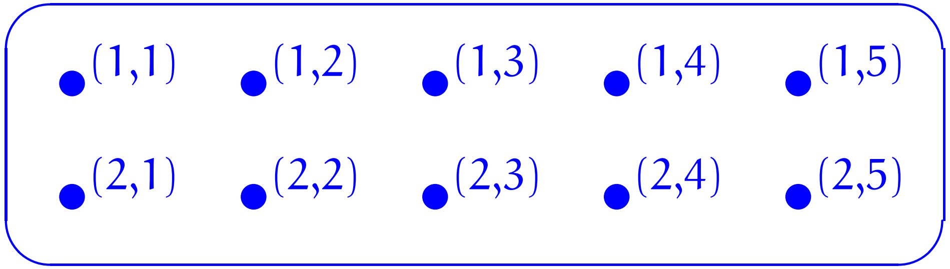
Sets

Objective

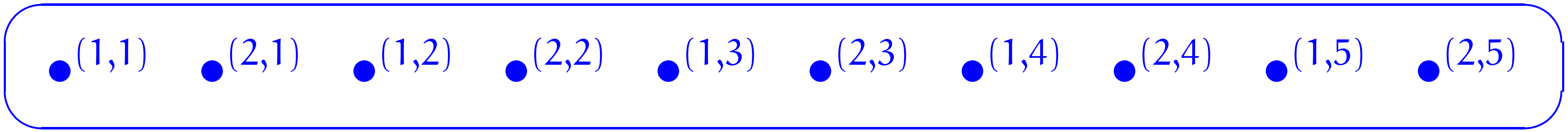
To introduce the basics of the theory of sets and some of its applications.

Sets

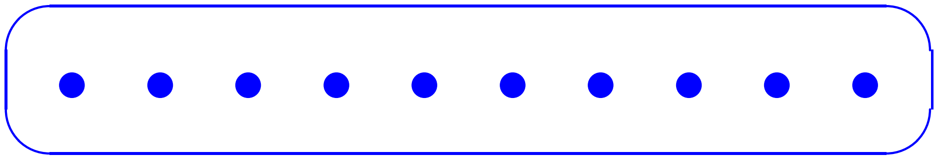
It has been said that a set is like a mental "bag of dots", except of course that the bag has no shape; thus,



may be a convenient way of picturing a certain set for some considerations, but what is apparently the same set may be pictured as



or even simply as



for other considerations.

Naive Set Theory

We are not going to be formally studying Set Theory here; rather, we will be *naively* looking at ubiquitous structures that are available within it.

$(\forall x.\ x \in A \Rightarrow x \in B)$

$\wedge (\forall x.\ x \in B \Rightarrow x \in A)$ $\Longleftrightarrow$ $\forall x.\ (x \in A \Rightarrow x \in B)$ & $(x \in B \Rightarrow x \in A)$

Extensionality axiom

Two sets are equal if they have the same elements.

Thus,

$$\forall \text{ sets } A, B.\ A = B \iff (\, \forall x.\, x \in A \iff x \in B \,) \quad .$$

Example:

$$\{0\} \neq \{0, 1\} = \{1, 0\} \neq \{2\} = \{2, 2\}$$

Subsets and supersets

$A \subseteq B$ A is a subset of B

Also B is a superset of A

Iff $\forall x.\ x \in A \Rightarrow x \in B$.

<u>NB</u> $A = B \iff (A \subseteq B \ \& \ B \subseteq A)$

Separation principle

For any set A and any definable property P, there is a set containing precisely those elements of A for which the property P holds.

Set comprehension

$$\{x \in A \mid \mathrm{P}(x)\} \subseteq A$$

$$a \in \{x \in A \mid P(x)\} \iff (a \in A \;\&\; P(a))$$

Russell's paradox

NB Separation does not allow the definition of a set unbounded

$$R = \{x \mid x \notin x\}$$

because it would yield

$$R \in R \iff R \notin R$$

$\underline{\text{NB}}$ $\forall$ sets A. $\emptyset \subseteq A$

Empty set

$$\emptyset \quad \text{or} \quad \{\,\}$$

defined by

$$\forall x.\, x \notin \emptyset$$

or, equivalently, by

$$\neg(\exists x.\, x \in \emptyset)$$

Cardinality

The *cardinality* of a set specifies its size. If this is a natural number, then the set is said to be *finite*.

Typical notations for the cardinality of a set S are $\#S$ or $|S|$.

