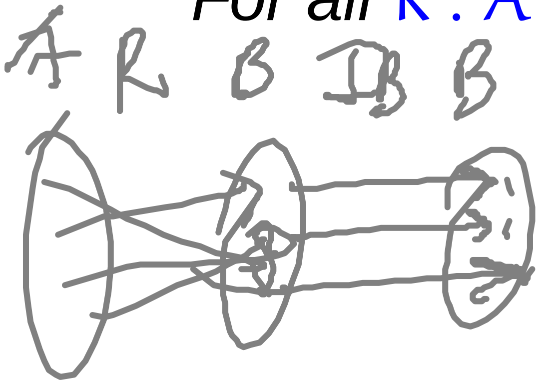
Exercise

Show $a\,[(T \circ S) \circ R]\,d$ iff $a\,[T \circ (S \circ R)]\,d$

- *Neutral element.*

  For all $R : A \rightarrow\!\!\!\!+ B$,

  $$R \circ I_A = R = I_B \circ R \, .$$



Relations and matrices

Definition 80

1. *For positive integers* m *and* n*, an* $(m \times n)$-*matrix* M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.

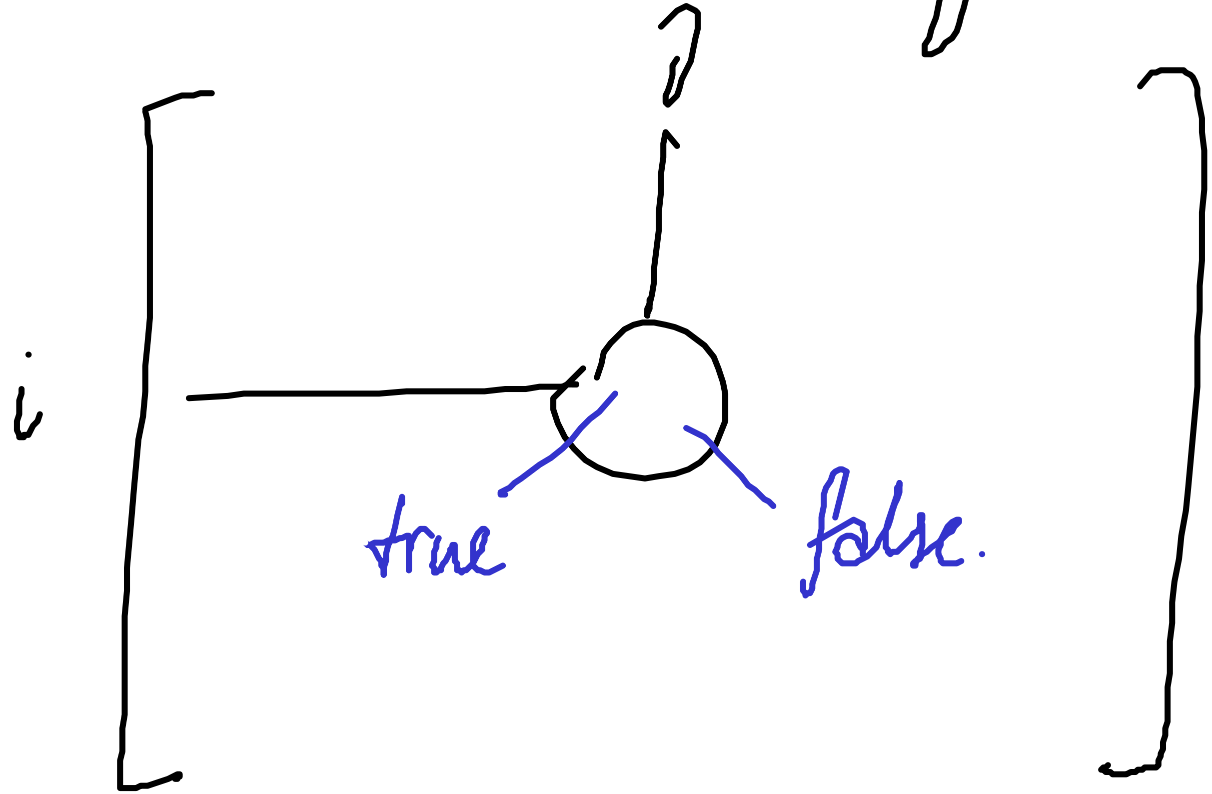
$(m \times n)$-matrix M $(n \times \ell)$-matrix L

$\rightsquigarrow$ $(m \times \ell)$-matrix $(L \cdot M)$

$$(L \cdot M)_{i,j} = \bigoplus_{k=0}^{n-1} L_{k,j} \odot M_{i,k}$$

Theorem 81 *Matrix multiplication is associative and has the identity matrix as neutral element.*

What if S is the Boolean semiring.

