Contextual preorder between PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \leq_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \implies \mathcal{C}[M_2] \Downarrow_\gamma V \ .$$

Result

$M_1 \leq_{ctx} M_2$ iff $[\![M_1]\!] \trianglelefteq M_2$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$, ~ def $\lhd_\gamma$

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \, (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \ .$$

At a function type $\tau \to \tau'$, ~ from $\lhd_{\tau \to \tau'}$ being logical

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \, (M_1 \, M \leq_{\mathrm{ctx}} M_2 \, M : \tau') \ .$$

<u>Applicative context:</u> $[\bullet] M$

Topic 8

Full Abstraction

Proof principle

For all types τ and closed terms $M_1, M_2 \in \mathrm{PCF}_\tau$,

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!] \implies M_1 \cong_{\mathrm{ctx}} M_2 : \tau \ .$$

Hence, to prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!] \ .$$

Full abstraction

A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.

- The domain model of PCF is *not* fully abstract.

  In other words, there are contextually equivalent PCF terms with different denotations.

$[\![T_1]\!], [\![T_2]\!] : (B_\bot \to (B_\bot \to B_\bot)) \to B_\bot$

Failure of full abstraction, idea

We will construct two closed terms

f definable, i.e. $f = [\![M]\!]$

$$T_1, T_2 \in \mathrm{PCF}_{(bool\to(bool\to bool))\to bool}$$

such that

$$T_1 \cong_{\mathrm{ctx}} T_2$$

$$[\![T_1]\!] f = [\![T_1]\!]([\![M]\!]) = [\![T_1 M]\!] = [\![T_2 M]\!] = [\![T_2]\!]([\![M]\!]) = [\![T_2]\!] f$$

and

$$[\![T_1]\!] \neq [\![T_2]\!]$$

iff $\exists f \in (B_\bot \to (B_\bot \to B_\bot)).\ [\![T_1]\!] f \neq [\![T_2]\!] f$

↳ such an f should be undefinable.

- We achieve $T_1 \cong_{\mathrm{ctx}} T_2$ by making sure that

$$\forall\, M \in \mathrm{PCF}_{bool\to(bool\to bool)}\,(\, T_1\, M \not\Downarrow_{bool}\ \&\ T_2\, M \not\Downarrow_{bool}\,)$$

  Hence,

$$[\![T_1]\!]([\![M]\!]) = \bot = [\![T_2]\!]([\![M]\!])$$

  for all $M \in \mathrm{PCF}_{bool\to(bool\to bool)}$.

- We achieve $[\![T_1]\!] \neq [\![T_2]\!]$ by making sure that

$$[\![T_1]\!](por) \neq [\![T_2]\!](por)$$

  for some *non-definable* continuous function

$$por \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot))\ .$$

Parallel-or function

is the unique continuous function $por : \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)$ such that

$$
\begin{aligned}
por\ true\ \perp &= true \\
por\ \perp\ true &= true \\
por\ false\ false &= false
\end{aligned}
$$

In which case, it necessarily follows by monotonicity that

$$
\begin{aligned}
por\ true\ true &= true & \qquad por\ false\ \perp &= \perp \\
por\ true\ false &= true & \qquad por\ \perp\ false &= \perp \\
por\ false\ true &= true & \qquad por\ \perp\ \perp &= \perp
\end{aligned}
$$

There is a denotational model of stable functions

continuous & (intuitively) there is minimal input to produce some output

Undefinability of parallel-or

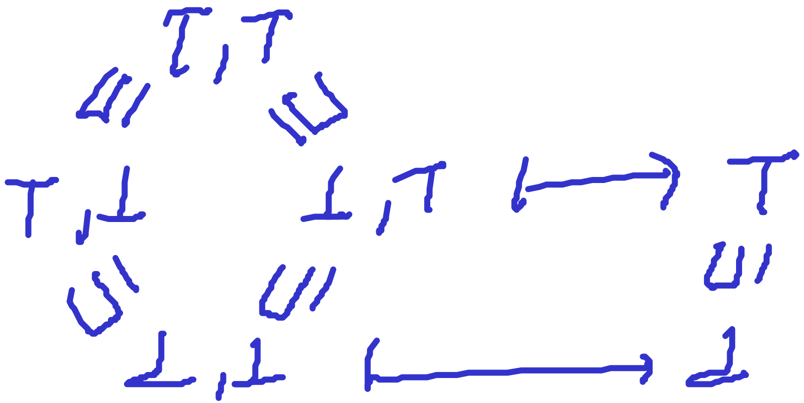
Proposition. *There is no closed PCF term*