Example:

$$\#\emptyset = 0$$

is the set of all subsets of U

Powerset axiom

$\boxed{\text{For any set, there is a set consisting of all its subsets.}}$

$\mathcal{P}(U)$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

Hasse diagrams

	#
$\emptyset$	0
$\mathcal{P}(\emptyset) = \{\emptyset\}$	1
$\mathcal{P}(\mathcal{P}\emptyset) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$	2
$\mathcal{P}(\mathcal{P}(\mathcal{P}\emptyset)) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$	4

<u>NB</u> $\mathcal{P}(A) \neq \emptyset$ because $\emptyset \in \mathcal{P}(A)$ & $A \in \mathcal{P}(A)$

Hasse diagrams

$\mathcal{P}(\{a,b\})$

$$\begin{tikzcd}
 & \{a,b\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{a\} \arrow[dr, dash] & & \{b\} \arrow[dl, dash] \\
 & \emptyset &
\end{tikzcd}$$

Proposition 70 *For all finite sets* U,

$$\#\,\mathcal{P}(\mathsf{U}) = 2^{\#\mathsf{U}} \quad .$$

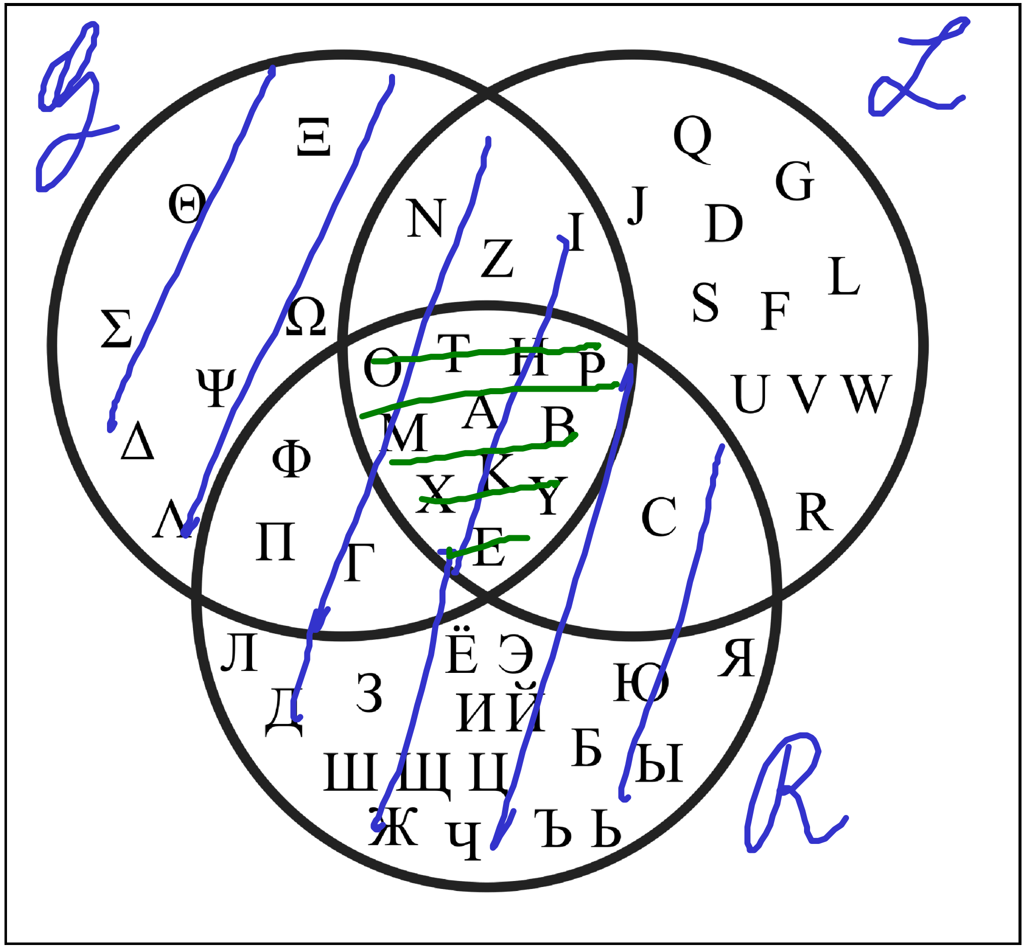
PROOF IDEA: $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$ $\qquad \#\mathcal{U} = n \in \mathbb{N}$

$$\left.\begin{array}{lcccc}
\emptyset & 0 & 0 & \cdots & 0 \\
\{x_1\} & 1 & 0 & \cdots & 0 \\
\vdots & & \vdots & & \\
\{x_1 - x_n\} & 1 & 1 & \cdots & 1
\end{array}\right\} 2^n$$