$M =$ 

Every M gives us a relation $\underline{rel}(M) : [m] \nrightarrow [n]$ where $[k] = \{0, 1, \ldots, k-1\}$ defined by

$$i \,(\underline{rel}\, M)\, j \iff M_{i,j} = \text{true}$$

Given a relation $R : [n] \nrightarrow [n]$

define $\underline{\mathrm{mat}}(R)$ by

$$\underline{\mathrm{mat}}(R)_{i,j} = \begin{cases} \text{true} & \text{if } iRj \\ \text{false} & \text{otw} \end{cases}$$

$$\begin{tikzcd}
M \arrow[r, mapsto] \arrow[d, equal] & \underline{\mathrm{rel}}(M) \arrow[dl, mapsto] \\
\underline{\mathrm{mat}}(\underline{\mathrm{rel}}(M)) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[r, mapsto] \arrow[d, equal] & \underline{\mathrm{mat}}(R) \arrow[dl, mapsto] \\
\underline{\mathrm{rel}}(\underline{\mathrm{mat}}\, R) &
\end{tikzcd}$$

$$R\,,\,S \longmapsto \underline{\text{mat}}(R)\,,\,\underline{\text{mat}}(S)$$

$$S \circ R \longmapsto \underline{\text{mat}}(S \circ R)$$

$$\| \text{ fact}$$

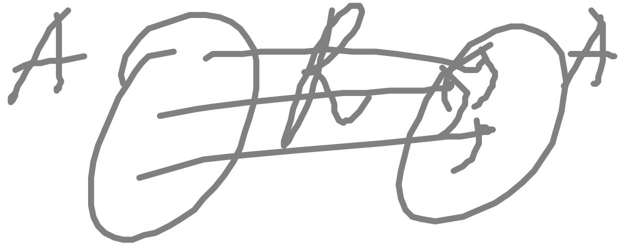
$$\underline{\text{mat}}(S) \cdot \underline{\text{mat}}(R)$$

$$Id_A \longmapsto \underline{\text{met}}(Id_A) = \begin{bmatrix} \text{true} & \text{false} \\ \text{false} & \text{true} \end{bmatrix}$$

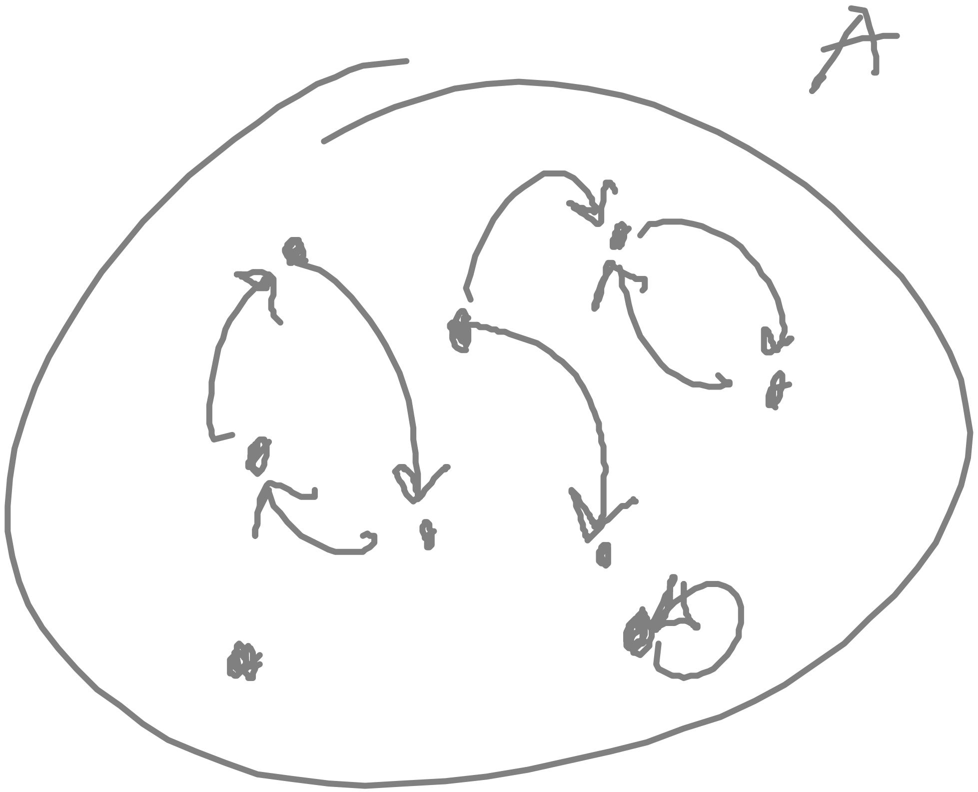
Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