$$P : bool \to (bool \to bool)$$

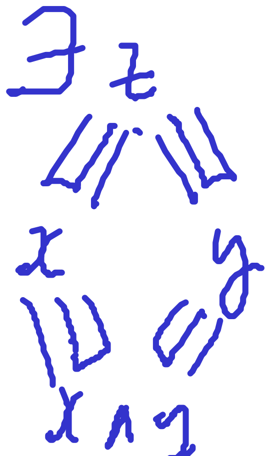
satisfying

$$[\![P]\!] = por : \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp) \ .$$

por is not stable



formally



$fx \quad fy$

$fx \wedge fy = f(x \wedge y)$

Parallel-or test functions

For $i = 1, 2$ define

$$
\begin{aligned}
T_i \stackrel{\text{def}}{=}\ & \mathbf{fn}\ f : \mathit{bool} \to (\mathit{bool} \to \mathit{bool})\ .\\
& \quad \mathbf{if}\ (f\ \mathbf{true}\ \Omega)\ \mathbf{then}\\
& \quad\quad \mathbf{if}\ (f\ \Omega\ \mathbf{true})\ \mathbf{then}\\
& \quad\quad\quad \mathbf{if}\ (f\ \mathbf{false}\ \mathbf{false})\ \mathbf{then}\ \Omega\ \mathbf{else}\ B_i\\
& \quad\quad \mathbf{else}\ \Omega\\
& \quad \mathbf{else}\ \Omega
\end{aligned}
$$

$$
\frac{\dfrac{\mathrm{fn}\,x.x \Downarrow \mathrm{fn}\,x.x \qquad \mathrm{fix}(\mathrm{fn}\,x.x)\Downarrow}{(\mathrm{fn}\,x.x)(\mathrm{fix}(\mathrm{fn}\,x.x))\Downarrow}}{\mathrm{fix}(\mathrm{fn}\,x.x)\Downarrow}
$$

where $B_1 \stackrel{\text{def}}{=} \mathbf{true}$, $B_2 \stackrel{\text{def}}{=} \mathbf{false}$,
and $\Omega \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\ x : \mathit{bool}\ .\ x)$.

$[\![\Omega]\!] = \bot$

Failure of full abstraction

Proposition.

$$T_1 \cong_{\mathrm{ctx}} T_2 : (bool \to (bool \to bool)) \to bool$$

$$[\![T_1]\!] \neq [\![T_2]\!] \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot)) \to \mathbb{B}_\bot$$

PCF+por

Expressions $M ::= \cdots \mid \mathbf{por}(M, M)$

Typing $$\frac{\Gamma \vdash M_1 : bool \quad \Gamma \vdash M_2 : bool}{\Gamma \vdash \mathbf{por}(M_1, M_2) : bool}$$

Evaluation

$$\frac{M_1 \Downarrow_{bool} \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{true}} \qquad \frac{M_2 \Downarrow_{bool} \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{true}}$$

$$\frac{M_1 \Downarrow_{bool} \mathbf{false} \quad M_2 \Downarrow_{bool} \mathbf{false}}{\mathbf{por}(M_1, M_2) \Downarrow_{bool} \mathbf{false}}$$

Plotkin's full abstraction result

The denotational semantics of PCF+por is given by extending that of PCF with the clause

$$[\![\Gamma \vdash \mathbf{por}(M_1, M_2)]\!](\rho) \stackrel{\text{def}}{=} por\big([\![\Gamma \vdash M_1]\!](\rho)\big)\big([\![\Gamma \vdash M_2]\!](\rho)\big)$$

This denotational semantics is fully abstract for contextual equivalence of PCF+por terms:

$$\Gamma \vdash M_1 \cong_{\mathrm{ctx}} M_2 : \tau \;\Leftrightarrow\; [\![\Gamma \vdash M_1]\!] = [\![\Gamma \vdash M_2]\!].$$

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