$\mathcal{G} \cap \mathcal{L} \cap \mathcal{R}$

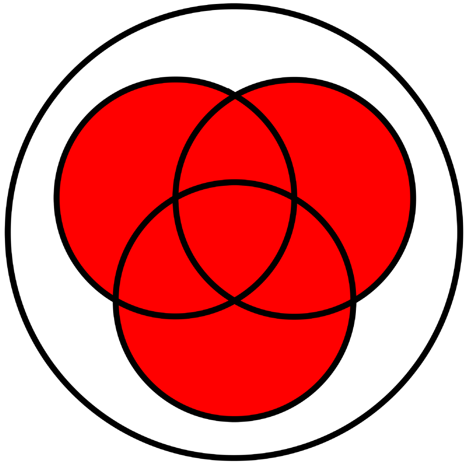
Venn diagrams[a]

$\mathcal{G} \cup \mathcal{R}$

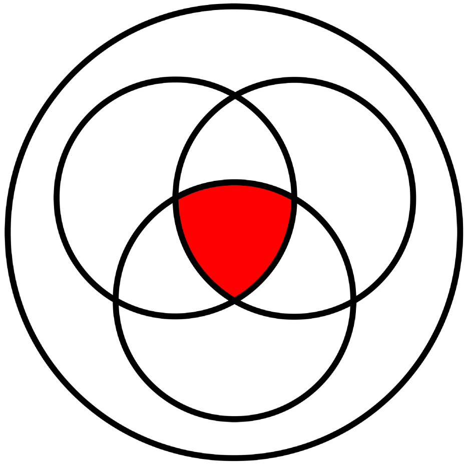


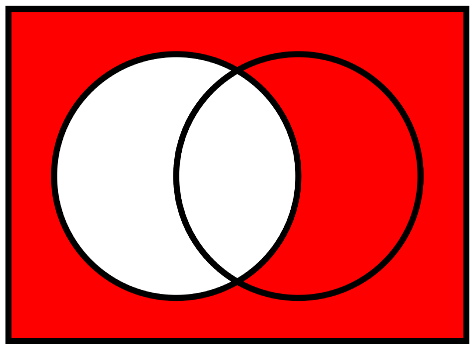
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$\left(\quad \mathcal{P}(U) \; , \quad \emptyset \; , \quad U \; , \quad \cup \; , \quad \cap \; , \quad (\cdot)^{c} \quad \right)$$

$$A \cup B \;=\; \{ x \in U \mid x \in A \vee x \in B \}$$

$$A \cap B \;=\; \{ x \in U \mid x \in A \;\&\; x \in B \}$$

$$A^{c} \;=\; \{ x \in U \mid \neg (x \in A) \}$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

$$A \cup A = A \qquad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathfrak{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A = A \qquad \mathcal{U} \cap A = A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$A \cap \emptyset = \emptyset \qquad A \cup \mathcal{U} = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are absorptive and distributive.

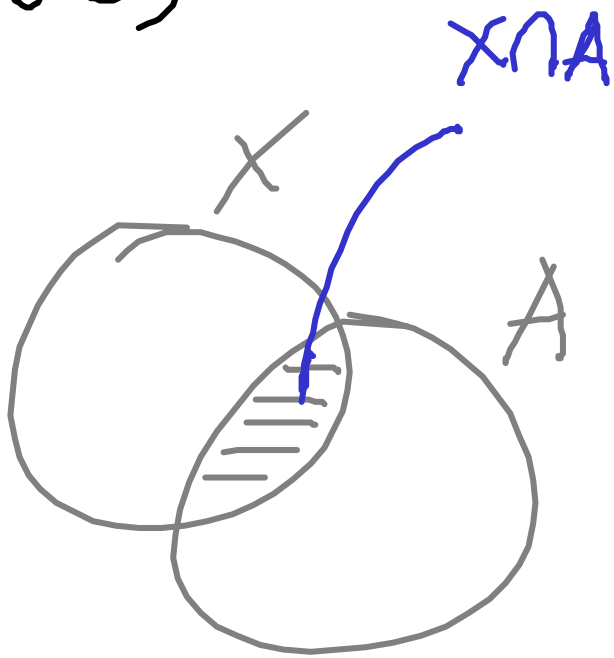
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

check the venn diagrams

$$X \cup (X \cap A) = X$$

$$X \cap (X \cup A) = X$$



- The complement operation $(\cdot)^c$ satisfies complementation laws.

$$X \cup X^c = \mathcal{U}$$

$$X \cap X^c = \emptyset$$