Directed graphs



Definition 82 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



$R : A \mathrel{-\!\!\!+\!\!\!\rightarrow} A$

$R \circ R : A \mathrel{-\!\!\!+\!\!\!\rightarrow} A$

$R \circ R \circ R : A \mathrel{-\!\!\!+\!\!\!\rightarrow} A$

$\vdots$

Corollary 84 *For every set A, the structure*

$$(\ \mathrm{Rel}(A)\ ,\ \mathrm{I}_A\ ,\circ\ )$$

is a monoid.

Definition 85 *For $R \in \mathrm{Rel}(A)$ and $n \in \mathbb{N}$, we let*

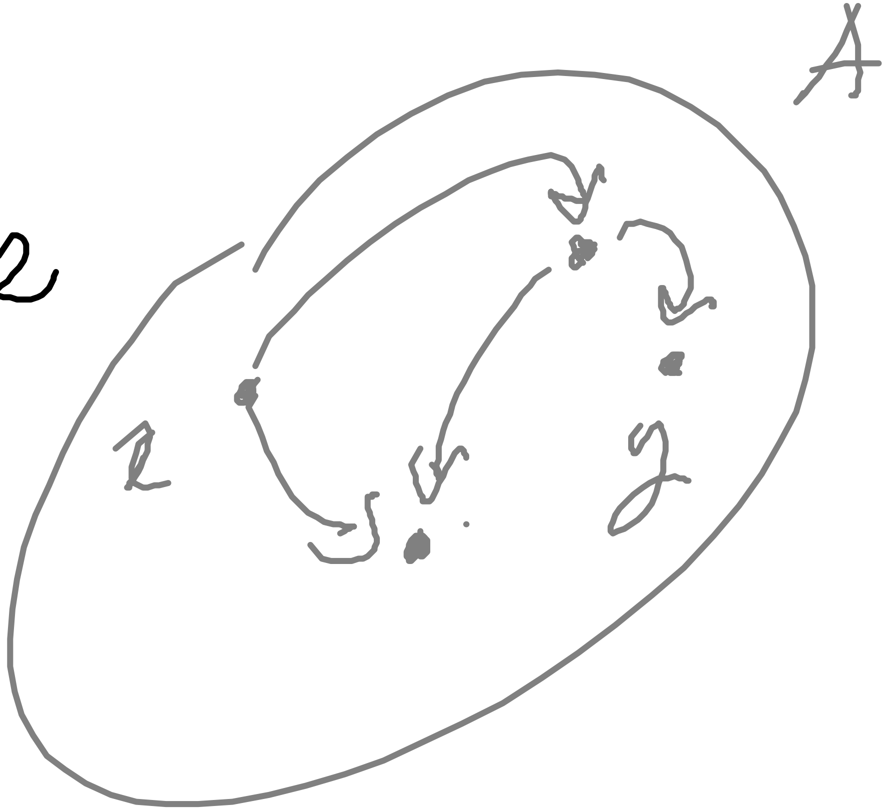
$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \in \mathrm{Rel}(A)$$

be defined as I_A for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$.

$R \circ R : A \to A$

$x \, (R \circ R) \, y$ iff $\exists z : xRz \ \& \ zRy$

iff There is a path of length 2 in the directed graph



<u>Claim</u> $x \, R^{\circ n} \, y$ iff There is a path of length n in the directed graph.

Paths

Proposition 86 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s \; R^{\circ n} \; t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF:

EXERCISE (by induction)

Definition 87 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \left\{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \right\} = \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 88 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with sourse* s *and target* t *in* R.