Discrete Mathematics

Exercise Sets

for Part I CST 2013/14

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Supervision 1: $\Longrightarrow$, $\forall$, &

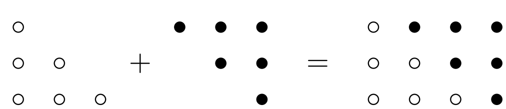
1. Solve Workouts 1–7. [Comment: You may postpone Workout 4.4 to after Workout 7.5.]

2. [Adapted from David Burton]

   (a) A natural number is said to be *triangular* if it is of the form $\sum_{i=0}^{k} i = 0+1+\cdots+k$, for some natural number k. For example, the first three triangular numbers are $t_0 = 0$, $t_1 = 1$, and $t_2 = 3$. Find the next three triangular numbers t_3, t_4, and t_5.

   (b) Find a formula for the k-th triangular number t_k.
   Hints:

   - Geometric approach: Observe that

   - Algebraic approach: Note that

     $$(n+1)^2 \;=\; \sum_{i=0}^{n}(i+1)^2 - \sum_{i=0}^{n} i^2 \;. \qquad (\dagger)$$

   (c) A natural number is said to be *square* if it is of the form k^2 for some natural number k.
   [Plutarch, circ. 100BC] Show that n is triangular iff $8n+1$ is square.

   (d) [Nicomachus, circ. 100BC] Show that the sum of every two consecutive triangular numbers is square.

   (e) [Euler, 1775] Show that, for all natural numbers n, if n is triangular, then so are $9 \cdot n + 1$, $25 \cdot n + 3$, and $49 \cdot n + 6$.

Supervision 2: $\exists$, $\vee$, $\neg$

1. Solve Workouts 8–9.

2. Solve Workouts 11–12.

3. [Euclid, circ. 300BC] Given any three positive real numbers, show that there exists a triangle whose sides have lengths of these sizes iff the sum of every two of those numbers is greater than the third.

Supervision 3: Number systems, the division theorem, modular arithmetic

1. Solve Workout 10.

2. Solve Workouts 13 and 14.

3. [Adapted from David Burton]

   A *decimal (respectively binary) repunit* is a natural number whose decimal (respectively binary) representation consists solely of 1's.

   (a) What are the first three decimal repunits? And the first three binary ones?

   (b) Show that no decimal repunit strictly greater than 1 is square, and that the same holds for binary repunits. Is this the case for every base?

Supervision 4: Greatest common divisor

1. Solve Workouts 15–18.

Supervision 5: Mathematical induction

1. Solve Workouts 19 and 20.

2. Prove that for all natural numbers $n \geq 3$, if n distinct points on a circle are joined in consecutive order by straight lines, then the interior angles of the resulting polygon add up to $180 \cdot (n-2)$ degrees.

3. Prove that, for any positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be tiled with L-shaped pieces consisting of 3 squares.

4. The set of *(univariate) polynomials* (over the rationals) on a variable x is defined as that of arithmetic expressions equal to those of the form $\sum_{i=0}^{n} a_i \cdot x^i$, for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in \mathbb{Q}$.

   (a) Show that if $p(x)$ and $q(x)$ are polynomials then so are $p(x) + q(x)$ and $p(x) \cdot q(x)$.

   (b) Deduce as a corollary that, for all $a, b \in \mathbb{Q}$, the linear combination $a \cdot p(x) + b \cdot q(x)$ of two polynomials $p(x)$ and $q(x)$ is a polynomial.

(c) Show that there exists a polynomial $p_2(x)$ such that, for every $n \in \mathbb{N}$, $p_2(n) = \sum_{i=0}^{n} i^2 = 0^2 + 1^2 + \cdots + n^2$.[1]

Hint: Note that for every $n \in \mathbb{N}$,

$$(n+1)^3 \quad = \quad \sum_{i=0}^{n}(i+1)^3 - \sum_{i=0}^{n} i^3 \ . \qquad (\ddagger)$$

(d) Show that, for every $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ such that, for all $n \in \mathbb{N}$, $p_k(n) = \sum_{i=0}^{n} i^k = 0^k + 1^k + \cdots + n^k$.

Hint: Generalise (†) and (‡).

[1]Chapter 2.5 of *Concrete Mathematics: A Foundation for Computer Science* by R.L. Graham, D.E. Knuth, and O. Patashnik looks at this in great detail.